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A Direct Quadrature Approach for Nonlinear Filtering

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***Abstract*—The nonlinear filtering problem consists of estimating states of nonlinear systems from noisy measurements and the corresponding techniques can be applied to a wide variety of civil or military applications. Optimal estimates of a general continuous-discrete nonlinear filtering problem can be obtained by solving the Fokker-Planck equation, coupled with a Bayesian update. This procedure does not rely on linearizations of the dynamical and/or measurement models. However, the lack of fast and efficient algorithms for solving the Fokker-Planck equation presents challenges in real time applications. In this paper, a direct quadrature method of moments is introduced which involves approximating the state conditional probability density function as a finite collection of Dirac delta functions. The weights and locations, i.e., abscissas, in this representation are determined by moment constraints and modified using the Baye's rule according to measurement updates. As compared with finite difference methods, the computational cost is lower without a compromising in accuracy. As demonstrated in two classical numerical examples, this approach appears to be promising in the field of nonlinear filtering.**

I. Introduction

THE Fokker-Planck equation (FPE), which is also known as the Kolmogorov forward equation, is first used by Fokker [1] and Planck [2] to explain the Brownian motion of particles. The equation can explain the behavior of a dynamic system that depicts the characteristic of the Brownian motion, and can be used for a wide variety of applications in many different fields such as ecology, genetics, economics, and engineering [3]. In a typical nonlinear filtering problem, the system is modeled as an n-dimensional continuous Itô stochastic differential equation (SDE)

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t),t) + \mathbf{G}(\mathbf{x}(t),t)\mathbf{w}(t) \quad t \ge t_0 \tag{1}$$

where $\mathbf{x} = [x_i]_{i=1,\dots,N_s} \in \Re^{N_s\times 1}$, $\mathbf{F} = [F_i]_{i=1,\dots,N_s} \in \Re^{N_s\times 1}$, and $\mathbf{G}(\mathbf{x}(t),t) \in \Re^{N_s\times N_w}$ are the state vector, state function, and diffusion matrix respectively. $\mathbf{w}(t) \in \Re^{N_w\times 1}$ is the vector of the zero-mean Gaussian process with an autocorrelation of $E[\mathbf{w}(t)\mathbf{w}(\tau)^T] = \mathbf{Q}(t)\delta(t-\tau)$. The measurement $\mathbf{y}(t_k)$ taken at discrete time instants t_k is defined as

$$\mathbf{y}(t_k) = \mathbf{h}(\mathbf{x}(t_k),t_k) + \mathbf{v}(t_k) \quad k = 1,2,\dots \tag{2}$$

where $\mathbf{h}(\mathbf{x}(t_k),t_k) \in \Re^{N_y\times 1}$ is a measurement function (either linear or nonlinear). The measurement noise $\mathbf{v}(t_k)$ is assumed to be a Gaussian white noise with a covariance matrix of $\mathbf{R}$ and independent of $\mathbf{x}(0)$, $\mathbf{w}(t)$, and $\mathbf{h}$.

The basic procedure of the nonlinear filtering technique is illustrated in Fig. 1. If the process described by the SDE is a Markovian diffusion process, the probability density function characterizing this process between measurements ($t_k < t < t_{k+1}$) is governed by the FPE [1, 2, 4] as

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{N_s}\frac{\partial[pF_i]}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\frac{\partial^2\left[p\left(\mathbf{GQG}^T\right)_{ij}\right]}{\partial x_i \partial x_j} \tag{3}$$

where $p = p(\mathbf{x}(t)|\mathbf{Y}(t_k))$ is the state conditional PDF and the measurement observation history is defined as $\mathbf{Y}(t) \triangleq \{\mathbf{y}_k, t_k \le t\}$. The first term on the right hand side (RHS) of the FPE is the drift term whereas the second term is the diffusion term.

Once the PDF function is found from Eq. (3), the measurement $\mathbf{y}(t_{k+1})$ made at the time instant t_{k+1} and the Bayes' formula are used together to update the conditional PDF $p(\mathbf{x}(t_{k+1})|\mathbf{y}_{k+1})$ as

$$p(\mathbf{x}(t_{k+1})|\mathbf{Y}(t_{k+1})) = \frac{p(\mathbf{x}(t_{k+1})|\mathbf{Y}(t_k))\,p(\mathbf{y}(t_{k+1})|\mathbf{x}(t_{k+1}))}{\int p(\xi(t_{k+1})|\mathbf{Y}(t_k))\,p(\mathbf{y}(t_{k+1})|\xi(t_{k+1}))\,d\xi} \tag{4}$$

Hence, using the updated conditional PDF, the minimum mean-square estimate (MMSE) estimates of any state variables or functions of state variables $\phi(\mathbf{x})$ can be obtained.

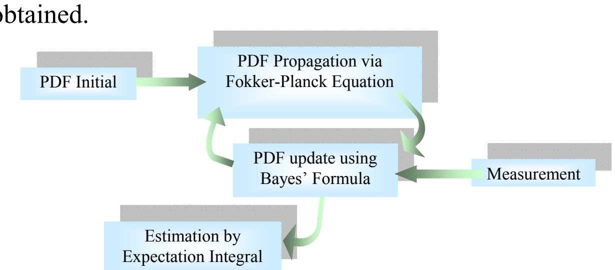

Fig. 1 FPE based nonlinear/linear filtering.

Extensive experiences in defense agencies and industries appear to indicate that the EKF is adequate in most of the applications. However, this happens only when the system is not highly nonlinear and measurements can be updated in a high frequency.

As one of the directions in nonlinear filtering, the sequential Monte Carlo type methods have been investigated

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[5], in which the particle filter is one of them [6]. This type of nonlinear filtering techniques is more adaptive to complex systems because of the flexibility nature of the Monte Carlo simulation. However, the problem of computational complexity still remains unsolved [7, 8] and in most of the time, parallel computing has to be relied on.

The central issue associated with the FPE and Bayes' based nonlinear filtering technique is the high computational cost [10-13]. Since it is difficult to obtain the exact solution of the FPE, numerical approximations, such as the finite difference method [14, 15], path integral method [16], and cell-mapping method [17], are typically used to evaluate the FPE between measurements. These methods are developed for systems with low dimensions and may not be appropriate for real time applications. In Daum's paper [8], the characteristics of nonlinear filtering with numerical approximation of FPE are discussed. Also it is noticed that the high computational cost may be avoided with adaptive grids such as the ones been used by Challa[13], Musick [18], and Yoon and Xu [19]. However, as shown in these papers, even with adaptive grids, the computational cost is still high even for low dimension problems and the accuracy is very sensitive to the domain selection.

In this paper, the direct quadrature method of moments (DQMOM), along with Bayesian update of the conditional state PDF, is formulated for solving the FPE based nonlinear filtering problems. This approach involves representation of the state conditional PDF in terms of a finite collection of Dirac delta functions, whose weights and locations (abscissas) are determined based on constraints due to evolution of moments and modified using Baye's rule for measurement update. Using a small number of scalars (in the Dirac delta function), the method is capable of handing problems described by a multi-variables FPE through a set of differential algebraic equations (DAEs) with good accuracy and less update rate.

This paper is organized as follows: The next section begins with descriptions of the DQMOM in solving the FPE. Then the procedure to obtain estimates through the Bayes' formula using weights and abscissas is discussed. Following this, two classical numerical examples are shown. Conclusions are summarized in the final section.

II. Direct Quadrature Method of Moments (DQMOM)

To easy the derivation, the FPE of the state conditional PDF (Eq. 1) between measurements is rewritten here

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{N_s}\frac{\partial[pF_i]}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\frac{\partial^2\left[p\left(\mathbf{GQG}^T\right)_{ij}\right]}{\partial x_i \partial x_j}, t_k < t < t_{k+1} \quad (5)$$

The DQMOM method, originally investigated by Marchisio and Fox for the population balance problem [20], is illustrated in terms of the nonlinear filtering. First, let us define the state conditional PDF as a summation of a multi-dimensional Dirac delta function

$$p\left(\mathbf{x}(t)\middle|\mathbf{Y}(t_k)\right) = \sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t_k))\prod_{j=1}^{N_s}\delta[x_j - \langle x_j\rangle_\alpha(\mathbf{Y}(t_k))] \quad (6)$$

where N is the number of nodes, $w_\alpha = w_\alpha(\mathbf{Y}(t_k))$, $\alpha = 1,...,N$ is the corresponding weight for node α, and $\langle x_j\rangle_\alpha = \langle x_j\rangle_\alpha(\mathbf{Y}(t_k)),\ \alpha=1,...,N;\ j=1,...,N_s$ is the property vector of node α (called "abscissas"). The weights and abscissas will be computed next. Substituting Eq. (6) into Eq. (5), then the left hand side (LHS) of Eq. (5) becomes

$$\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\{\sum_{\alpha=1}^{N} w_\alpha \prod_{j=1}^{N_s}\delta[x_j - \langle x_j\rangle_\alpha]\} = \sum_{\alpha=1}^{N}\left(\frac{\partial w_\alpha}{\partial t}\right)\prod_{j=1}^{N_s}\delta(x_j - \langle x_j\rangle_\alpha) \\ &- \sum_{\alpha=1}^{N} w_\alpha \sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta[x_k - \langle x_k\rangle_\alpha]\frac{\partial \delta_{j\alpha}}{\partial\langle x_j\rangle_\alpha}\frac{\partial\langle x_j\rangle_\alpha}{\partial t} \\ &= \sum_{\alpha=1}^{N}\prod_{j=1}^{N_s}\delta_{j\alpha}\left(\frac{\partial w_\alpha}{\partial t}\right) - \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s} w_\alpha \delta_{k\alpha}\delta'_{j\alpha}\frac{\partial\langle x_j\rangle_\alpha}{\partial t}\end{aligned} \quad (7)$$

where $\delta_{j\alpha} \triangleq \delta[x_j - \langle x_j\rangle_\alpha]$ and $\delta'_{j\alpha} \triangleq \partial\delta_{j\alpha}/\partial\langle x_j\rangle_\alpha$.

If the weighted abscissas $\varsigma_{j\alpha} \triangleq w_\alpha\langle x_j\rangle_\alpha$ is introduced, after some manipulations, Eq. (7) can be rewritten as

$$\begin{aligned}\frac{\partial p}{\partial t} &= \sum_{\alpha=1}^{N}\left[\prod_{j=1}^{N_s}\delta_{j\alpha}\left(\frac{\partial w_\alpha}{\partial t}\right) + \sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha \delta_{k\alpha}\delta'_{j\alpha}\frac{\partial w_\alpha}{\partial t}\right] \\ &- \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\frac{\partial\varsigma_{j\alpha}}{\partial t}\end{aligned} \quad (8)$$

Notice that w_α, $\varsigma_{j\alpha}$, and $\delta_{j\alpha}$ are functions of time, thus

$$\begin{aligned}\frac{\partial p}{\partial t} &= \sum_{\alpha=1}^{N}\left[\prod_{j=1}^{N_s}\delta_{j\alpha}\left(\frac{dw_\alpha}{dt}\right) + \sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha \delta_{k\alpha}\delta'_{j\alpha}\frac{dw_\alpha}{dt}\right] \\ &- \sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\frac{d\varsigma_{j\alpha}}{dt}\end{aligned} \quad (9)$$

Let us define

$$dw_\alpha / dt \triangleq a_\alpha,\ \alpha = 1,...,N \quad (10)$$

and

$$d\varsigma_{j\alpha}/dt \triangleq b_{j\alpha},\ j=1,...,N_s;\ \alpha=1,...,N \quad (11)$$

Eq. (9) (LHS of Eq. (5)) can be further simplified as

$$\begin{aligned}\frac{\partial p}{\partial t} &= \sum_{\alpha=1}^{N}\left[\prod_{j=1}^{N_s}\delta_{j\alpha} + \sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha \delta_{k\alpha}\delta'_{j\alpha}\right]a_\alpha \\ &- \sum_{\alpha=1}^{N}\left[\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\right]b_{j\alpha}\end{aligned} \quad (12)$$

Now let the right hand side (RHS) of Eq. (5) defined to be

$$S_{\mathbf{x}}(\mathbf{x}) = -\sum_{i=1}^{N_s}\frac{\partial pF_i}{\partial x_i} + \sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\frac{\partial^2[1/2p(GQG^T)_{ij}}{\partial x_i \partial x_j} \quad (13)$$

The FPE can be written in terms of the multi-variable Dirac delta function as

$$\sum_{\alpha=1}^{N}\left(\prod_{j=1}^{N_s}\delta_{j\alpha}\right)a_\alpha+\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha\delta_{k\alpha}\delta'_{j\alpha}a_\alpha$$
$$-\sum_{\alpha=1}^{N}\left[\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\right]b_{j\alpha}=S_{\mathbf{x}}(\mathbf{x}) \quad (14)$$

There are total $N(1+N_s)$ parameters (in Eq. 14) need to be found to construct the conditional PDF $p\left(\mathbf{x}(t)\middle|\mathbf{Y}(t)\right)$: $a_\alpha, \alpha=1,...N$ and $b_{j\alpha}, j=1,...,N_s, \alpha=1,...,N$. In general, DQMOM method applies an independent set of moments that user wish to control to construct $N(1+N_s)$ DAEs.

Given the following three Dirac delta function properties

$$\int_{-\infty}^{+\infty}x^k\delta(x-\langle x\rangle_\alpha)dx=\langle x\rangle_\alpha^k \quad (15)$$

$$\int_{-\infty}^{+\infty}x^k\delta'(x-\langle x\rangle_\alpha)dx=-k\langle x\rangle_\alpha^{k-1} \quad (16)$$

$$\int_{-\infty}^{+\infty}x^k\delta''(x-\langle x\rangle_\alpha)dx=k(k-1)\langle x\rangle_\alpha^{k-2} \quad (17)$$

The $k_1,k_2,...,k_{N_s}$ moment of the Eq. (14) can be derived as followed

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\prod_{j=1}^{N_s}\delta_{j\alpha}a_\alpha\right)\prod_{l=1}^{N_s}dx_l$$
$$+\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha\delta_{k\alpha}\delta'_{j\alpha}a_\alpha\right)\prod_{l=1}^{N_s}dx_l$$
$$-\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\left[\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\delta'_{j\alpha}\right]b_{j\alpha}\right)\prod_{l=1}^{N_s}dx_l \quad (18)$$
$$=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left[S_{\mathbf{x}}(\mathbf{x})\right]\prod_{l=1}^{N_s}dx_l$$

The first term in the LHS of Eq. (18) can be simplified as

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\sum_{\alpha=1}^{N}\prod_{j=1}^{N_s}\delta_{j\alpha}a_\alpha\right)dx_1...dx_{N_s}$$
$$=\sum_{\alpha=1}^{N}\left(\prod_{j=1}^{N_s}\langle x_j\rangle_\alpha^{k_j}\right)a_\alpha \quad (19)$$

whereas the second term in the LHS of Eq. (18) is derived to be

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{N_s}x_m^{k_m}\left(\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}\prod_{k=1,k\neq j}^{N_s}\langle x_j\rangle_\alpha\delta_{k\alpha}\delta'_{j\alpha}a_\alpha\right)dx_1...dx_{N_s}$$
$$=-\sum_{\alpha=1}^{N}\left(\sum_{j=1}^{N_s}k_j\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right)a_\alpha \quad (20)$$

In the same way, the third term of the LHS in Eq. (18) can be derived as

$$-\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}b_{j\alpha}\int_{-\infty}^{+\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\delta'_{j\alpha}\left(\prod_{k=1,k\neq j}^{N_s}\delta_{k\alpha}\right)dx_1...dx_{N_s}$$
$$=\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}k_j\langle x_j\rangle_\alpha^{k_j-1}\prod_{k=1,k\neq j}^{N_s}\langle x_k\rangle_\alpha^{k_k}b_{j\alpha} \quad (21)$$

The $k_1,...,k_{N_s}$ moment of the RHS of Eq. (18) are derived to be

$$\bar{S}_{k_1,...k_{N_s}}=-\sum_{i=1}^{n}\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left(\frac{\partial pF_i}{\partial x_i}\right)dx_1...dx_{N_s}$$
$$+\int_{-\infty}^{\infty}x_1^{k_1}\cdots x_{N_s}^{k_{N_s}}\left[\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\frac{1}{2}\frac{\partial^2[p(GQG^T)_{ij}]}{\partial x_i\partial x_j}\right]dx_1...dx_{N_s} \quad (22)$$
$$=\bar{S}^1_{k_1,...k_{N_s}}+\bar{S}^2_{k_1,...k_{N_s}}$$

where

$$\bar{S}^1_{k_1,...k_{N_s}}=\sum_{i=1}^{N_s}\sum_{\alpha=1}^{N}k_iw_\alpha(t)\langle x_1\rangle_\alpha^{k_1}\cdots\langle x_{i-1}\rangle_\alpha^{k_{i-1}}\langle x_i\rangle_\alpha^{k_i-1}$$
$$\langle x_{i+1}\rangle_\alpha^{k_{i+1}}\cdots\langle x_{N_s}\rangle_\alpha^{k_{N_s}}\cdot F_i(\langle x_1\rangle_\alpha,...\langle x_{N_s}\rangle_\alpha) \quad (23)$$

When $i\neq j$, $\bar{S}^2_{k_1,...k_{N_s}}$ is derived as

$$\bar{S}^2_{k_1,...k_{N_s}}=\sum_{i=1}^{N_s}\sum_{j=1}^{N_s}\sum_{\alpha=1}^{N}w_\alpha k_ik_j\left(\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right)/\langle x_i\rangle_\alpha\langle x_j\rangle_\alpha$$
$$\cdot[D(\mathbf{x})]_{ij}\Big|_{\langle x_1\rangle_\alpha,...\langle x_{N_s}\rangle_\alpha} \quad (24)$$

whereas when $i=j$, $\bar{S}^2_{k_1,...k_{N_s}}$ is derived to be

$$\bar{S}^2_{k_1,...k_{N_s}}=\sum_{\alpha=1}^{N}w_\alpha k_i(k_i-1)\left(\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right)/\langle x_i\rangle_\alpha^2[D(\mathbf{x})]_{ij}\Big|_{\langle x_1\rangle_\alpha,...\langle x_{N_s}\rangle_\alpha} \quad (25)$$

Notice that $D(\mathbf{x})\triangleq(1/2)GQG^T$. Thus, the $N(1+N_s)$ DAEs can be constructed using a set of independent moments constraints $k_1,...,k_{N_s}$ as

$$\sum_{\alpha=1}^{N}\left[\left(1-\sum_{j=1}^{N_s}k_j\right)\prod_{k=1}^{N_s}\langle x_k\rangle_\alpha^{k_k}\right]a_\alpha$$
$$+\sum_{\alpha=1}^{N}\sum_{j=1}^{N_s}k_j\langle x_j\rangle_\alpha^{k_j-1}\prod_{k=1,k\neq j}^{N_s}\langle x_k\rangle_\alpha^{k_k}b_{j\alpha}=\bar{S}_{k_1,...k_{N_s}} \quad (26)$$

For example, if the number of states is $N_s=2$ and the number of nodes used in the in the multi-dimensional Dirac delta function is $N=2$, there will be $N(1+N_s)=6$ unknown parameters in Eq. (26). In order to solve these six DAEs, the following six moments constraints

$$(k_1,k_2)=(0,0),(1,0),(0,1),(2,0),(1,1),(0,2) \quad (27)$$

can be applied such that there are enough equations for solving $a_\alpha, \alpha=1,2$ and $b_{j\alpha}, j=1,2; \alpha=1,2$ explicitly.

Note: Typically, the precision of the estimation and the computational cost will be higher when the number of nodes increases. The selected moment constraint $k_1,k_2,...,k_{N_s}$ will guarantee the PDF approximated by the Eq. (6) has exact value for this moment of the PDF. For typical estimation problem, the accuracy of the first moment (e.g. minimum mean-square estimate (MMSE) estimates of any state variables or functions of state variables $\phi(\mathbf{x})$ can be obtained) is automatically guaranteed.

For simplicity, Eq. (26) can be rewritten in a matrix form as

$$\mathbf{A}\boldsymbol{\mu} = \mathbf{s} \tag{28}$$

where the unknown parameters are

$$\boldsymbol{\mu} \triangleq [a_1, a_2, ..., a_N, b_{11}, b_{12}, .. b_{1N}, ... b_{N_s 1}, b_{N_s 2}, ... b_{N_s N}]^T \in \Re^{N(1+N_s)\times 1} \tag{29}$$

and matrix $\mathbf{A}$ can be derived from Eq. (26) as a nonlinear function of the abscissas. The moment constraints are

$$\mathbf{s} = [\bar{S}_{0,...0}, \bar{S}_{1,...0}, ...]^T \in \Re^{N(N_s+1)\times 1} \tag{30}$$

As compared with the widely used finite difference methods used in [8, 13, 16, and 19], with the help of the DQMOM scheme, the partial differential equation is reduced to a set of differential algebraic equations and the computational cost is reduced.

III. Bayes' Formula by DQMOM

Once the weights and abscissas in the "predictor" PDF is found through the DQMOM (Eq. 28) propagation, the "updated" conditional PDF can be found using the new measurement $\mathbf{y}(t_{k+1})$ made at the time instant t_{k+1} and the Bayes' formula (Eq. 4). Substitute Eq. (6) into Eq. (4), the DQMOM based Bayes' equation can be derived as

$$\begin{aligned} & p(\mathbf{x}_{k+1}|\mathbf{Y}(t_{k+1})) \\ &= \frac{p(\mathbf{y}(t_{k+1})|\mathbf{x}(t_{k+1}))\sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t_k))\prod_{j=1}^{N_s}\delta[x_j - \langle x_j\rangle_\alpha(\mathbf{Y}_k(t_k))]}{\int p(\mathbf{y}(t_{k+1})|\boldsymbol{\xi}(t_{k+1}))\sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t_k))\prod_{j=1}^{N_s}\delta[\xi_j - \langle x_j\rangle_\alpha(\mathbf{Y}(t_k))]d\boldsymbol{\xi}} \\ &= \frac{\sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t_k))p(\mathbf{y}(t_{k+1})|\mathbf{x}(t_{k+1}))\prod_{j=1}^{N_s}\delta[x_j - \langle x_j\rangle_\alpha(\mathbf{Y}(t_k))]}{\sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t_k))\,p(\mathbf{y}(t_{k+1})|\langle x_1\rangle_\alpha, ..., \langle x_{N_s}\rangle_\alpha)} \end{aligned} \tag{31}$$

The new weights in the update step (i.e. after accounting for measurements at $t = t_{k+1}$) for the DQMOM are obtained by renormalizing the old weights as

$$w_\alpha(\mathbf{Y}(t_{k+1})) = \frac{w_\alpha(\mathbf{Y}(t_k))p(\mathbf{y}(t_{k+1})|\langle x_1\rangle_\alpha, ..., \langle x_{N_s}\rangle_\alpha)}{\sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t_k))p(\mathbf{y}(t_{k+1})|\langle x_1\rangle_\alpha, ..., \langle x_{N_s}\rangle_\alpha)} \tag{32}$$

while the abscissas are unchanged as

$$x_\alpha(\mathbf{Y}(t_{k+1})) = x_\alpha(\mathbf{Y}(t_k)),\ \alpha = 1, ..., N_s \tag{33}$$

Hence, using the computed state conditional PDF value of $p(\mathbf{x}(t)|\mathbf{Y}(t)), t = t_1, t_2, ...$, the MMSE estimates of the state variables or functions of state variables $\phi(\mathbf{x}, t)$ at each time step can be obtained as

$$\begin{aligned} \hat{\phi}(\mathbf{x}(t)) &= E[\mathbf{x}(t)|\mathbf{Y}(t)] \\ &= \sum_{\alpha=1}^{N} w_\alpha(\mathbf{Y}(t))\phi(\langle x_1\rangle_\alpha, ..., \langle x_{N_s}\rangle_\alpha) \end{aligned} \tag{34}$$

IV. Numerical Examples

In this section, the performance of the proposed nonlinear filter is demonstrated through two numerical examples: univariate nonstationary growth and 2D bearing-only tracking problems. All the codes are written in Matlab and run in a Dell Precision T7400 desktop (Intel Dual Core CPUs with 3.16 GHz, 3.25 GB RAM). The Runge-Kutta 4th order method is used for integration.

Example 1: Univariate Nonstationary Growth Model

First, the method is applied in a continuous time version of a nonstationary problem, modified from the discrete-time univariate nonstationary growth model (UNGM) [21-23], where both the state dynamics and measurement models are nonlinear. The process equation of the UNGM is

$$x_n = \alpha x_{n-1} + \beta \frac{x_{n-1}}{1 + x_{n-1}^2} + \gamma \cos[1.2(n-1)] + w_n, n = 1, 2, ... \tag{35}$$

and the measurement model is

$$z = x^2/20 + v \tag{36}$$

A continuous time version of the process equation is derived as

$$\dot{x} = \alpha^* x + \beta^* \frac{x}{1+x^2} + \gamma^* \cos\left[\frac{1.2(t - t_0)}{\Delta t}\right] + w,\ t \ge t_0 = 1 \tag{37}$$

where $\alpha^* = (\alpha - 1)/\Delta t$, $\beta^* = \beta/\Delta t$, and $\gamma^* = \gamma/\Delta t$ based upon the first order Euler scheme. $\alpha = 0.5$, $\beta = 10$, and $\gamma = 8$ are used in the simulation. The time step used in the conversion from the discrete time model [21] to the continuous time model is $\Delta t = 0.1s$. w is a t-distribution with a degrees of freedoms of 10 and $v \sim N(0, 0.01)$. A step size of 0.1 seconds is applied in the propagation of the corresponding SDE and FPE equations, whereas the measurements are updated at different sampling rates to show the consistence in estimation precision of the state.

In this example, two nodes are selected. Therefore, there is a total of four unknowns (i.e. two weights and two abscissas) to characterize the conditional PDF and $k = 0, 1, ...3$.

A set of fifty Monte Carlo runs have been used for testing the algorithm. The measurement update rates are set at every 0.1, 0.2, 0.4, 0.8, and 1.6 seconds respectively. As shown in Fig. 2 (a) through Fig. 2 (c), the estimated state tracks the actual state history for all the cases under different update rates.

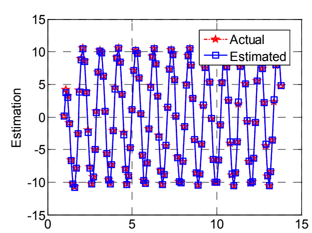

Fig. 2 (a) Estimated state history using the DQMOM

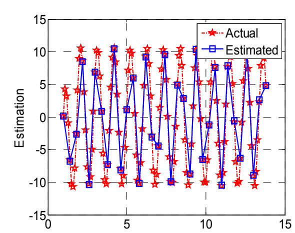

Fig. 2 (b) Estimated state history using the DQMOM approach (update rate at 0.1 seconds).

approach (update rate at 0.4 seconds).

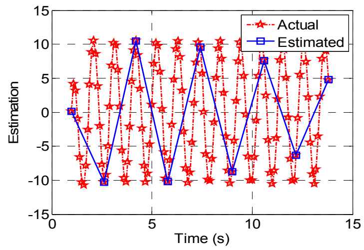

Fig. 2 (c) Estimated state history using the DQMOM approach (update rate at 1.6 seconds).

The mean square error of the estimation procedure based on DQMOM has a consistent precision in the rage of 0.218 to 0.260 for update rates between 0.1s and 1.6 s (shown in Table 1). With reference to the signal amplitude, which is around 10 according to the simulation results as shown in Figs. 2(a) – 2(c), the percentage of the state estimation error is only about 2.5%. As expected, we find that the computational cost of estimation based on DQMOM (shown in Table 1), also decreases monotonically as the measurement updates become less frequent.

Table 1 DQMOM for the UNGM

Update delay (s)	MSE	Error Percentage	Computational Cost (in seconds)
0.1	0.254	2.54%	8.23
0.2	0.260	2.60%	4.79
0.4	0.248	2.48%	3.18
0.8	0.218	2.18%	2.55
1.6	0.246	2.46%	2.33

Example 2: Bearing-only Tracking Problem

A simplified version of the passive bearing only tracking problem is adopted from [8] as the second example. The motion of the sensor platform follows $x_p = 4t$ and $y_p = 20$, whereas the motion of the target is governed by

$$\frac{d\boldsymbol{x}(t)}{dt} = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \qquad (38)$$

where the initial condition of the target is $\boldsymbol{x}(t_0) = [80,1]^T$, and the process noise is a Gaussian with a zero mean and a covariance of $Q = 10^{-4}$. The measurement model is

$$y(t_k) = \tan^{-1}\frac{y_p(t_k)}{x_1(t_k) - x_p(t_k)} + w_s(t_k) \qquad (39)$$

where the sensor noise is assumed to have zero mean with a variance of $R = (4^o)^2$.

The step size used in the FPE and the ODE propagations $\Delta t = 0.01s$. The measurements updated rate varies at different sampling rates at 0.1 seconds, 0.5 seconds, and 1 second. Two nodes are chosen and there is a total of six unknowns to characterize the conditional PDF and

$$(k_1, k_2) = (0,0),(1,0),(0,1),(2,0),(1,1),(0,2) \qquad (40)$$

The results are compared with two other methods: EKF, and Alternating Directional Implicit (ADI) methods [24]. ADI method is an implicit method for parabolic and elliptic equations. The major advantage of the ADI method compared with other implicit methods used in FDMs is that it involves only tridiagonal systems, and the matrix inversion can be achieved efficiently by the Thomas' algorithm [25]. To achieve a similar estimation precision in ADI as those in DQMOM, the spatial difference is chosen to be 0.01 for both the position and velocity states.

A set of fifty Monte Carlo runs have been used for testing the algorithm. As shown in Fig. 3 (a), when the measurement update rate is set every 0.1 seconds, the estimation errors from DQMOM and ADI filters have a comparable precision on the order of 0.5 in the stationary state (also shown in Table 1), whereas the estimation error from the EKF is roughly bounded by 2.5. As the update delay increases as shown in Fig. 3(b) (0.5 seconds) and Fig. 3 (c) (1.0 second), the EKF becomes unstable, whereas the estimation error bound of the ADI increases to 0.7. However, the results from the DQMOM are consistent with the case of update rate of 0.1 seconds. Also notice that in Fig. 3(a), if in order to achieve roughly the same estimation precision, the update rate of the EKF is required to be 0.01 seconds and the computational cost is high as shown in Table 3 (2.09 seconds for a 20-second simulation time).

To achieve the same precision, the ADI needs approximately 14 seconds and the time spent is almost constant for the range of update rates considered (Table 3). The time taken in the DQMOM approach is much less that that of the ADI and as the update frequency decreases, the speed of the DQMOM increases without compromising in the estimation precision (as shown in Table 3).

Table 2 Bounds of the estimation error

Update delay (s)	EKF	ADI	DQMOM
0.01	0.4	-	-
0.1	2.5	0.5	0.25
0.5	4	0.5	0.3
1.0	5	0.5	0.25

Table 3 Computational cost (in seconds)

Update delays (s)	EKF	ADI	DQMOM
0.01	2.09	-	-
0.1	0.02	14.07	11.44
0.5	0.0042	14.42	2.6
1.0	0.0023	14.11	1.47

Therefore, simulation results for this numerical example also lead us to conclusions that are similar to those observed from the analysis of numerical example 1. While, the error in estimation based on EKF type approaches increases with decrease in update rate, owing to increase in errors due to linearization of dynamics, the DQMOM approach can give better estimation performance than EKF especially at low update rates. Besides, the computational cost of estimation based on the DQMOM approach was also found to be significantly lower than ADI for comparable levels of accuracy (as in numerical example 1).

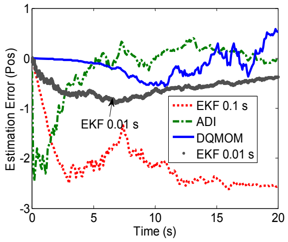

Fig. 3 (a) Estimation error (update rate at 0.1 seconds).

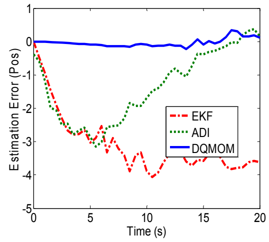

Fig. 3 (b) Estimation error (update rate at 0.5 seconds).

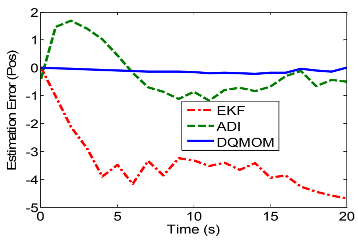

Fig. 3 (c) Estimation error (update rate at 1 second).

V. Conclusion

A new approach for solving the nonlinear filtering problem is proposed using the direct quadrature method of moments coupled with Bayesian update of the conditional PDF. Through this new approach, the FPE (a PDE) can be transformed into a set of DAEs in terms of Dirac delta functions. As compared with EKF type methods, which are used widely in nonlinear filtering problems, the nonlinear dynamics is not required to be linearized. Also, if compared with finite difference type methods, the computational cost is dramatically reduced without a compromising in estimation accuracy. Two numerical examples, univariate nonstationary growth and 2D bearing only tracking problems, are tested for optimal estimation and compared with the methods of EKF and ADI, and the results appear to be promising in the field of nonlinear filtering theory.

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