2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Model Predictive Control of Parabolic PDE Systems with Dirichlet Boundary Conditions via Galerkin Model Reduction

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Abstract— **We propose a framework to solve a closed-loop, optimal tracking control problem for a parabolic partial differential equation (PDE) via diffusivity, interior, and boundary actuation. The approach is based on model reduction via proper orthogonal decomposition (POD) and Galerkin projection methods. A conventional integration-by-parts approach during the Galerkin projection fails to effectively incorporate the considered Dirichlet boundary control into the reduced-order model (ROM). To overcome this limitation we use a spatial discretization of the interior product during the Galerkin projection. The obtained low dimensional dynamical model is bilinear as the result of the presence of the diffusivity control term in the nonlinear parabolic PDE system. We design a closed-loop optimal controller based on a nonlinear model predictive control (MPC) scheme aimed at bating the effect of disturbances with the ultimate goal of tracking a nominal trajectory. A quasi-linear approximation approach is used to solve on-line the quadratic optimal control problem subject to the bilinear reduced-order model. Based on the convergence properties of the quasi-linear approximation algorithm, the asymptotical stability of the closed-loop nonlinear MPC scheme is discussed. Finally, the proposed approach is applied to the current profile control problem in tokamak plasmas and its effectiveness is demonstrated in simulations.**

I. Introduction

In this work, we focus on a 1-D parabolic PDE with diffusivity, interior, and boundary control inputs over $\Omega = \{(x,t) : |0 \le x \le 1,\ 0 \le t \le T\}$, which is governed by

$$\begin{aligned}\frac{\partial\theta(x,t)}{\partial t} = & u_1(t)\Big[h_0(x)\frac{\partial\theta^2(x,t)}{\partial x^2} + h_1(x)\frac{\partial\theta(x,t)}{\partial x} + h_2(x)\theta(x,t)\Big] \\ & + \quad h_3(x)u_2(t),\end{aligned} \tag{1}$$

with nonhomogeneous Dirichlet boundary conditions

$$\theta(0,t) = 0, \quad \theta(1,t) = k_3 u_3(t), \tag{2}$$

where $\theta(x,t)$ represents the system state; $u_1(t)$, $u_2(t)$ and $u_3(t)$ denote the diffusivity, interior and boundary controls respectively and $\forall\, t$, $u_1(t) > 0$, $u_2(t) > 0$ and $u_3(t) > 0$; $h_0(x)$, $h_1(x)$, $h_2(x)$ and $h_3(x)$ are functions of the space coordinate and $\forall\, x \in [0,1]$, $h_0(x) > 0$; and k_3 is a constant coefficient. The control objective is to make $\theta(x,t)$ track a prescribed spatiotemporal profile for any arbitrary initial condition $\theta_0(x)$, minimizing at the same time the control effort.

This work was supported in part by a grant from the Commonwealth of Pennsylvania, Department of Community and Economic Development, through the Pennsylvania Infrastructure Technology Alliance (PITA), and the NSF CAREER award program (ECCS-0645086).

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The control of parabolic diffusion-reaction partial differential equations (PDE) such as (1) has been extensively studied using *interior control* (defines a feedback control law for $u_2(t)$ in (1)), usually making use of model reduction techniques (see [1] and references therein) or *boundary control* (defines a feedback control law for $u_3(t)$ in (1)) (see [2] and references therein). Control through $u_1(t)$ in (1), named *diffusivity control* here, has been rarely considered before. However, the diffusivity coefficient is not necessarily fixed or uncontrollable in some applications. For example, the diffusivity control problem arises in the control of the current density profile in magnetically confined fusion plasmas [3], where physical actuators such as plasma total current, line-averaged density and non-inductive total power are used to steer the plasma current density to a desired profile in a designated time period. By modulating these physical actuators it is possible not only to vary the amount of non-inductive current driven into the system (interior control) and the total plasma current (boundary control) but also to modify the resistivity of the plasma (diffusivity control). Another example can be found in the area of flow control [4]. In [4], a saturated flow through a one-dimensional idealized tube packed with soil is considered. The soil contains contaminant samples and a fluid is pumped through the tube (from left to right) to remove the contaminants. The velocity of the fluid pumped into the tube is considered as the control variable which appears as the convective coefficient in the convective-diffusive PDE system governing the contaminant concentration. In terms of controllability, it has been demonstrated that bilinear controls can improve the controllability obtained by just using either interior or boundary controls (see, e.g., [5] and references therein). We propose in this paper a nonlinear model predictive control (MPC) scheme that makes use of the three types of actuation to solve the optimal tracking control problem described above.

Model predictive control, also referred to as moving horizon control or receding horizon control, has become an attractive feedback strategy. In the last two decades, several formulations have been developed for linear and nonlinear systems [6], [7] finding many successful applications, particularly in the process industry [8]. The use of MPC schemes for the control of PDE systems is part of the literature in this field [9], [10]. In prior work, accurate high order dimension models are usually accounted for. The drawback of the use of infinite-dimensional PDE models is that the computational burden associated with the on-line optimization procedure may make the implementation of this control strategy simply unfeasible for not sufficiently slow dynamical systems.

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Several methods have been proposed to deal with the infinite-dimensionality and the complex computational requirements associated with feedback control of PDE systems. Dufour and coworkers [11] adapted MPC with internal model control (IMC) structure where the nonlinear PDE system (solved off-line) and a linearized PDE system (solved during the on-line optimization task) are both used in order to decrease the computational burden. A MPC scheme for output control of hyperbolic PDE systems based on the method of characteristics has been proposed by Shang [12]. Model reduction by inertial manifold theory and partition of the eigenspectrum of the PDE operator has been proposed by Christofides and coworkers [13], and a MPC scheme for linear parabolic PDE systems is presented in [14].

In this paper we use proper orthogonal decomposition (POD) [15] and Galerkin methods to obtain a low dimensional dynamical model for the nonlinear PDE system (1) with the ultimate goal of reducing the computational burden associated with the optimization procedure. Dirichlet boundary control cannot be effectively incorporated into the reduced-order model by following a conventional integration-by-parts approach during the Galerkin projection. Specific difficulties in Dirichlet boundary control problems result from the fact that they are not of variational type [16]. Inspired by [17], we overcome this problem by using a spatial discretization of the interior product during the Galerkin projection and employing the end-point-separation method.

The obtained low dimensional dynamical model is bilinear as the result of the presence of the diffusivity control term in the nonlinear parabolic PDE system (1). To quickly compute the solution of the optimal control problem associated with the MPC scheme and subject to this bilinear system, we follow a successive approach based on a quasi-linear approximation algorithm [18], [19]. As shown in [20], a general MPC scheme does not guarantee closed-loop stability because a finite-horizon criterion is not designed to guarantee an asymptotical property such as stability. Closed-loop stability can only be obtained by suitable tuning of the design parameters such as prediction horizon, control horizon and weighting matrices. Therefore, stability of the proposed nonlinear MPC scheme is discussed.

This paper is organized as follows. In Section II, we discuss the model reduction based on POD/Galerkin projection and the end-point-separation method to integrate the Dirichlet boundary control into the reduced order model. After obtaining a low dimensional bilinear system, we state in Section III the optimal tracking control problem. We propose in Section IV an infinite-horizon nonlinear MPC scheme, where a quasi-linear approximation method is used to solve the associated open-loop optimal control problem on-line. In Section V, we discuss feasibility and stability of the proposed nonlinear MPC scheme. Section VI illustrates the effectiveness of the proposed feedback controller in addressing the current profile control problem in tokamaks. Finally, conclusions and future work are presented in Section VII.

II. MODEL REDUCTION WITH END-POINT-SEPARATION

A. POD Modes

We simulate the parabolic PDE system on the grid $\mathscr{Q}_{ij}=(x_i,t_j)$, where i,j are integers with $1\le i\le m; 1\le j\le n$. The set $\mathscr{V}=\mathrm{span}\{\theta_1,\cdots,\theta_n\}\subset\mathbb{R}^m$ refers to a data ensemble consisting of the snapshots $\{\theta_j\}_{j=1}^n$ obtained from the simulation. We let $\{\varphi_k\}_{k=1}^d$ be the orthonormal basis of the data ensemble $\mathscr{V}$, where $d=\dim\mathscr{V}\le m$. The goal of the POD method is to find a subset of the orthonormal basis $\{\varphi_k\}_{k=1}^d$ such that for some predefined $1\le l\le d$ the reconstruction error for the snapshots is minimized, i.e.,

$$\min_{\{\varphi_k\}_{k=1}^l}\frac{1}{n}\sum_{j=1}^n\left\|\theta_j-\sum_{k=1}^l(\theta_j,\varphi_k)\varphi_k\right\|^2, \tag{3}$$

subject to $(\varphi_i,\varphi_j)=\delta_{ij}, 1\le i\le l, 1\le j\le i$, where $\|\theta\|=\sqrt{\theta^T\theta}$ and $(\cdot,\cdot)$ denotes the inner product. The solution of (3) can be found in the literature, e.g., [15].

B. Galerkin Projection

Let $V_{POD}=\{\varphi_1,\varphi_2,\varphi_3,\varphi_4,...,\varphi_l\}$ be the set of obtained POD modes. Using the l POD modes, we approximate the system state as

$$\theta(x,t)\approx\theta^l(x,t)=\sum_{i=1}^l\alpha_i(t)\phi_i(x), \tag{4}$$

where continuous POD basis functions $\phi_i(x)\in C^2([0,1])\cap L^2([0,1])$ are obtained by interpolating the POD modes φ_i (vectors). We substitute this expression in (1) to obtain

$$\begin{aligned}&\sum_{i=1}^l\dot{\alpha}_i(t)\phi_i(x)\\ =&h_0(x)u_1(t)\sum_{i=1}^l\alpha_i(t)\frac{\partial^2\phi_i}{\partial x^2}+h_1(x)u_1(t)\sum_{i=1}^l\alpha_i(t)\frac{\partial\phi_i}{\partial x}\\ &+h_2(x)u_1(t)\sum_{i=1}^l\alpha_i(t)\phi_i(x)+\sum_{i=1}^l h_3(x)u_2(t).\end{aligned} \tag{5}$$

We write the *weak form* of equation (5) by multiplying both sides by $\phi_k(x)$, for $k=1,2,...,l$, and integrating over the spatial domain $[0,1]$, i.e.,

$$\begin{aligned}&\sum_{i=1}^l\dot{\alpha}_i(t)<\phi_i(x),\phi_k(x)>=u_2(t)<h_3(x),\phi_k(x)>\\ &+u_1(t)\sum_{i=1}^l\alpha_i(t)<h_0(x),\phi_i''(x),\phi_k(x)>\\ &+u_1(t)\sum_{i=1}^l\alpha_i(t)<h_1(x),\phi_i'(x),\phi_k(x)>\\ &+u_1(t)\sum_{i=1}^l\alpha_i(t)<h_2(x),\phi_i(x),\phi_k(x)>,\end{aligned} \tag{6}$$

where

$$<g_1g_2...g_n>\triangleq\int_0^1 g_1g_2...g_n\,dx\approx\ll g_1g_2...g_n\gg^N, \tag{7}$$

$$\ll g_1 g_2 ... g_n \gg^N \triangleq \sum_{n=1}^{N} g_1(n\Delta x) g_2(n\Delta x) ... g_n(n\Delta x). \quad (8)$$

Here Δx is the spatial interval size and $N+1$ is the number of grid points ($N\Delta x = 1$) considered for the numerical approximation of the interior product. The grid is partitioned as

$$\vec{x} = [[0 \quad \Delta x \quad 2\Delta x \quad ... \quad (N-1)\Delta x]^T \quad 1]^T \triangleq [\vec{x}_o^T \ 1]^T. \quad (9)$$

Since the POD modes $\phi_i(x)$ are orthonormal to each other, i.e., $< \phi_i(x), \phi_j(x) >= \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta function, from (6) $\dot{\alpha}$ can be approximated by

$$\begin{aligned}
&\dot{\alpha}_k(t) = u_2(t) \ll h_3, \phi_k \gg^N \\
&+u_1(t)\sum_{i=1}^{l} \alpha_i(t)[\ll h_0, \phi_i'', \phi_k \gg^{N-1} + h_0(1)\phi_i''(1)\phi_k(1)] \\
&+u_1(t)\sum_{i=1}^{l} \alpha_i(t)[\ll h_1, \phi_i', \phi_k \gg^{N-1} + h_1(1)\phi_i'(1)\phi_k(1)] \\
&+u_1(t)\sum_{i=1}^{l} \alpha_i(t)[\ll h_2, \phi_i, \phi_k \gg^{N-1} + h_2(1)\phi_i(1)\phi_k(1)].
\end{aligned} \quad (10)$$

C. Inclusion of Boundary Control in Reduced-Order Model

We include the Dirichlet boundary control into the equation (10) without applying "integration-by-parts". Using (4), we rewrite the boundary condition as

$$\theta|_{x=1} = \sum_{i=1}^{l} \alpha_i(t)\phi_i(1) = k_3 u_3(t). \quad (11)$$

From (11) we can write $\alpha_k(t)\phi_k(1) = k_3 u_3(t) - \sum_{i=1}^{l}(1-\delta_{i,k})\alpha_i(t)\phi_i(1)$. By substituting $\alpha_k(t)\phi_k(1)$ into $u_1(t)h_0(1)\sum_{i=1}^{l}\alpha_i(t)\phi_k(1)\phi_i''(1)$ in (10), we obtain

$$\begin{aligned}
&u_1(t)h_0(1)\sum_{i=1}^{l}\alpha_i(t)\phi_k(1)\phi_i''(1) \\
&= u_1(t)h_0(1)\alpha_k(t)\phi_k(1)\phi_k''(1) \\
&+ u_1(t)h_0(1)\sum_{i=1}^{l}(1-\delta_{i,k})\alpha_i(t)\phi_k(1)\phi_i''(1) \\
&= u_1(t)h_0(1)k_3u_3(t)\phi_k''(1) \\
&+ u_1(t)h_0(1)\sum_{i=1}^{l}\alpha_i(t)[\phi_k(1)\phi_i''(1) - \phi_i(1)\phi_k''(1)].
\end{aligned}$$

We follow similar procedure for the terms $u_1(t)h_1(1)\sum_{i=1}^{l}\alpha_i(t)\phi_k(1)\phi_i'(1)$ and $u_1(t)h_2(1)\sum_{i=1}^{l}\alpha_i(t)\phi_k(1)\phi_i(1)$ in (10) to write

$$\begin{aligned}
&\dot{\alpha}_k(t) = u_1(t)\Bigg\{\sum_{i=1}^{l}\alpha_i(t)[\ll h_0\phi_i''\phi_k \gg^N - h_0(1)\phi_i(1)\phi_k''(1)] \\
&+ h_0(1)k_3u_3(t)\phi_k''(1) + \sum_{i=1}^{l}\alpha_i(t)[\ll h_1, \phi_i', \phi_k \gg^N \\
&- h_1(1)\phi_i(1)\phi_k'(1)] + h_1(1)k_3u_3(t)\phi_k'(1) + h_2(1)k_3u_3(t)\phi_k(1) \\
&+ \sum_{i=1}^{l}\alpha_i(t) \ll h_2, \phi_i, \phi_k \gg^{N-1}\Bigg\} + u_2(t) \ll h_3, \phi_k \gg^N .
\end{aligned}$$

Using the notation

$$\begin{aligned}
\Gamma_{ki} =& \ll h_0, \phi_i'', \phi_k \gg^N - h_0(1)\phi_i(1)\phi_k''(1) \\
&+ \ll h_1, \phi_i', \phi_k \gg^N - h_1(1)\phi_i(1)\phi_k'(1) + \ll h_2, \phi_i, \phi_k \gg^{N-1}, \\
\Phi_k =& \ll h_3, \phi_k \gg^N, \\
\Pi_k =& h_0(1)k_3\phi_k''(1) + h_1(1)k_3\phi_k'(1) + h_2(1)k_3\phi_k(1),
\end{aligned}$$

and redefining the control vector as

$$u = (v_1, v_2, v_3)^T = (u_1, u_2, u_1u_3)^T, \quad (12)$$

we obtain a matrix representation for the reduced order model,

$$\frac{d\bar{\alpha}}{dt} = \Gamma\bar{\alpha}v_1(t) + \Phi v_2(t) + \Pi v_3(t), \quad (13)$$

where $\bar{\alpha}(t) = (\alpha_1, ..., \alpha_l)^T \in \mathbb{R}^l$, $\Gamma \in \mathbb{R}^{l\times l}$, $\Phi, \Pi \in \mathbb{R}^{3\times 1}$ and $v_i \in \mathbb{R}^1$, for $i = 1,2,3$. The vector $\bar{\alpha}(t)$ is the finite dimensional approximation of $\theta(x,t)$, w.r.t. the associated POD modes.

III. TRACKING CONTROL DESIGN

In this section, a feedback control law is proposed for the optimal tracking problem around a predefined open-loop control trajectory. The optimal controller mainly focuses on improving the system response when the whole control process is perturbed.

We let $v^o(t) = [v_1^o \ v_2^o \ v_3^o]^T$ be a set of open-loop control trajectories, which are computed off-line, and $\alpha^o(t)$ be the open-loop state trajectory associated with the open-loop control $v^o(t)$, with a nominal initial state α_0^o. The open-loop state trajectory satisfies

$$\frac{d\alpha^o}{dt} = \Gamma\alpha^o v_1^o(t) + \Phi v_2^o(t) + \Pi v_3^o(t), \quad (14)$$

with initial condition $\alpha^o(t_0) = \alpha_0^o$.

Let us define

$$e(t) = \alpha(t) - \alpha^o(t), \qquad v^c(t) = v(t) - v^o(t), \quad (15)$$

where $v(t) = [v_1 \ v_2 \ v_3]^T$ is the overall control input and $v^c(t) = [v_1^c \ v_2^c \ v_3^c]^T$ is the to-be-designed closed-loop control, which is appended to the open-loop control $v^o(t)$. Then, we can write

$$\frac{d\alpha^o}{dt} + \frac{de}{dt} = \Gamma(\alpha^o + e)(v_1^o + v_1^c) + \Phi(v_2^o + v_2^c) + \Pi(v_3^o + v_3^c). \quad (16)$$

By substituting (14) into (16), we obtain

$$\frac{de}{dt} = A(t)e + B(e)u = f(e,u), \quad (17)$$

where, $A(t) = \Gamma v_1^o(t) \in \mathbb{R}^{l\times l}$, $B(e) = [\Gamma(e+\alpha^o) \quad \Phi \quad \Pi] \in \mathbb{R}^{l\times 3}$, $u(t) = v^c(t) = [v_1^c \ v_2^c \ v_3^c]^T \in \mathbb{R}^{3\times 1}$ subject to input constraints of the form: $u(t) \in U$, $\forall t \geq 0$ where $U := \{u \in \mathbb{R}^3 | \ |u_i| \leq u_i^{max}\}$, $i = 1,2,3$.

IV. Nonlinear Model Predictive Control

In general, the model predictive control (MPC) problem is formulated as solving on-line at time t a finite horizon open-loop optimal control problem subject to system dynamics and constraints involving states and controls. In order to incorporate some feedback mechanism, the open-loop input function obtained from the optimization process is implemented only until the next measurement becomes available at time $t+\delta$, which is used to update the optimization process. One of the key questions in nonlinear MPC is certainly whether a finite horizon nonlinear MPC strategy does lead to stability of the closed-loop, which is an asymptotical property [20]. We propose in this work a MPC scheme with infinite prediction horizon t_p. The feasibility of implementing an infinite-horizon scheme is indeed a consequence of employing a quasi-linear approximation approach to the bilinear optimal control problem defined in each step of the MPC scheme.

The open-loop optimal control problem at time t with measured initial state $\bar{e}(t)$ is formulated as

$$\min_{u(\cdot)} J = \frac{1}{2}e^T(t_f)\bar{P}e(t_f) + \frac{1}{2}\int_t^{t_f} e^T(\tau)Qe(\tau) + u^T(\tau)Ru(\tau)d\tau, \tag{18}$$

subject to the system dynamics (17), and where $t_f = t + t_p$.

By introducing the lagrange multiplier $\lambda(t) \in \mathbb{R}^{l\times 1}$, we can define the Hamiltonian

$$\begin{aligned} H(e,u,\lambda) =& \frac{1}{2}e^T(t)\Omega(t)e(t) + \frac{1}{2}u^T(t)R(t)u(t) \\ &+ \lambda^T(t)[A(t)e(t) + B(e)u(t)]. \end{aligned} \tag{19}$$

And by invoking the principle of optimality, the open-loop optimal problem reduces to solving a nonlinear two-point-boundary-value (TPBV) problem,

$$\begin{aligned} \frac{de}{d\tau} &= \frac{\partial H}{\partial \lambda} = A(\tau)e + B(e)u \\ \frac{d\lambda}{d\tau} &= -\frac{\partial H}{\partial e} = -Qe - A(\tau)^T\lambda - u^T\frac{\partial B^T(e)}{\partial e}\lambda, \end{aligned} \tag{20}$$

with boundary conditions $e(t) = \bar{e}(t),\ \lambda(t_f) = \bar{P}e(t_f)$.

The solution of this nonlinear TPBV problem is usually computationally demanding. To quickly compute the solution of the optimal problem (18), we follow a successive approach based on a quasi-linear approximation algorithm [18], [19]. We replace the bilinear system (17) with a sequence of linear systems. By expanding our problem (17) up to first-order around the previous iteration trajectories $e^k(\tau)$ and $u^k(\tau)$, the system takes the form

$$\dot{e}^{k+1} = A(\tau)e^{k+1} + B^k(\tau)u^{k+1}, \tag{21}$$

where k is the iteration number and $B^k(\tau) = B(e)|_{e^k(\tau)}$, with initial condition $e^{k+1}(t) = \bar{e}(t)$. The cost function is

$$\begin{aligned} J^{k+1} =& \frac{1}{2}(e^{k+1})^T(t_f)\bar{P}e^{k+1}(t_f) \\ &+ \frac{1}{2}\int_t^{t+t_p}(e^{k+1})^T(\tau)Qe^{k+1}(\tau) + (u^{k+1})^T(\tau)Ru^{k+1}(\tau)d\tau. \end{aligned} \tag{22}$$

For each iteration k, we have a standard linear quadratic optimal control defined by (21)-(22).

We assume that the linear system (21) is always controllable, i.e., the pair $(A(t), B^k(e(t)))$ is controllable $\forall t$. Then, the optimal control at iteration k is given by

$$u^{k+1} = -R^{-1}(B^k(e))^TP^{k+1}e^{k+1}. \tag{23}$$

The matrix $P^{k+1} \in \mathbb{R}^{l\times l}$ is governed by the Riccati matrix differential equation

$$\dot{P} = -A^TP^{k+1} - P^{k+1}A - Q + P^{k+1}B^kR^{-1}(B^k)^TP^{k+1}, \tag{24}$$

with $P(t_f) = \bar{P}$, which is derived from a (20)-like TPBV problem for e^{k+1} and λ^{k+1}, assuming $\lambda^{k+1}(\tau) = P^{k+1}(\tau)e^{k+1}(\tau)$, and taking into account that $B^k = B^k(\tau)$ and $\frac{\partial((B^k)^T(e^k(\tau))}{\partial e^{k+1}} = 0$.

Due to the stability issues discussed above, we extend the prediction horizon to infinite, i.e., $t_p \to \infty$. Assuming convergence, the Riccati differential equation (24) reduces to the Riccati algebraic equation

$$0 = -A^TP^{k+1} - P^{k+1}A - Q + P^{k+1}B^kR^{-1}(B^k)^TP^{k+1}. \tag{25}$$

The iterative procedure is stopped when convergence (as shown in [19]) is achieved under given error tolerance. The solution of the open-loop optimal control problem (18) with $t_f \to \infty$, and subject to the bilinear system dynamics (17), is given by

$$u^*(\tau) = -R^{-1}B^T(e^*)P^*e^*, \tag{26}$$

where $*$ denotes the converged values of the iteration. The optimal trajectory $e^*(t)$ driven by $u^*(t)$ is

$$\frac{de^*}{d\tau} = (A - B^*R^{-1}(B^*)^TP^*)e^*. \tag{27}$$

V. Asymptotic Stability Property

According to MPC fundamentals, the open-loop optimal control problem given by equations (17) and (18) will be solved repeatedly, updated with new measurements $\bar{e}(t)$. The closed-loop control $\bar{u}(\cdot)$ is defined by

$$\bar{u}(\tau) = u^*(\tau; \bar{e}(t), t, t_f = \infty), \qquad \tau \in [t, t+\delta]. \tag{28}$$

where $u^*(\cdot)$ in (26) is the solution of the open-loop optimal problem (18) when $t_f \to \infty$. In this section, we study the stability properties of the closed-loop system

$$\dot{e}(t) = f(e(t), \bar{u}(t)). \tag{29}$$

Lemma 1: For the nominal system (17) with no disturbance, the feasibility of the open-loop control problem (18) subject to equations (17) at time $t = t_0$ ($t_0 \geq 0$) implies its feasibility for all $t > t_0$. Here, feasibility of the optimal problem means that there exists at least one (not necessarily optimal) control input trajectory $u(\cdot) : [t, t_f = \infty] \to U$, such that the value of the cost functional (18) is bounded.

Proof: It is assumed that at time $t = t_0$, with measured initial condition $\bar{e}(t_0)$, an optimal solution $u^*(\cdot) : [t_0, t_f = \infty] \to U$ to the optimal control problem given by equations (17) and (18) exists and is found. Since by assumption there are no disturbances and we only consider the nominal system driven by the optimal control input $u^*(\tau; \bar{e}(t_0))$, $\tau \in [t_0, t_0+\delta]$, the state measurement at time $t_0+\delta$ is $\bar{e}(t_0+\delta) = e^*(t_0+\delta; \bar{e}(t_0))$. Therefore, to solve the open-loop optimal control problem at $t_0+\delta$ with the initial condition $e(t_0+\delta)=\bar{e}(t_0+\delta)$, a feasible candidate control input $u(\cdot)$ on $[t_0+\delta, t_f=\infty]$ may be chosen as

$$u(\tau)=u^*(\tau;\bar{e}(t_0),t_0,t_f=\infty] \text{ for } \tau\in[t_0+\delta,t_f=\infty], \quad (30)$$

where $u^*(\cdot)$ is the optimal control input at time t_0. Thus, the nominal state $e(t)$ follows the optimal trajectory $e^*(t)$ in (27). Then, the argument can be repeated as $t\to\infty$. ■

Theorem 2: Suppose that the open-loop control problem (18) subject to (17) is feasible at $t=0$. Then in the absence of disturbances, the closed-loop system with the model predictive control (26) is nominally asymptotically stable. Let $\mathbf{X}\subseteq\mathbb{R}^l$ denote the set of all the initial states satisfying the assumption, then $\mathbf{X}$ is the attraction region of the closed-loop system.

Proof: According to Lemma 1, feasibility of the open-loop control problem at each time $t>0$ is guaranteed by the assumption in the theorem. For $\bar{e}(t)=0$, the optimal solution to the optimization problem (18) is $u^*(\cdot;\bar{e}(t),t,t_f=\infty):[t,t_f=\infty]\to 0$, i.e., $\bar{u}^*(\tau)=0,\ \forall\ \tau\in[t,t+\delta]$. Due to $f(0,0)=0$ in (17), then $\bar{e}(t)=0$ is an equilibrium of the closed-loop system (29).

The key point of this proof is that in the absence of disturbance, driven by control $\bar{u}(t)$, the closed-loop states $e(t)$ will always follow an open-loop optimal trajectory $e^*(t)$ in (27) controlled by the corresponding $u^*(t)$ in (26).

We define a function $V(e(t))=e^T(t)P^*(t)e(t)$, where for any given $\bar{e}(0)\in\mathbf{X}$, $P^*(t)$ is the solution of the algebraic Riccati equation (25) after the quasi-linear approximation algorithm converges. Then, $V(e)$ has the following properties:

(1) $V(0)=0$ and $V(e)>0$ for $e\neq 0$,

(2) along the trajectory of the closed-loop system starting from $\bar{e}(0)\in\mathbf{X}$,

$$\begin{aligned}\dot{V}(e)=&\dot{e}^TP^*e+e^TP^*\dot{e}\\ =&e^T[A-BR^{-1}B^TP^*]^TP^*e+e^TP^*[A-BR^{-1}B^TP^*]e\\ =&-e^T(Q+P^*BR^{-1}B^TP^*)e.\end{aligned}$$

Since Q and R are positive definite, $\dot{V}(e)$ is negative definite. Therefore, the closed-loop system (29) is asymptotically stable. Note that stability does not depend on the optimality of the solution but on the convergence of the quasi-linear approximation scheme. ■

VI. SIMULATION STUDY

In this section, the proposed approach is applied to the current profile control problem in tokamak plasmas and its effectiveness is demonstrated in simulations.

A. Current Profile Evolution Model

A key goal in the control of a magnetic fusion reactor is to maintain current profiles that are compatible with a high fraction of the self-generated non-inductive current as well as with magnetohydrodynamic (MHD) stability at high plasma pressure. This enables high fusion gain and noninductive sustainment of the plasma current for steady-state operation. It is possible to use the poloidal component B_{pol} of the helicoidal magnetic lines confining the plasma in a tokamak to define nested toroidal surfaces corresponding to constant values of the poloidal magnetic flux. The poloidal flux ψ at a point P is the total flux through the surface S bounded by the toroidal ring passing through P, i.e., $\psi=\int B_{pol}dS$. The dynamics of the poloidal flux ψ is governed in normalized cylindrical coordinates by a nonlinear parabolic partial differential equation (PDE) usually referred to as the magnetic diffusion equation, where the spatial coordinate corresponds to the minor radius of the torus [3],

$$\frac{\partial\psi}{\partial t}=f_1(\hat{\rho})u_1(t)\frac{1}{\hat{\rho}}\frac{\partial}{\partial\hat{\rho}}\left(\hat{\rho}f_4(\hat{\rho})\frac{\partial\psi}{\partial\hat{\rho}}\right)+f_2(\hat{\rho})u_2(t), \quad (31)$$

with boundary conditions

$$\left.\frac{\partial\psi}{\partial\hat{\rho}}\right|_{\hat{\rho}=0}=0, \quad \left.\frac{\partial\psi}{\partial\hat{\rho}}\right|_{\hat{\rho}=1}=k_3u_3(t). \quad (32)$$

and where

$$\begin{aligned}f_1(\hat{\rho}) &= \frac{k_{eff}Z_{eff}}{k_{Te}^{3/2}\mu_o\rho_b^2}\frac{1}{\hat{F}^2(\hat{\rho})(T_e^{profile}(\hat{\rho}))^{3/2}} && (33)\\ f_2(\hat{\rho}) &= -R_o\hat{H}\mu_o\rho_b^2\hat{F}^2(\hat{\rho})k_{NIpar}j_{NIpar}^{profile}(\hat{\rho})f_1(\hat{\rho}) && (34)\\ &= \frac{k_{eff}Z_{eff}R_ok_{NIpar}}{k_{Te}^{3/2}}\frac{\hat{H}(\hat{\rho})j_{NIpar}^{profile}(\hat{\rho})}{(T_e^{profile}(\hat{\rho}))^{3/2}} && (35)\\ k_3 &= \frac{\mu_o}{2\pi}\frac{R_o}{\hat{G}\big|_{\hat{\rho}=1}\hat{H}\big|_{\hat{\rho}=1}} && (36)\\ f_4(\hat{\rho}) &= \hat{F}\hat{G}\hat{H}, && (37)\end{aligned}$$

and

$$u_1(t)=\left(\frac{\bar{n}(t)}{I(t)\sqrt{P_{tot}}}\right)^{3/2},\ u_2(t)=\frac{\sqrt{P_{tot}(t)}}{I(t)},\ u_3(t)=I(t).$$

The non-inductive current drive power $P_{tot}(t)$, the spatially averaged density $\bar{n}(t)$, and the total plasma current $I(t)$ are considered as the physical actuators of the system. The spatial functions $\bar{j}_{NI}^{profile}$ and $T_e^{profile}$ denote the non-inductive current drive and electron temperature reference profiles. $\hat{F},\hat{G},\hat{H}$ are geometric factors, $\hat{\rho}=\rho/\rho_b$ is the normalized radius, ρ_b is the radius of last closed flux surface, R_o is the plasma geometric center, μ_o is the vacuum permeability, $k_{eff}=4.2702\cdot10^{-8}$ $(\Omega m(kev)^{3/2})$, $Z_{eff}=1.5$, $k_{Te}=1.7295\cdot10^{10}$ $(m^{-3}A^{-1}W^{-1/2})$, and $k_{NIpar}=1.2139\cdot10^{18}$ $(m^{-9/2}A^{-1/2}W^{-5/4})$.

Since the current density in tokamak is proportional to the spatial derivative of the magnetic flux ψ, we define the to-be-controlled variable as $\theta(\hat{\rho},t)=\frac{\partial\psi}{\partial\hat{\rho}}$. The control objective is to drive $\theta(\hat{\rho},t)$ from any arbitrary initial profile to a prescribed target or desirable profile $\theta^{des}(\hat{\rho})$ at some time T by tracking a predefined trajectory. To simplify notation, we replace $\hat{\rho}$ with x hereafter. We rewrite (31) as

$$\frac{\partial\psi}{\partial t}=g_1(x)u_1(t)\left(g_2(x)\theta(x,t)\right)'+g_3(x)u_2(t) \quad (38)$$

with $g_1(x)=f_1(x)\frac{1}{x}$, $g_2(x)=xf_4(x)$, $g_3(x)=f_2(x)$.

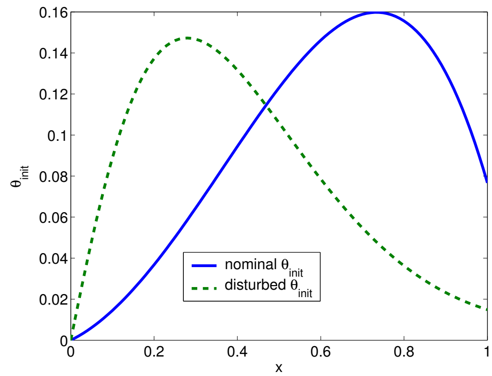


Fig. 1. Comparison of initial θ profiles.

By differentiating both sides of equation (38) w.r.t x, we obtain

$$\begin{aligned}\frac{\partial\theta(x,t)}{\partial t} =& g_1(x)g_2(x)u_1(t)\theta''(x,t)+(g_1'(x)g_2(x)\\ &+2g_1(x)g_2'(x))u_1(t)\theta'(x,t)+(g_1'(x)g_2'(x)\\ &+g_1(x)g_2''(x))u_1(t)\theta(x,t)+g_3'(x)u_2(t),\end{aligned} \tag{39}$$

with boundary conditions (32) rewritten as $\theta(0,t)=0$, $\theta(1,t)=k_3u_3(t)$. By defining $h_0(x)=g_1(x)g_2(x)$, $h_1(x)=g_1'(x)g_2(x)+2g_1(x)g_2'(x)$, $h_2(x)=g_1'(x)g_2'(x)+g_1(x)g_2''(x)$, $h_3(x)=g_3'(x)$, we recover (1)-(2).

B. Simulations

In this section, we present simulation results showing the effectiveness of the proposed optimal control algorithm in a disturbance rejection problem. The nominal initial profile θ_{init} shown in Fig. 1 (solid blue line) has been considered for the synthesis of an off-line optimal controller via Extremum Seeking [21]. The time evolution of the control inputs $v^o(t)$ obtained from the off-line optimization procedure are shown in Fig. 2 (blue dashed lines). Fig. 3-(a) illustrates the nominal space-time profile of $\theta(t,\hat{\rho})$, which is driven by the control inputs $v^o(t)$ without any disturbance.

We first simulate the original PDE system (1) by using a finite difference scheme on the grid $\mathscr{Q}_{ij}=(x_i,t_j)$, where i,j are integers with $1\le i\le m; 1\le j\le n$ ($m=101$, $n=121$), and then a data ensemble is created with snapshots of $\theta(t,\hat{\rho})$. We next extract POD modes from the created data ensemble. With the eight most dominant POD modes, we construct a low dimensional dynamical system governed by the ordinary differential equation (ODE) system (13). Before computing the nonlinear MPC scheme based on the reduced-order model, we assess the effectiveness of the reduced-order model in approximating the original PDE system. Fig. 3-(b) shows the approximation error as function of time and space. The order of the error demonstrates that the reduced-order model based on only eight POD modes can successfully approximate the PDE system.

In each open-loop optimal control problem of the nonlinear MPC scheme, we choose $Q(t)\equiv Q=100\mathbb{I}^{l\times l}$ ($\mathbb{I}$ is an identity matrix, $l=8$), and

$$R(t)\equiv R=diag\left\{\frac{200}{max(v_1^{ol})},\frac{2}{max(v_2^{ol})},\frac{200}{max(v_3^{ol})}\right\},$$

for the cost functional (18), where $max(v_i^{ol})$ stands for the maximum value of the open-loop control $v_i^{ol}(t)$. We use the proposed quasi-linear approximation scheme to compute the optimal control. After several iterations, the solution of the Riccati matrix equation converges, and the controller is implemented according to (26). In order to test the infinite-horizon nonlinear MPC scheme, we use $\delta=0.1s$ as the measurement sampling time. Each of these intervals is discretized in steps of $0.01s$ to solve the algebraic Riccati equation (25).

We consider now a disturbed initial profile θ_{init}, as shown in Fig. 1 (dashed green line), and compare the performances of both open-loop and closed-loop controllers in the presence of this disturbance. In addition, a process disturbance $d(x,t)$ is added to test the performance of the nonlinear MPC scheme. The disturbed PDE model is given as,

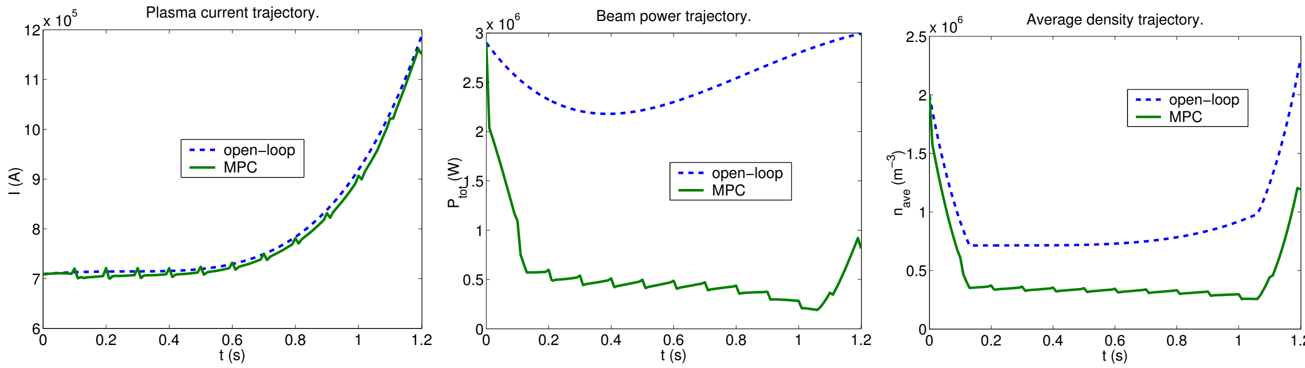
$$\begin{aligned}\frac{\partial\theta(x,t)}{\partial t} =& h_0(x)\theta''(x,t)u_1(t)+h_1(x)\theta'(x,t)u_1(t)\\ &+h_2(x)\theta(x,t)u_1(t)+h_3(x)u_2(t)+d(x,t)\end{aligned}$$

where $d(x,t)=0.1\sin(\pi x)\cos(t)$.

For our particular problem, the convergence rate of the quasi-linear approximation scheme is quite fast. Simulations indicate that given an initial error $e(0)\in\mathbf{X}$, the scheme will converge after 2-3 iterations. Fig. 3-(c) shows the differences between the final-time profiles $\theta(x,T)$, for $T=1.2s$, obtained with both the open-loop and the closed-loop controllers and the desired target profile $\theta^d(x)$. Both final-time profiles are obtained considering disturbances in both the initial profile and the state. In the case of the open-loop controller, the control input trajectories shown in Fig. 2, and computed for the nominal initial profile, are used. In the case of the closed-loop controller, the control input trajectories are shown in Fig. 2. It is possible to note from Fig. 3-(c) that the closed-loop controller can reduce the matching error caused by the disturbances. It is also possible to note that the matching by the closed-loop controller is not perfect. However, this does not imply a limitation of the closed-loop controller but a consequence of the imposed constraints for the actuators (the matrix R is selected to keep the actuator trajectories within physical ranges) and the final time T. If the constraints of actuators can be reduced or the actuators are allowed to act longer, (e.g. $T>1.2s$), the control effect is more observable. Fig 3-(c) shows that better performance could be achieved by setting T equal to $4.8s$ (longer plasma discharge).

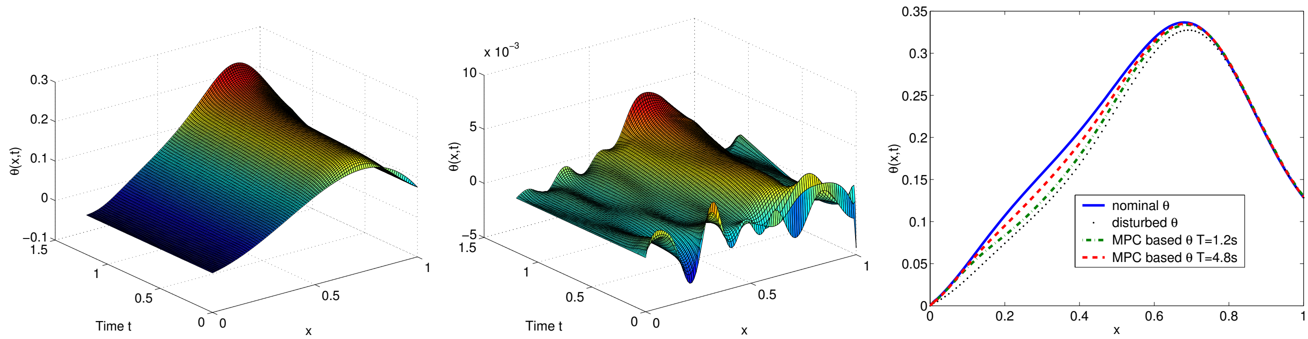
VII. CONCLUSIONS AND FUTURE WORKS

In this paper, we consider a nonlinear parabolic PDE system with nonhomogeneous Dirichlet boundary conditions. Using this PDE model, and the POD/Garlekin technique combined with the end-point-separation approach, we derive a low dimensional dynamical system which integrates the Dirichlet boundary control. To overcome the disturbances in the system, we propose a nonlinear MPC scheme using the convergent successive approach to solve on-line an open-loop infinite-horizon optimal tracking control problem based on the reduced order system. A simulation study is carried out



(a) (b) (c)

Fig. 2. Open-loop and MPC-based optimal tracking control trajectories: (a) $I(t)$, (b) P_{tot}, (c) $\bar{n}(t)$.



(a) (b) (c)

Fig. 3. (a) Evolution of θ produced by the original PDE model, (b) difference between the evolutions of θ predicted by the original PDE and reduced ODE models, (c) final-time θ matching comparison.

for the current profile control problem in tokamak plasmas, showing that the proposed controller can overcome to some extent perturbations both in the initial conditions and in the process. The asymptotic stability of the MPC approach is proved for the reduced-order model. The experimental validation of this controller at the DIII-D tokamak is part of our research plans.

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