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Adaptive Control of Linear Periodic Systems

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Abstract— **This paper deals with the adaptive control of linear systems with periodically time-varying parameters (referred to as LTP systems) in discrete-time. While a large literature exists on the adaptive control of linear time-invariant (LTI) systems, practically very little is known regarding adaptation when the parameters vary periodically. This may be attributed to the fact that adaptive control can be attempted only after the properties of the corresponding deterministic systems (i.e systems with all parameter known) are well understood and design procedures are well established. In contrast to LTI systems, even though LTP systems have been studied for decades, their properties are not well understood and design procedures for stabilizing and controlling them are not straightforward. However, as discussed in this paper, there are compelling reasons for dealing with the adaptive control of such systems. In the following chapters, an attempt is made to answer several questions related to LTP systems, some of which are listed below: How much of adaptive control theory of LTI systems can be extended to the LTP case? What accounts for the difficulties encountered in such extensions? To what extent are the methods used in the two cases similar, and when are they different? Further, when they are different, what reasons can be attributed for the differences? Throughout the paper many comments are included to improve its readability.**

I. Introduction

Ordinary differential equations with periodic coefficients have a long history in mathematics and physics going back to the contributions of Faraday, Mathieu, Floquet, Rayleigh, Hill and Poincare [3-8] in the nineteenth century. They form an important intermediate class of systems bridging the time-invariant realm to the general time-varying one. With major advances in control theory, systems theory, and signal processing in the second half of the twentieth century, there was renewed interest in the study of periodic systems in both continuous and discrete time. One of the reasons for this interest in periodic systems was the appearance of a larger number of industrial processes which work in a periodic regime in steady-state conditions, often leading to periodically varying parameters. These include helicopters [9-12], engines [13-14], and satellites [15-18] As in the case of linear time-invariant systems, parametric uncertainties are invariably present in such systems which, in turn, call for adaptive solutions. However, while control methods have been investigated extensively for deterministic systems where the periodic parameters are known, the corresponding adaptive control problems have not received the same degree of attention, and it is safe to say that very few results currently exist in this area.

The main results related to the adaptive control of discrete LTP systems are included in this paper. The discrete SISO LTP system considered is described by

$$\begin{aligned} \Sigma_d: \quad x(k+1) = &\quad A(k)x(k) + b(k)u(k) \quad A(k) \in \mathbb{R}^{n\times n} \\ y(k) = &\quad c^T(k)x(k) \quad b(k), c(k) \in \mathbb{R}^n \end{aligned} \tag{1}$$

where $A(k)$, $b(k)$, and $c(k)$ are N-periodic matrix and vectors respectively, and N is specified. In the time-invariant case the following increasingly complex problems:

a) adaptive stabilization, regulation, and tracking with all state variables accessible, and
b) adaptive control of LTI systems with only the input and output of the plant accessible.

II. Stabilization - State Vector Accessible

For LTI systems, controllability assures stabilizability. This is generally demonstrated by showing that a nonsingular transformation exists which transforms the system into controllable canonical form. For LTP systems such a transformation does not in general exist, However, using the Riccati equation [24] it has been shown that a periodic time-varying gain $g(k)$ exists such that $A(k)+b(k)g^T(k)$ is asymptotically stable. It is this result together with adaptive laws derived for static systems in [23] that we use to solve the adaptive stabilization problem in this section.

The general problem of adaptively stabilizing a dynamical system can be stated in terms of a plant of the form described by

$$\begin{aligned} \Sigma_d: \quad x(k+1) = &\quad A(k)x(k) + b(k)u(k) \\ y(k) = &\quad c^T(k)x(k) \end{aligned} \tag{2}$$

where $A(k)$ is an unknown and unstable matrix, $b(k)$ is an unknown, bounded, and N-periodic vector, and $c(k)$ is specified and N-periodic. The aim is to determine a bounded control input u so that all the signals in the system remain bounded and

$$\lim_{k\to\infty} y(k) = 0, \quad \text{or equivalently,,} \quad \lim_{k\to\infty} x(k) = 0. \tag{3}$$

To meet this control objective, we assume:

I. the pair ($A(k)$, $b(k)$) is controllable and the pair ($c(k)$, $A(k)$) is observable.
II. $c^T(k+1)b(k)$ is sign definite and is assumed to be positive.
III. $|c^T(k+1)b(k)| < \eta$, and η is known.
IV. the state vector $x(k)$ is accessible.

Comment: If ($A(k)$, $b(k)$) is controllable, it has been shown [24] that a vector $g(k)$ exists such that $A(k) + b(k)g^T(k)$ is asymptotically stable. If $y(k)$ can be regulated around zero, the observability of ($c(k)$, $A(k)$) assures that the state vector $x(k)$ will be bounded, and that $\lim_{k\to\infty} |x(k)| = 0$. Assuming $g(k)$ has the form

$$g(k) = -\frac{c^T(k+1)A(k)}{c^T(k+1)b(k)} + a_m \frac{c^T(k+1)}{c^T(k+1)b(k)} \tag{4}$$

where $|a_m| < 1$, the plant Σ_d is stabilized and can be written as

$$y(k+1) = a_m y(k). \tag{5}$$

When $A(k)$ and $b(k)$ are unknown, the control input u is chosen to have the form

$$u(k) = \hat{g}^T(k)x(k) \tag{6}$$

where $\hat{g}(k)$ is the adjustable parameter. The adaptation of $\hat{g}(k)$ should be such that $\hat{g}(k)$ evolves to the periodic gain $g(k)$. Defining the parameter error $\tilde{g}$ as

$$\tilde{g}(k) = \hat{g}(k) - g(k), \tag{7}$$

the plant Σ_d together with the controller in (6) can be described by

$$\begin{aligned} y(k+1) = \; & c^T(k+1)[A(k)x(k) + b(k)g^T(k)]x(k) \\ & + c^T(k+1)b(k)\tilde{g}^T(k)x(k) \\ = \; & a_m y(k) + c^T(k+1)b(k)\tilde{g}^T(k)x(k). \end{aligned} \tag{8}$$

or equivalently

$$y(k+1) - a_m y(k) = c^T(k+1)b(k)\tilde{g}^T(k)x(k) \tag{9}$$

The rule for updating $\hat{g}(k)$ is given by

$$\hat{g}(k+N) = \hat{g}(k) - \frac{\epsilon(k)}{\eta}x(k) \tag{10}$$

where $\epsilon(k) = y(k+1) - a_m y(k)$. Since $(c(k), A(k))$ is observable, the convergence of output $y(k)$ to the origin implies that $\lim_{k\to\infty} x(k) = 0$.

III. Regulation and Tracking - State Vector Accessible

Regulation and tracking are fundamental problems of both control theory and adaptive control theory. In Section II the problems of stabilization of an LTP system was studied under the assumption that $(A(k), b(k))$ is controllable. Given the system Σ_d in (2), the problems of regulation and tracking are treated in Sections III-A- III-C respectively when the plant state vector $x(k)$ is assumed to be accessible.

A. Output Set Point Regulation

In this section the problem of regulating $y(k)$ around any non-zero value y^* is addressed. It is known that if the input to a stable LTI system is a constant the output in general has a constant value. The exception is when the plant transfer function $H(z)$ ($H(z) = c^T(zI - A)^{-1}b$) has a zero at 1, or equivalently $c^T(I - A)^{-1}b = 0$. This implies that in general , by scaling a constant input, the output can be regulated around any desired constant value. In the LTP case, the problem is obviously more complex. A constant input does not result in a constant output, and the input needed to maintain the output constant is not immediately evident even when the parameters are known. Hence determining conditions under which the output can be regulated around a constant value in the periodic case need to be derived.

1) Algebraic Part: In the previous section, it was shown that if $\{A(k), b(k)\}$ is controllable, the time-periodic system could be stabilized using a periodic feedback parameter, so that the output y is regulated around $y(k) \equiv 0$. Therefore, without loss of generality, $A(k)$ is assumed to be stable in the rest of this section. Further, from Floquet theory [5] it is known that a periodic non-singular matrix $P(k)$ exists such that the transformation $\bar{x}(k) = P(k)x(k)$, results in the equations

$$\bar{\Sigma}_d : \quad \begin{aligned} \bar{x}(k+1) = \; & \bar{A}\bar{x}(k) + \bar{b}(k)v(k) \\ y(k) = \; & \bar{c}(k)^T\bar{x}(k) \end{aligned} \tag{11}$$

where $\bar{b}(k) = P(k)b(k)$, $\bar{c}^T(k) = c^T(k)P^{-1}(k)$, and $\bar{A} = P(k+1)A(k)P^{-1}(k)$ is a constant matrix.

Comment: Given $\bar{A}$ is stable and time-invariant and $\bar{b}(k)$ is N-periodic, we have

$$y(k) = \bar{c}^T(k)x(k) = \bar{c}^T(k)\bar{A}^{k-1}x(1) + \sum_{\tau=1}^{k} \bar{A}^{k-\tau-1}\bar{b}(\tau)u(\tau).$$

It is evident from the above equation that $u(\tau)$ must be N-periodic for regulation of $y(k)$ around a nonzero constant.

From equation (11) it follows that

$$\begin{aligned} y(k+1) = \; & \bar{c}^T(k+1)R[\bar{A}^{N-1}\bar{b}(k)u(k) \\ & + \bar{A}^{N-2}\bar{b}(k-1)u(k-1) \\ & + \cdots + \bar{b}(k+1)u(k+1)] \\ y(k+2) = \; & \bar{c}^T(k+2)R[\bar{A}^{N-1}\bar{b}(k+1)u(k+1) \\ & + \bar{A}^{N-2}\bar{b}(k)u(k) \\ & + \cdots + \bar{b}(k-N)u(k-N)] \\ & \vdots \\ y(k+N) = \; & \bar{c}^T(k)R[\bar{A}^{N-1}\bar{b}(k-1)u(k-1) \\ & + \bar{A}^{N-2}\bar{b}(k-2)u(k-2) \\ & + \cdots + \bar{b}(k)u(k)] \end{aligned} \tag{12}$$

where $R = (I - \bar{A}^N)^{-1}$. Note that for any initial state, only N values of the input $u(k), u(k+1), \cdots, u(k+N-1)$ need to be determined.

If $y(k+1) = y(k+2) = \cdots = y(k+N) = y^*$ (a constant), Equation (12) can be modified as

$$M \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-N+1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} y^* \tag{13}$$

where matrix M is defined in Equation (14)

Using a result from linear algebra, it can be shown that if the following condition is satisfied:

$$[1\ 1\ \cdots\ 1]^T \in \text{image } M, \tag{15}$$

then there exists at least one solution $u(k)$ to the regulation problem. Further, a unique solution $u(k)$ exists if matrix M is nonsingular.

In summary, we have the following theorem:

Theorem 1: The output of Σ_d can be regulated around a constant value y^*, if $(A(k), b(k))$ is controllable and $[1\ 1\ \cdots\ 1]^T \in$ image M. A unique input sequence $\{u(k)\}$ exists, if and only if the matrix M defined in (14) is nonsingular.

Comment: Unlike LTI systems, set-point regulation of an LTP system requires a periodic control input. Also, note that the solution to the regulation problem may not be unique.

2) Analytical Part: Having established the conditions for the existence of solutions for the deterministic problem, we turn our attention to the analytical part.

Relative degree: The importance of the relative degree in the adaptive control of LTI systems is well known. For an LTI system $d = n - m$ (the number of poles - the number of zeros of the transfer function) represents the relative degree. It is the number of time instants after which an input affects the output. By definition, since the system is time-invariant, this applies to every instant of time and hence is well defined. In the time-periodic case, since the parameters are time-varying, the question arises whether a well defined relative degree exists. From equation (2) it follows that

$$\begin{aligned} y(k+1) = \;& c^T(k+1)x(k+1) \\ = \;& c^T(k+1)A(k)x(k) + c^T(k+1)b(k)u(k). \end{aligned} \tag{16}$$

If $c^T(k+1)b(k) \neq 0$, the input $u(k)$ affects the output y at time $(k+1)$. For a well defined relative degree, it therefore follows that the above condition should hold for all k. Since the system is time-periodic, this implies

$$c^T(k+1)b(k) \neq 0 \qquad \forall k_0 \leq k < k_0 + N. \tag{17}$$

Similarly for a well defined relative degree 2, we have

$$\begin{aligned} y(k+2) = \;& c^T(k+2)\{A(k+1)a(k)x(k) \\ & +A(k+1)b(k)u(k) + b(k+1)u(k+1)\} \end{aligned} \tag{18}$$

since

$$\left.\begin{aligned} c^T(k+2)b(k+1) &= 0 \\ c^T(k+2)A(k+1)b(k) &\neq 0 \end{aligned}\right. \forall k_0 \leq k < k_0 + N$$

This can therefore be generalized to the case of any relative degree $d > 1$ and can be stated as follows.

$$\begin{aligned} c^T(k+j) \prod_{i=k+j-1}^{k+1} A(i)b(k) \quad &= 0, \quad (j < d) \\ c^T(k+d) \prod_{i=k+d-1}^{k+1} A(i)b(k) \quad &\neq 0. \end{aligned} \tag{19}$$

In the following discussions, we shall assume that the plant to be controlled has a well defined relative degree. Also, for convenience in the discussions and practical purposes, we shall first assume that the relative degree $d = 1$, and then extend the results to cases where $d > 1$.

In the earlier discussion it was assumed that the matrix $A(k)$ was asymptotically stable. In the general case, $A(k)$ may be unstable and may have to be stabilized on-line using state feedback. This problem may now be stated as follows:

Problem Statement: A linear time-periodic system is described by equation (2). It is not known a priori that $A(k)$ is stable. The conditions for stabilizability and output regulation are satisfied by the plant. Determine a control input of the form

$$u(k) = g^T(k)x(k) + v(k) \tag{20}$$

such that $\lim_{k\to\infty} y(k) = y^*$, a specified constant. To meet this control objective, we assume

I. the sign of $c^T(k+1)b(k)$ is known, and without loss of generality $c^T(k+1)b(k) > 0,\ \forall\ k \in \mathbb{Z}^+$.
II. $|c^T(k+1)b(k)| < \eta$, and η is known.
III. the state vector $x(k)$ is accessible.

We assume that periodic parameters $g^*(k) \in \mathbb{R}^n$ and $v^*(k) \in \mathbb{R}$ exist (and are unknown) such that the desired objective is achieved. To realize the parameters adaptively, $g(k)$ and $v(k)$ are adjusted on-line. Defining $g(k) - g^*(k) = \tilde{g}(k)$, $v(k) - v^*(k) = \tilde{v}(k)$, the regulation error $e(k+1) = y(k+1) - y^*$ can be expressed as

$$e(k+1) = c^T(k+1)b(k)[\tilde{g}(k)x(k) + \tilde{v}(k)]. \tag{21}$$

Since sgn$(c^T(k+1)b(k)) > 0$ and $|c^T(k+1)b(k)| < \eta$, $\forall$ $k \in \mathbb{Z}^+$, the adaptive laws can be derived as

$$\begin{aligned} g^T(k+N) \quad &= g^T(k) - \tfrac{\gamma}{\eta}e(k+1)x^T(k) \\ v(k+N) \quad &= v(k) - \tfrac{\gamma}{\eta}e(k+1). \end{aligned} \tag{22}$$

Comment: In the above solution it was tacitly assumed that the relative degree of the plant is well defined and is unity. The same procedure can also be extended to cases where $d > 1$. In such a case, the adaptive law has the form

$$\begin{aligned} g^T(k+N) \quad &= g^T(k) - \tfrac{\gamma}{\eta}e(k+d)x^T(k) \\ v(k+N) \quad &= v(k) - \tfrac{\gamma}{\eta}e(k+d). \end{aligned} \tag{23}$$

$$M = \begin{bmatrix} \bar{c}^T(k+1)R\bar{A}^{N-1}\bar{b}(k) & \bar{c}^T(k+1)R\bar{A}^{N-2}\bar{b}(k-1) & \cdots & \bar{c}^T(k+1)R\bar{b}(k+1) \\ \bar{c}^T(k+2)R\bar{A}^{N-2}\bar{b}(k) & \bar{c}^T(k+2)R\bar{A}^{N-3}\bar{b}(k-1) & \cdots & \bar{c}^T(k+2)R\bar{A}\bar{b}(k+1) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}^T(k)R\bar{b}(k) & \bar{c}^T(k)R\bar{A}^{N-1}\bar{b}(k-1) & \cdots & \bar{c}^T(k)R\bar{A}\bar{b}(k+1) \end{bmatrix} \tag{14}$$

Comment:Note that a well defined relative degree has to be assumed in the analytical part, while it is not required in the algebraic part.

B. Tracking an N-Periodic Signal $y^*(k)$

The general problem of determining a control input $u(k)$ so that the output of a given system follows an arbitrary desired output $y^*(k)$ asymptotically is referred to as the tracking problem. The method discussed in the previous section to track a constant signal y^* can be directly extended to a periodic signal, where the period of $y^*(k)$ is also N-periodic. However, as described below the consequences are somewhat different.

Modify Eq (13) as

$$M \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-N+1) \end{bmatrix} = \begin{bmatrix} y^*(1) \\ y^*(2) \\ \vdots \\ y^*(N) \end{bmatrix}. \tag{24}$$

It can be shown, in a straightforward way, that a unique solution of $u(k)$ exists if and only if matrix M is invertible.

Comment: Unlike the regulation problem, where the right handside of equation (24) was a constant vector, all of whose elements are unity, the r.h.s. in the present case consists of arbitrary vectors. Hence, the solution of tracking any N-periodic signal $y^*(k)$, if it exists, must be unique.

C. Tracking a General Signal $y^*(k)$

We have thus far considered the regulation problem and the problem of tracking an N-periodic signal that is mathematically equivalent to it. We now proceed to consider the more general case where the output $y^*(k)$ to be followed is any arbitrary signal, and as in the previous cases, the state of the system Σ_d is assumed to be accessible. This in turn, leads to the concepts of the normal form of the system and zero dynamics, which are central to both the statement as well as the solution of the general tracking problem. To facilitate the discussion, we express the plant in normal form as shown below.

Normal Form and Zero Dynamics: Consider the linear periodic system Σ_d with well defined relative degree d. Let $z_1 = y$, then

$$z_1^+(k) = c^T(k+1)x^+ = c^T(k+1)A(k)x + c^T(k+1)b(k)u$$

If $c^T(k+1)b(k) = 0$, define $z_2 = c^T(k+1)A(k)x$, then

$$z_2^+ = c^T(k+2)A(k+1)A(k)x + c^T(k+2)A(k+1)b(k)u$$

Similarly, defining $z_d = c^T(k+d-1)A(k+d-2)\cdots A(k)x$ we have

$$\begin{aligned} z_3^+ = \; & c^T(k+d)A(k+d-1)\cdots \\ & A(k)x + c^T(k+d)A(k+d-1) \\ & \cdots A(k+1)b(k)u \end{aligned}$$

Let $T(k)$ be a nonsingular matrix such that $c^T(k)$, $c^T(k+1)A(k), \cdots$, are its first three rows. The periodic transformation

$$z = T(k)x$$

will transform the system into the normal form

$$\begin{aligned} z_i(k+1) &= z_i^+(k), \quad \forall 1 \le i \le d-1 \\ z_d^+(k) &= P(k)\bar{z} + Q(k)\eta + l(k)u(k) \\ \eta^+(k) &= R(k)\bar{z} + S(k)\eta + w(k)u \\ y(k) &= z_1(k) \end{aligned} \tag{25}$$

where $\bar{z} = [z_1(k), z_2(k), \cdots, z_d(k)]^T$, $\eta = [z_{d+1}(k), z_{d+2}(k), \cdots, z_n(k)]^T$, $P(k) \in \mathbb{R}^d$, $Q(k) \in \mathbb{R}^{n-d}$, $R(k) \in \mathbb{R}^{(n-d)\times d}$, $S(k) \in \mathbb{R}^{(n-d)\times(n-d)}$, $l(k) = c^T(k+d)A(k+d-1)\cdots A(k+1)b(k) \in \mathbb{R}$. As in LTI systems, it can be shown that $T(k)$ can be chosen so that $w(k) = 0$, and this is assumed in the following discussions.

Now we consider the case when $y^*(k)$, and $z_i(k)$ are identically zero. Then

$$\eta^+(k) = S(k)\eta, \tag{26}$$

which defines the zero dynamics of the LTP system Σ_d. If $S(k)$ is asymptotically stable, then $\eta(k)$ tends to zero. In such a case, given a bounded desired signal $y^*(k)$, $\eta(k)$ is a bounded vector for all $k \in \mathbb{Z}^+$.

Our interest is for $y(k) = z_1(k)$ to follow a desired trajectory $y^*(k)$. From (25) it follows that if the system has a well defined relative degree, i.e. $l(k)$ is nonzero $\forall$ $k \in \mathbb{Z}^+$, a $u(k)$ exists such that

$$u(k) = l(k)^{-1}(y^*(k+d) - P(k)\bar{z} - Q(k)\eta) \tag{27}$$

However to assure that $u(k)$ is bounded, the zero dynamics of the plant must be asymptotically stable. Under such an assumption it follows that

$$\begin{aligned} u(k) &= l^{-1}(k)(y^*(k+d) - \bar{P}(k)x(k)) \\ &= g^{*T}(k)x(k) + g_0^*(k)y^*(k+d) \end{aligned} \tag{28}$$

exists which achieves the objective of tracking if $x(k)$ is accessible.

Theorem 2: If the linear periodic system Σ_d has a well defined relative degree, and if the zero dynamics (26) is asymptotically stable, the a control law (28) exists such that the output of Σ_d follows any desired signal $y^*(k)$.

Adaptive Control: If $y^*(k+d)$ is provided as a desired signal at time k. $u(k)$ can be generated adaptively by choosing

$$u(k) = g^T(k)x(k) + g_0(k)y^*(k+d) \tag{29}$$

and adjusting $g^T(k)$ and $g_0(k)$ adaptively. Once again, it can be shown that the adaptive laws

$$\begin{aligned} g(k+N) &= g(k) - \gamma e(k+d)x(k) \\ g_0(k+N) &= g_0(k) - \gamma e(k+d)y^*(k+d) \end{aligned} \tag{30}$$

have the same form as in the previous cases.

IV. Existence of Solutions - Input and Output Accessible

In Section III, the problems of adaptive stabilization , regulation, and tracking were solved assuming that the state vector of the system is accessible. In this section, the same problems are considered assuming that adaptation has to be carried out only using input-output data.

A. Re-construction of the State

Let Σ_d be defined by the state equation (1), and let $x(k)$, the state vector at any time k not be accessible. It can be demonstrated that if the system is observable, the state can be reconstructed from the values of the inputs and outputs. In particular, $x(k)$ can be expressed as

$$x(k) = T_1(k)Y_{(k-nN+1,k-1)} + T_2(k)U_{(k-nN+1,k-1)} \tag{31}$$

where T_1 and $T_2 \in \mathbb{R}^{n\times(nN)}$ are N-periodic matrices. In Section III the periodic system Σ_d is controlled by a periodic state feedback law If, in place of the state variable $x(k)$, its reconstructed value from input and output is used, then Σ_d can also be stabilized.

B. Input-Output Representation - Periodic ARMA Model

Our objective here is to represent the state of the system Σ_d as a function of the past values of the output y and input u. From equation (31) we have

$$\begin{aligned} \Sigma_{io}:\ y(k) &= c^T(k)x(k) \\ &= c^T(k)T_1(k)Y_{(k-nN+1,k-1)} \\ &\quad + c^T(k)T_2(k)U_{(k-nN+1,k-1)} \\ &\triangleq \sum_{i=1}^{nN}\alpha_i(k)y(k-i) + \sum_{i=1}^{nN}\beta_i(k)u(k-i) \\ &= \theta^T(k)\omega(k) \end{aligned} \tag{32}$$

where the N-periodic vector $\theta(k) \in \mathbb{R}^{2nN} = [\alpha_1,\ldots,\alpha_{nN},\beta_1,\ldots,\beta_{nN},]^T$ and the repression vector is $\omega(k) = [y(k-1),\ldots,y(k-nN),u(k-1),\ldots,u(k-nN)]^T$.

Comment: Unlike an n-th order LTI system, which can always be represented as an n-th order difference equation, the order of input-output representation of an n-th order LTP systems depends upon its period as well.

The above representation Σ_{io} is derived only when Σ_d is observable and its relative degree $d = 1$. Suppose that Σ_d has relative degree $d > 1$. It can be shown that $\beta_1(k),\cdots,\beta_{d-1}(k) = 0$, and $\beta_d(k) \neq 0 \quad \forall\ k \in \mathbb{Z}^+$. Therefore $u(k),\cdots,u(k-d+1)$ do not affect $y(k)$. Using (32), we can recursively replace these future outputs with past outputs, and system Σ_d with relative degree d can be shown to be the alternate form

$$\begin{aligned} y(k+d) = &\ \bar{\beta}_0(k)u(k) \\ &+ \sum_{i=1}^{nN}\bar{\alpha}_i(k)y(k-i) + \sum_{j=1}^{nN}\bar{\beta}_j(k)u(k-j) \\ = &\ \bar{\beta}_0(k)u(k) + \bar{\theta}^T(k)\bar{\omega}(k) \end{aligned} \tag{33}$$

where $\theta(k) \in \mathbb{R}^{2nN} = [\bar{\alpha}_1,\ldots,\bar{\alpha}_{nN},\bar{\beta}_1,\ldots,\bar{\beta}_{nN},]$

C. General Tracking Problem

From the discussion in Section IV-B, it follows that, if the desired value $y^*(k+d)$ is given at time k, a desired control law can be determined as

$$\begin{aligned} u^*(k) = &\ \frac{1}{\beta_0(k)}\Big(y^*(k+d) - \theta(k)\bar{\omega}(k)\Big) \\ = &\ g^{*T}(k)\bar{\omega}(k) + g_0^*(k)y^*(k+d) \end{aligned} \tag{34}$$

The role of zero dynamics was discussed in Section III. We now look at its counterpart in the input-output representation. When the output $y(k)$ is identically zero for all k, the zero dynamics in the input output representation has the form:

$$\begin{aligned} u(k) = &\ \tfrac{1}{\beta_1}(\beta_2 u(k-1) + \beta_3 u(k-2) + \\ &\cdots + \beta_{nN}u(k-nN+1)) \end{aligned} \tag{35}$$

determines the evolution of $u(k)$ which is an nNth order difference equation, and if it is asymptotically stable, so is the zero dynamics associated with $\eta(k)$.

Theorem 3: If system Σ_d is observable and has a well defined relative degree, it has an input output representation (32). If the zero dynamics (35) is asymptotically stable, then using the control law controller (34), the output $y(k)$ will follow any reference signal $y^*(k)$.

At this stage either a direct method or an indirect method may be used for the computation of the control input. In the former, the fact that the plant is observable is used to reconstruct the state, which, in turn, is used as in the previous section to generate the control input adaptively.

In indirect adaptive control, a model of the plant is set up as

$$\hat{y}(k+1) = \hat{\beta}_0(k)u(k) + \hat{\theta}(k)\omega(k) \tag{36}$$

where $\hat{\theta}(k)$ is the estimate of the periodic parameter. $u(k)$ is computed at instant k as

$$\begin{aligned} u(k) = & \frac{1}{\hat{\beta}_0(k)}\Big(y^*(k+1) - \hat{\theta}(k)\bar{\omega}(k)\Big) \\ = & \ g^T(k)\bar{\omega}(k) + g_0(k)y^*(k+d) \end{aligned} \tag{37}$$

(Taking the usual precautions when the estimate $\hat{\beta}_0(k)$ is close to zero). Defining the parameter errors $g(k)-g^*(k) = \tilde{g}(k)$, $g_0(k) - g_0^*(k) = \tilde{g}_0(k)$, the tracking error $e(k+1) = y(k+1) - y^*$ can be expressed as

$$e(k+1) = \tilde{g}(k)\bar{\omega}(k) + \tilde{g}_0(k)y^*(k+1)] \tag{38}$$

As in Section III, the adaptive laws can be chosen as

$$\begin{aligned} g(k+N) = & \ g(k) - \gamma e(k+1)\bar{\omega}(k) \\ g_0(k+N) = & \ g_0(k) - \gamma e(k+1)y^*(k+1). \end{aligned} \tag{39}$$

In this section, the input-output representation of a periodic discrete-time system is established, based on the results of observability. The problem of tracking a general desired signal considered in SectionIII, is extended to the case when only input and output are accessible using such a representation.

V. Conclusions

The problems of adaptive control in discrete linear periodic systems are considered. It is shown that most of the results of classical adaptive control can be derived under suitable assumptions concerning the plant in the discrete case.

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