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Robust nonlinear model predictive controller design based on multi-scenario formulation

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***Abstract*— We propose a robust nonlinear model predictive control (NMPC) algorithm based on multi-scenario nonlinear programming (NLP). We show that this multi-scenario formulation is Input-to-State practical stable (ISpS). Moreover, the proposed algorithm yields less steady-state offset than the min-max strategy.**

I. INTRODUCTION

Robust nonlinear model predictive control (NMPC) has received considerable attention. Issues related to nominal stability and feasibility have been extensively addressed [1]. In general, NMPC with asymptotic stability does not guarantee robust stability. In this case, robust design strategies that explicitly account for uncertainties within the controller formulation are necessary. A robust NMPC strategy is the min-max NMPC formulation, which computes the best control policy based on the worst expected realization of the uncertainties [2]. This formulation may dramatically increase the computational cost of the on-line NMPC problem, and may yield large offset for controlled variables.

In this work, we propose a robust NMPC design strategy based on a multi-scenario NLP formulation, from which the calculated control sequence is feasible for the entire uncertainty region. The proposed method is interesting for several reasons. Theoretically, this multi-scenario based strategy is Input-to-State practical Stable (ISpS) under some mild assumptions, and more importantly, it yields less offset than the min-max formulation. Moreover, the proposed method can potentially be implemented on industrial-size applications because the formulation can be decomposed and computed in parallel computing architectures [3], and it can be extended to the recently proposed as-NMPC framework [4] to reduce on-line computational delay.

II. PROBLEM FORMULATION

In this work, the dynamics of a plant will be described by the following discrete-time model,

$$x(k+1) = f(x(k), u(k), \phi(k)), \;\; k \geq 0 \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the system state, $u(k) \in \mathbb{R}^{n_u}$ is the current control variable and $\phi(k) \in \mathbb{R}^{n_\phi}$ is an uncertainty signal which models plant-model mismatches at time step $k \geq 0$. For the sake of clarity, we consider only state-independent uncertainty signals in this work, i.e. $\phi(k)$ is not a function of $x(k)$ and $u(k)$. In general, only partial information of $\phi(k)$ is available, e.g. its feasible region Ω and its probability distribution, etc. The control and state variables of the plant are required to fulfill constraints, e.g. $x(k) \in \mathbb{X}, \; u(k) \in \mathbb{U}$. Without losing generality, we assume that the given plant (1) has an equilibrium point at the origin, i.e. $f(0,0,0) = 0$.

Given $x(k)$, the current state value at time step k, an NMPC formulation can be described in the following discretized form [1]:

$$\min_{z(j),v(j)} \; J(x(k)) := F(z(N)) + \sum_{j=0}^{N-1} \varphi(z(j), v(j), \theta(j)) \tag{2a}$$

$$s.t. \; z(j+1) = f(z(j), v(j), \theta(j)), \;\; z(0) = x(k) \tag{2b}$$

$$z(j) \in \mathbb{X}, \;\; z(N) \in \mathbb{X}_f, \;\; v(j) \in \mathbb{U}, \;\; \theta(j) \in \mathbf{\Omega} \tag{2c}$$

$$j = 0, \ldots N-1 \tag{2d}$$

where N is the time horizon, $\theta \in \Omega$ is the uncertainty parameter. The computed control $v(j) \in \mathbb{R}^{n_u}$ and predicted state $z(j) \in \mathbb{R}^{n_x}$ are enforced to satisfy the constraints $v(j) \in \mathbb{U}$, $z(j) \in \mathbb{X}$ and terminal constraint $z(N) \in \mathbb{X}_f$. The cost function $J(x(k))$ comprises the stage cost $\varphi(\cdot,\cdot,\cdot)$ and terminal penalty cost $F(\cdot)$. When the solution sequence $(z^*(l), v^*(l))$ is available, the first control $u(k) = v^*(0)$ is injected into the plant.

A. *Multi-scenario Formulation*

If the presence of uncertainties does not cause any loss of feasibility (e.g. no state or control constraints in the system), the NMPC formulation (2) enjoys inherent robustness. Otherwise, the calculated control v may violate constraints in the plant, losing stability of the closed-loop system. To avoid this situation, we solve a multi-scenario problem :

$$\min_{z_l(j),v(j)} \; V(x(k)) := \sum_{l=1}^{M} w_l J_l(x(k))$$

$$= \sum_{l=1}^{M} w_l \left\{ F(z_l(N)) + \sum_{j=0}^{N-1} \varphi(z_l(j), v(j), \theta_l) \right\} \tag{3a}$$

$$s.t. \; z_l(j+1) = f(z_l(j), v(j), \theta_l), \;\; z_l(0) = x(k)$$

$$j = 0, \ldots N-1, \;\; l = 1, \ldots M \tag{3b}$$

$$z_l(j) \in \mathbb{X}, \;\; z_l(N) \in \mathbb{X}_f, \;\; v(j) \in \mathbb{U}, \;\; \theta_l \in \mathbf{\Omega} \tag{3c}$$

with M different scenarios, where l is the index of scenarios and w_l are weights associated to each scenario, satisfying $0 \leq w_l \leq 1$, $\sum_{l=1}^{M} w_l = 1$. θ_l and $z_l(j)$ are uncertainty and state variables in scenario l. Consequently, the calculated control sequence $v(j)$ is feasible for all the uncertainties $\theta_l, \forall l$.

B. Robust Stability

For the sake of brevity, we leave the definitions and assumptions required for the analysis in an extended report [5]. Consider $B_{\delta l}$ as M closed balls, $B_{\delta l} \triangleq \{|\vartheta_l - \theta_l| \leq \delta_l, \vartheta_l \in \Omega\}$, $\forall l \in 1 \ldots M$ centered around θ_l with radius δ_l. The balls $B_{\delta l}$ are defined such that an NMPC formulated with θ_l as nominal model parameter is robustly stable within this ball. The robust stability of multi-scenario NMPC can be established from the following theorem.

Theorem 1: If the plant uncertainty parameter vector lies in the union of balls centered around the uncertainty parameters in the controller, i.e. $\phi(j) \in \bigcup_{l=1}^{M} B_{\delta l}, \forall j$, then under Assumption 1 in [5], with $K \geq L_\epsilon$, the cost function $V(x(k)) = \sum_{l=1}^{M} w_l \left[F_l(z_l(N)) + \sum_{j=0}^{N-1} \varphi(z_l(j), v(j), \theta_l)\right]$ is an ISpS-Lyapunov function for plant (1), and the resulting closed-loop system is ISpS stable.

The proof of Theorem 1 is given in [5].

This Theorem guarantees the closed-loop stability if the plant uncertainty ϕ is close to one of uncertainty parameters θ_l. The robust stability will improve if the majority of the uncertainty parameters in the controller are close to that in the plant. In practice, if the multi-scenario NMPC appears to be unstable, we can add more scenarios to make sure that ϕ is in the neighborhood of θ_l.

III. ILLUSTRATIVE EXAMPLE

We consider a simulated NMPC scenario with a nonlinear CSTR model, represented by the following differential equations:

$$\frac{dz_c}{dt} = \frac{z_c - 1}{\theta} + k_0 z_c \exp\left[\frac{-E_a}{z_t}\right] \quad (4a)$$

$$\frac{dz_t}{dt} = \frac{z_t - z_t^f}{\theta} - k_0 z_c \exp\left[\frac{-E_a}{z_t}\right] + \alpha v(z_t - z_t^{cw}) \quad (4b)$$

This system involves two states $z = [z_c, z_t]$ corresponding to dimensionless concentration and temperature, and one control v corresponding to the cooling water flow rate. The states are required to satisfy $0 \leq z_c \leq 0.4$, $0.6 \leq z_t \leq 0.9$, while the control needs to satisfy $250 \leq v \leq 450$. The model parameters are $z_t^{cw} = 0.38$, $z_t^f = 0.395$, $E_a = 5$, $\alpha = 1.95 \times 10^4$ and $k_0 \in [200, 400]$ is deemed as an uncertainty parameter in the plant. The system is moved from a stable steady state to an unstable steady state at time step 25. The multi-scenario NMPC is formulated with horizon equals to 10, sampling time as 0.5. Here we choose three scenarios with $k_1 = 220$, $k_2 = 300$, $k_3 = 380$ and the corresponding weights are $w_1 = w_3 = 0.3$, $w_2 = 0.4$.

We consider two cases. 1) $k_0 = 300$ in the plant, if an standard NMPC is utilized with $k_0 = 400$ in the controller, the closed-loop system will be unstable. Moreover, the minmax formulation gives 28% steady state offset for z_c. The dotted lines in Figure 1 show that the closed-loop system is stable with the multi-scenario formulation and the offset of the controlled variables is small. 2) $k_0 = 200$ in the plant, the solid lines in Figure 1 show that the closed-loop system is stable, though it presents relatively large offset for z_c due to significant plant-model mismatch. In addition, without plotting the result here, we see that with $k_0 = 400$ in the plant, the multi-scenario formulation is able to stabilize the closed-loop system. As a result, the multi-scenario formulation guarantees the robust stability of this system within the entire uncertainty region of k_0.

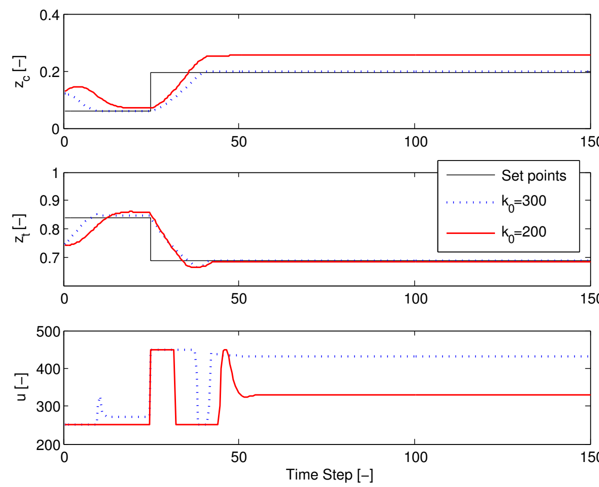

Fig. 1. Application multi-scenario NMPC.

IV. CONCLUSIONS AND FUTURE WORK

In this work, a robust NMPC design based on a multi-scenario NLP formulation is proposed. The robust stability of the proposed method can be established. Moreover it generally yields less offset than the min-max formulation. The proposed method is illustrated through a CSTR example.

For the future work, the proposed method will be implemented on large-scale applications. In order to reduced the associated computational burden and online feedback delay, the parallel decomposition algorithm [3] and advanced step NMPC [4] will be studied and integrated with the multi-scenario formulation.

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