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Stable Certainty Equivalence Adaptive Control using Normalized Parameter Adjustment Laws

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Abstract— **Parameter estimation problem in dynamical systems can be addressed using the static adaptive observers to generate parameter estimates on line. However, if such estimates are used in a Certainty Equivalence Adaptive Control (CEAC) law, the stability problem arises. In this paper it is shown that the CEAC law, in which parameter estimates are generated using static observers and several adaptive laws with normalization, results in a stable system in which the tracking control objective is achieved. The adaptive laws include gradient algorithms with normalization and projection, and least squares with covariance resetting and exponential forgetting. This is followed by an analysis of the case when dynamic observers and adaptive laws with normalization are used to generate parameter estimates. It is shown that with such adaptive laws overall system stability can be guaranteed provided that the observer is sufficiently fast and that its gain is chosen to be larger than a calculable worst-case bound.**

I. Introduction

A standard parameter adjustment law that is used both in the direct and indirect adaptive control context is the so called gradient algorithm [2]. It arises in the Lyapunov analysis of the overall adaptive control system, and results in a stable system in which the tracking control objective is achieved asymptotically. However, since its primary objective is not to accurately identify the parameters but rather to assure system stability, it may also result in large transients and inaccurate state estimates. This is the case in both direct and indirect adaptive control.

On the other hand, there is a whole spectrum of parameter adjustment algorithms whose primary objective is to arrive at as accurate parameter estimates as possible for a given level of excitation in the system. These arise in the context of system identification (see e.g. [1], [3]) and include gradient with normalization and projection, and several variants of the celebrated least-squares (LS) algorithm (pure LS, LS with covariance resetting, LS with exponential forgetting). One common feature of all these algorithms is that they represent a class of adjustment laws with normalization, i.e. they are commonly divided by a suitably chosen normalization term to assure the boundedness of the adaptive law.

To the best of authors knowledge, adaptive algorithms with normalization, in conjunction with dynamic observers, have not been used extensively to address the adaptive control problem. One of the existing results [4] is based on dividing the error equation with a normalization term, and using a gradient adaptive algorithm with normalization to adjust the parameters. Then a term roughly corresponding to "nonlinear damping" in nonlinear control is used in the error model to assure signal boundedness. One of the questions that arise in this context is whether such a term is necessary to demonstrate the overall stability. In addition, least squares-based algorithms have not been used in this context. Another approach to using normalized adaptive laws arising from the use of a logarithmic Lyapunov function is presented in [5]. However, the approach is applicable only to a very limited class of linear plants with unknown matrix A and known matrix B.

In the context of indirect adaptive control using parameter adjustment laws with normalization, the following problems can be formulated:

1. If a Certainty Equivalence Adaptive Control (CEAC) law (i.e. indirect adaptive control) is used to control the plant, under which conditions can adaptive algorithms with normalization be used to generate parameter estimates while assuring the overall system stability?

2. How can the system be parameterized to enable the use of static observers and adaptive algorithms with normalization?

3. Can adaptive algorithms with normalization be used in the context of dynamic observers and under which conditions?

In this paper we address these questions and show the following:

- In the case of static observers, proper filtering of the plant state equation results in an expression that enables the use of static observers to estimate the parameters; any parameter adjustment law that satisfies standard properties plus a property of boundedness of the term multiplying the estimation error (i.e. the regressor divided by the normalization term), multiplied by the regressor signal, results in a stable system.
- In the case of dynamic observers, an additional condition needs to be imposed on the observer gain to assure the overall system stability.

II. Problem Statement

In this section the focus is on a first order plant whose dynamics is described by:

$$\dot{x} = ax + bu, \tag{1}$$

where a and b are unknown, and the lower bound on b, denoted by $\underline{b} > 0$, is known.

The objective is to design a Certainty-Equivalence Adaptive Control (CEAC) signal $u(t)$ such that $\lim_{t\to\infty}[x(t) - x_m(t)] = 0$, where $x_m(t)$ is the output of the reference model

$$\dot{x}_m = -a_m x_m + b_m r, \quad (2)$$

and where $a_m, b_m > 0$ and r is a bounded reference input. The CEAC control law is chosen in the form:

$$u = \frac{1}{\hat{b}}[-(\hat{a} + a_m)x + b_m r]. \quad (3)$$

Since the control law is given, now the objective is to design an observer and adaptive law for generating estimates of a and b in a way that assures the overall system stability.

Let $e = \hat{x} - x$. It is well known that a dynamic series-parallel observer and gradient adaptive algorithms of the form:

$$\dot{\hat{x}} = \hat{a}x + \hat{b}u - \lambda e$$

$$\dot{\hat{a}} = -\gamma_1 ex, \quad \dot{\hat{b}} = -\gamma_2 eu$$

assure overall system stability and achieve the control objective [2]. In the above expressions, $\lambda > 0$ denotes the observer gain, while $\gamma_i > 0$ denote adaptive gains. In addition, precautions need to be taken to prevent $\hat{b}(t)$ from crossing zero which would invalidate the CEAC law.

With proper choice of λ and γ_i, the above adaptive laws can achieve rapid stabilization of the system and ensure that the control objective is met. However, these laws are not designed with a primary objective of achieving accurate estimation of the unknown parameters, and the resulting steady-state values (if they exist) of the estimates may be far from the true ones if there is not sufficient excitation in the system. On the other hand, several adaptive algorithms with normalization have been shown to result in "good" estimates even when signals in the system are not persistently exciting.

Since one of the objectivies of this paper is to study the suitability of such adaptive algorithms in the CEAC law, several possible algorithms with normalization for estimating a and b are discussed next.

III. Stable Adaptive Control using Static Observers and Normalized Adjustment Laws

The following filtered variables are introduced first:

$$x_F = \frac{1}{s + \lambda_F} \cdot x, \; u_F = \frac{1}{s + \lambda_F} \cdot u. \quad (4)$$

Then the resulting filtered plant equation is of the form:

$$x = (\lambda_F + a)x_F + bu_F, \quad (5)$$

modulo an exponentially decaying term due to initial conditions. The above equation is rewritten as: $x = \lambda_F x_F + \theta^T\omega$, where $\theta = [a \; b]^T$ and $\omega = [x_F \; u_F]^T$. Now the observer is chosen as:

$$\hat{x} = \lambda_F x_F + \hat{\theta}^T\omega, \quad (6)$$

resulting in the error equation of the form: $\epsilon = \tilde{\theta}^T\omega$, where $\epsilon = \hat{x} - x$ denotes the estimation error, and $\tilde{\theta} = \hat{\theta} - \theta$ denotes the parametric error.

Let the CEAC law (3) be used to control the plant, where the estimator is chosen as in (6), while the parameter adjustment law is to be determined. The following theorem states that this law can be fairly general as long as it satisfies several properties listed below.

Theorem 1: *If the adaptive law ensures the following:*

(a) $\hat{\theta} \in \mathcal{L}^\infty$.

(b) $\dfrac{\epsilon}{\sqrt{1+\omega^T\omega}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$.

(c) $\dot{\hat{\theta}} = \dot{\tilde{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$, *and*

(d) $\dfrac{\dot{\hat{\theta}}^T\omega}{e}$ *is bounded for all values of arguments,*

then all the signals in the system are bounded and $\lim_{t\to\infty}[x(t) - x_m(t)] = 0$.

Proof: The observer (6) is first differentiated to obtain:

$$\dot{\hat{x}} = -\lambda_F^2 x_F + \lambda_F x + \dot{\hat{\theta}}^T\omega - \lambda_F(\hat{x} - \lambda_F x_F) + \hat{a}x + \hat{b}u,$$

which, after substituting the control law (3), reduces to

$$\dot{\hat{x}} = \dot{\hat{\theta}}^T\omega - \lambda_F\epsilon - a_m x + b_m r.$$

Let $e_m = \hat{x} - x_m$. Subtracting (2) from the above eq.yields:

$$\dot{e}_m = -a_m e_m + \dot{\hat{\theta}}^T\omega - (\lambda_F - a_m)\epsilon, \quad (7)$$

Since $\hat{\theta}$ is bounded and $\dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$, it follows that the system (7) is a linear time varying system with bounded parameters in which the signals can grow at most exponentially. Hence analysis based on the growth rates of signals in the system (see [2]) can be used to study the signal boundedness. Based on the fact that $\epsilon/\sqrt{1+\omega^T\omega} \in \mathcal{L}^\infty \cap \mathcal{L}^2$, it follows that $\epsilon = \beta(t)\sqrt{1+\omega^T\omega}$, where $\beta \in \mathcal{L}^2$.

Let $\bar{\omega} = [x \; u]^T$ grow in an unbounded fashion. Since $\omega = (1/(s+\lambda_F)\bar{\omega}$, it follows that $\omega = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. In addition, since $\dot{\omega} = -\lambda_F\omega + \bar{\omega}$, it follows that $\dot{\omega} = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. Since $\epsilon = \tilde{\theta}^T\omega$ and $\tilde{\theta}$ is bounded, it follows that $\epsilon = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. Taking a derivative of ϵ yields:

$$\dot{\epsilon} = \dot{\tilde{\theta}}^T\omega - \lambda_F\epsilon + \tilde{\theta}^T\bar{\omega}.$$

Hence $\dot{\epsilon} = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. From (7) it now follows that $e_m = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$ and $\dot{e}_m = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. Since $e_m = \hat{x} - x_m$ and x_m is bounded, it follows that $\hat{x} = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$ and $\dot{\hat{x}} = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. Since $\epsilon = \hat{x} - x$, it follows that $x = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. Similarly, it can be shown that $u = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$. Hence $\bar{\omega} = \mathcal{O}(sup_{\tau\leq t}\|\bar{\omega}(\tau)\|)$, and all the signals grow at the same rate, i.e. $sup_{\tau\leq t}|\epsilon(\tau)| \sim sup_{\tau\leq t}\|\bar{\omega}(\tau)\|$.

On the other hand, since $\epsilon = \beta(t)\sqrt{1+\omega^T\omega}$, where $\beta \in \mathcal{L}^2$, and $\dot{\epsilon} = \mathcal{O}(sup_{\tau\leq t}\|\omega(\tau)\|)$, it follows that $\epsilon = \boldsymbol{o}(sup_{\tau\leq t}\|\omega(\tau)\|)$ [2], i.e. ϵ and $\|\omega\|$ grow at different rates, which is a contradiction. Hence all the signals in the closed-loop system are bounded.

It can now be readily shown using the standard arguments [2] that $lim_{t\to\infty}e_m(t) = lim_{t\to\infty}[x(t) - x_m(t)] = 0$. ■

Comment: The main reason for using the signal growth rate arguments for stability analysis is that the estimator does not guarantee that the estimation error will be bounded. This is in contrast with the case when dynamic observers and adjustment laws without normalization are used.

It will next be shown that several parameter adjustment laws satisfy the conditions (a)-(d) of Theorem 1.

IV. Normalized Parameter Adjustment Laws

A. Normalized Gradient

This adaptive law is of the form:

$$\dot{\hat{\theta}} = \dot{\tilde{\theta}} = -\frac{-\Gamma\omega\epsilon}{1+\omega^T\omega} \tag{8}$$

where $\Gamma = \text{diag}([\gamma_1 \ \gamma_2])$.

Properties of the estimator: The following tentative Lyapunov function is chosen:

$$V(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}. \tag{9}$$

Its derivative along the solutions of the system yileds:

$$\dot{V}(\tilde{\theta}) = -\frac{\epsilon^2}{1+\omega^T\omega} = -\frac{\omega\omega^T\tilde{\theta}}{1+\omega^T\omega} \leq 0, \ \forall\tilde{\theta}\neq 0. \tag{10}$$

Hence $\hat{\theta}\in\mathcal{L}^\infty$. It can be readily shown that conditions (b) and (c) of Theorem 1 also hold. Based on (8) it is also seen that the condition (d) holds. Hence gradient algorithm with normalization can be used to estimate the parameters while ensuring the closed-loop system stability.

B. Normalized Gradient with Projection

Let an m-vector θ satisfy $\theta \in \mathcal{S}_\theta = \{\theta_i : \underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i, \ i = 1,2,...,m\}$. Then the normalized adaptive law with projection is expressed in the form:

$$\begin{aligned}\dot{\hat{\theta}}_i = \dot{\tilde{\theta}}_i &= \text{Proj}_{\hat{\theta}\in\mathcal{S}_\theta}\{-\frac{\gamma_i\omega_i e}{1+\omega^T\omega}\}, \ i=1,2,...,m \\ &= -\frac{\gamma_i\omega_i e}{1+\omega^T\omega}(1-\text{sign}(e\omega_i)\cdot\varphi(\hat{\theta}_i))\end{aligned} \tag{11}$$

where $\hat{\theta}_i(0)\in\mathcal{S}_\theta$, and

$$\varphi(\hat{\theta}_i) = \begin{cases} \dfrac{\overline{\theta}_i - 2\theta_i + \underline{\theta}_i}{2(\overline{\theta}_i - \underline{\theta}_i)} & \text{if } \theta_i(t)\in\partial\mathcal{S}_\theta \\ 0 & \text{if } \theta_i(t)\in\mathcal{S}_\theta\backslash\partial\mathcal{S}_\theta \end{cases}.$$

The following proposition is useful for the proof of stability.

Proposition 1: Adaptive law (11) assures that

$$\tilde{\theta}^T\dot{\tilde{\theta}} \leq -\frac{\Gamma\tilde{\theta}\omega e}{1+\omega^T\omega},$$

for all values of arguments.

The proof of this proposition is standard for adaptive algorithms with projection and is omitted here.

Now the same tentative Lyapunov function (9) is used, resulting in

$$\dot{V}(\tilde{\theta}) \leq -\frac{\epsilon^2}{1+\omega^T\omega} \leq 0, \ \forall\tilde{\theta}\neq 0. \tag{12}$$

Hence the gradient adaptive algorithm with projection has the same (a)-(c) properties as the one without projection. From (11) it also follows that the condition (d) is satisfied. The use of this algorithm also assures that $\hat{b}(t)\geq\underline{b}, \ \forall t\geq t_o$.

C. Least Squares Algorithm

As it is well known, the Recursive Least Squares (RLS) algorithm is derived as a solution to the minimization of the following cost [4]:

$$\begin{aligned} J(\hat{\theta}) &= \frac{1}{2}\int_0^t e^{-\beta(t-\tau)}\frac{(\hat{x}(\tau)-\hat{\theta}^T(\tau)\omega(\tau))^2}{1+\omega^T(\tau)\omega(\tau)}d\tau \\ &\quad + \frac{1}{2}e^{-\beta t}(\hat{\theta}-\hat{\theta}(0))^T Q_0(\hat{\theta}-\hat{\theta}(0)), \end{aligned}$$

where $Q_0 = Q_0^T > 0$, and $\beta\geq 0$. Since the above cost is convex, any local minimum is also global and results in:

$$\begin{aligned} \nabla J(\hat{\theta}) &= e^{-\beta t}Q_0(\hat{\theta}-\hat{\theta}(0)) \\ &\quad - \int_0^t e^{-\beta(t-\tau)}\frac{\hat{x}(\tau)-\hat{\theta}^T(\tau)\omega(\tau)}{1+\omega^T(\tau)\omega(\tau)}\omega d\tau, \end{aligned}$$

which yields the *batch least squares algorithm*:

$$\hat{\theta} = P(t)\Big[e^{-\beta t}Q_0\hat{\theta}(0) + \int_0^t e^{-\beta(t-\tau)}\frac{\hat{x}(\tau)\omega(\tau)}{1+\omega^T(\tau)\omega(\tau)}d\tau\Big]$$

where

$$P(t) = \Big[e^{-\beta t}Q_0 + \int_0^t e^{-\beta(t-\tau)}\frac{\omega(\tau)\omega^T(\tau)}{1+\omega^T(\tau)\omega(\tau)}d\tau\Big]^{-1}.$$

Since Q is positive definite and $\omega\omega^T$ is positive semi-definite, $P(t)$ exists at every instant. It can be readily shown that P satisfies the following differential equation:

$$\dot{P} = \beta P - P\frac{\omega\omega^T}{1+\omega^T\omega}P, \ P(0) = P_0 = Q_0^{-1}. \tag{13}$$

The adjustment law for $\hat{\theta}$ is now of the form:

$$\dot{\hat{\theta}} = -\frac{P\omega\epsilon}{1+\omega^T\omega}. \tag{14}$$

1) ***Pure Least Squares:*** In this case $\beta = 0$. It is first noted that $\dot{P}\leq 0$, i.e. $P(t)\leq P_0$. Since $P\geq 0$, it follows that $P(t)$ is bounded for all time.

Now a tentative Lyapunov function is chosen as:

$$V(\tilde{\theta}) = \frac{1}{2}\tilde{\theta}^T P^{-1}\tilde{\theta}. \tag{15}$$

Noting that $\dot{P}^{-1} = \frac{\omega\omega^T}{1+\omega^T\omega}$, it follows that

$$\dot{V}(\tilde{\theta}) = -\frac{\epsilon^2}{2(1+\omega^T\omega)} \leq 0, \ \forall\tilde{\theta}\neq 0.$$

Since $P(t)$ is bounded for all time, it follows that the pure LS algorithm satisfies conditions (a)-(d) of the Theorem 1.

2) ***Least Squares with Projection:*** Using the same tentative Lyapunov function as in the case of pure least squares and the results of Proposition 1 it follows that

$$\dot{V}(\tilde{\theta}) \leq -\frac{\epsilon^2}{2(1+\omega^T\omega)} \leq 0, \ \forall\tilde{\theta}\neq 0.$$

Hence the least squares algorithm with projection has the same properties as the pure least squares and satisfies the conditions (a)-(d) of the Theorem 1.

3) ***Least Squares with Covariance Resetting:*** One of the main drawbacks of the pure LS algorithm is a possibility of

$\lim_{t\to\infty} P(t) = 0$ which slows down the adaptation in some directions. To avoid this, the following modification, known as LS with covariance resetting, is commonly used:

$$\dot{P} = -P\frac{\omega\omega^T}{1+\omega^T\omega}P,\ P(t_R^+) = P_0 = a_0 I \tag{16}$$

where t_R is the resetting time for which $\lambda_{min}(P(t)) \le a_1$ where a_1 is a prespecified threshold, and $a_0 > a_1 > 0$. With this modification it follows that $a_0 I \ge P(t) \ge a_1 I$ for all time, which implies that $P(t)$ is positive definite and bounded for all $t \ge 0$.

Using the tentative Lyapunov function (15) results in

$$\dot{V}(\tilde{\theta}) = -\frac{\epsilon^2}{2(1+\omega^T\omega)} \le 0,$$

everywhere except at the resetting instants.

At the points of discontinuity of P one has:

$$V(t_R^+) - V(t_R) = \frac{1}{2}\tilde{\theta}^T(P^{-1}(t_R^+) - P^{-1}(t_R))\tilde{\theta}.$$

Since $P^{-1}(t_R^+) = \frac{1}{a_0}I$ and $P^{-1}(t_R) \ge \frac{1}{a_0}I$, it follows that $V(t_R^+) - V(t_R) \le 0$. Hence V is a non-increasing function of time for all $t \ge 0$. This implies that V is bounded and $\lim_{t\to\infty} V(t) = V_\infty < \infty$. Since the points of discontinuity form a set of measure zero, it follows that $\epsilon/\sqrt{1+\omega^T\omega} \in \mathcal{L}^2$. P^{-1} is bounded since $a_1^{-1}I \ge P^{-1} \ge a_0^{-1}I$. Bounded V and P^{-1} imply that $\tilde{\theta}$ is bounded; it follows that $\epsilon/\sqrt{1+\omega^T\omega} \in \mathcal{L}^\infty$, which implies that $\dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$. Hence properties (a)-(c) from Theorem 1 hold. Resetting does not change the expression for $\dot{\hat{\theta}}$. Hence the property (d) holds as well.

It can be readily shown that this algorithm satisfies these properties even when projection modification is used.

4) ***Least Squares with Exponential Forgetting:*** In this case $\beta > 0$ and $P(t)$ can grow without bound. A common modification to the LS algorithm is given below [4].

$$\begin{aligned}\dot{\hat{\theta}} &= -\frac{P\omega\epsilon}{1+\omega^T\omega}\\ \dot{P} &= \begin{cases}\beta P - P\dfrac{\omega\omega^T}{1+\omega^T\omega}P, & \text{if } \|P(t)\| \le \bar{p}\\ 0, & \text{elsewhere}\end{cases}\end{aligned}$$

where $P(0) = P_0 = P_0^T > 0$, $\|P_0\| \le \bar{p}$, and $\bar{p}$ is an upper bound on the norm of P. This modification keeps $P(t)$ bounded for all time and results in the same estimator properties as in the case of least squares and hence satisfies the properties (a)-(d) from Theorem 1.

It can be concluded that all of the algorithms analyzed in this section satisfy properties (a)-(d) of Theorem 1 and corresponding parameter estimates can, therefore, be used in the CEAC control law to guarantee closed-loop stability.

V. Stable Adaptive Control using Dynamic Observers and Normalized Adjustment Laws

Next question that will be addressed in this paper is whether adaptive laws with normalization can be used in the context of dynamic observers. It will be shown that in such a case an additional condition on the observer gain needs to be imposed to assure system stability. This heuristically makes sense since the dynamic observers are generally slower than the static ones (since the state estimate is generated by a differential equation, while in the case of static observers the estimate is calculated instantaneously); adaptive laws with normalization are slower than the ones without normalization and, therefore, the dynamic observer needs to be sufficiently fast to assure system stability.

In this section the same plant equation (1) is considered. Let $\theta = [a\ b]^T$. It is assumed that $\theta \in \mathcal{S}_\theta = \{\theta : \underline{\theta}_i \le \theta_i \overline{\theta}_i,\ i = 1,2\}$, where $\underline{\theta}_2 > 0$. The CEAC controller is of the form:

$$u = \frac{1}{\hat{b}}[-(\hat{a}+a_m)x + b_m r], \tag{17}$$

where the bound on $\dot{r}$ is assumed known, i.e. $|\dot{r}| \le \bar{r}$.

The dynamic observer for the above plant is now chosen as:

$$\dot{\hat{x}} = \hat{a}x + \hat{b}u - \lambda e, \tag{18}$$

where $\lambda > 0$ denotes the observer gain, and $e = \hat{x} - x$.

Let the adaptive law be of the form:

$$\dot{\tilde{\theta}} = \dot{\hat{\theta}} = \text{Proj}_{\hat{\theta}\in\mathcal{S}_\theta}\{-\frac{\Gamma\bar{\omega}e}{1+\omega^T\omega}\} \tag{19}$$

where $\Gamma = \text{diag}[\gamma_1\ \gamma_2]$, $\gamma_i > 0$, $\bar{\omega} = [x\ u]^T$, and $\omega = [x\quad u\quad \hat{x}/\sqrt{\lambda}]^T$.

The key in proving the stability with the above adaptive law is to study the term $\omega^T\dot{\omega}/(1+\omega^T\omega)$. The term $\omega^T\dot{\omega}$ is of the form:

$$\omega^T\dot{\omega} = x(ax+bu) + u\dot{u} - \hat{x}e + \hat{x}(\hat{a}x+\hat{b}u)/\lambda, \tag{20}$$

while the derivative of the CEAC law yields:

$$\begin{aligned}|\dot{u}| &= \Big|\frac{e\bar{\omega}^T\Gamma\bar{\omega}}{\hat{b}(1+\omega^T\omega)} + \frac{1}{\hat{b}}[-(\hat{a}+a_m)(ax+bu) + b_m\dot{r}]\Big|\\ &\le [(|\hat{x}|+|x|)\lambda_{max}(\Gamma) + ((\overline{a}+a_m)(\overline{a}|x|+\overline{b}|u|)+b_m\bar{r})]/\underline{b}\\ &= c_1|x| + c_2|u| + c_3|\hat{x}| + c_4.\end{aligned}$$

Hence:

$$\begin{aligned}|\omega^T\dot{\omega}| \le{}& \overline{a}x^2 + \overline{b}|x||u| + c_1|x||u| + c_2u^2 + c_3|\hat{x}||u| + c_4|u|\\ &+\hat{x}^2 + (1+\overline{a}/\lambda)|\hat{x}||x| + \overline{b}|\hat{x}||u|/\lambda.\end{aligned}$$

It now follows that:

$$\frac{|\omega^T\dot{\omega}|}{1+\omega^T\omega} \le d_1 + \frac{d_2}{\lambda},\ \forall\omega, \tag{21}$$

where $d_i > 0,\ i = 1,2$, can be readily calculated.

Now the following theorem is considered:

Theorem 2: *Let the plant* (1) *be controlled by the CEAC law* (17) *where the state and parameter estimates are generated using the observer* (18) *and adaptive law* (19)*. Then, if the observer gain satisfies* $\lambda > (d_1 + \sqrt{d_1^2 + 4d_2})/2$*, where* d_i *are given by* (21)*, all the signals in the system are bounded, and* $lim_{t\to\infty}[x_1(t) - x_{m1}(t)] = 0$.

Proof: Let $\theta = [a\ b]^T$, and $\hat{\theta} = [\hat{a}\ \hat{b}]^T$. Upon subtracting (1) from the observer dynamics (18), the error equation is obtained as:

$$\dot{e} = -\lambda e + \tilde{\theta}^T \bar{\omega}, \tag{22}$$

where $e = \hat{x} - x$, $\bar{\omega} = [x \ u]^T$, and $\tilde{\theta} = \hat{\theta} - \theta$.

Let a coordinate transformation ζ be defined as:

$$\zeta = \frac{e}{\sqrt{1 + \omega^T \omega}}.$$

It follows that:

$$\zeta\dot{\zeta} = \frac{-e^2 \omega^T \dot{\omega}}{(1 + \omega^T \omega)^2} + \frac{-\lambda e^2 + \tilde{\theta}\bar{\omega}e}{(1 + \omega^T \omega)}.$$

Now the following Lyapunov function is chosen:

$$V(\zeta, \phi) = \frac{1}{2}(\zeta^2 + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}),$$

where $\Gamma = \text{diag}[\gamma_1 \ \gamma_2]$. The derivative of V along the solutions of the system yields

$$\dot{V}(\zeta, \phi) \leq \frac{-e^2}{1 + \omega^T \omega}(\lambda - \frac{\omega^T \dot{\omega}}{1 + \omega^T \omega}).$$

The inequality follows from applying the adaptive law with projection (19). Using expression (21) it follows that:

$$\dot{V} \leq \frac{-e^2}{1 + \omega^T \omega}(\lambda - d_1 - \frac{d_2}{\lambda}) \leq 0,$$

i.e. $\dot{V} \leq 0$ since $\lambda > (d_1 + \sqrt{d_1^2 + 4d_2})/2$. It can now be concluded that $\zeta \in \mathcal{L}^\infty \cap \mathcal{L}^2$, $\tilde{\theta} \in \mathcal{L}^\infty$. It also follows from (19) that $\dot{\tilde{\theta}} = \dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$.

To prove the overall stability, the CEAC law (17) is substituted into the observer (18) to obtain:

$$\dot{\hat{x}} = -\lambda e - a_m x + b_m r. \tag{23}$$

It follows that $\dot{e}_m = -a_m e_m - (\lambda - a_m)e$, where $e_m = \hat{x} - x_m$. It is seen that if λ is chosen as $\lambda = a_m$, e_m is bounded which implies that $\hat{x}$ is bounded, and $\lim_{t\to\infty} e_m(t) = 0$. However, choosing $\lambda = a_m$ could violate the condition of the Theorem 2. Hence arguments based on the growth rates of signals can be used along the same lines as in the proof of Theorem 1 to demonstrate system stability. ■

Comments:

• Unlike the static observer case, in the case of a dynamic observer an additional condition is imposed on the observer gain to counteract the slowing down of the response due to the normalization in the adaptive law.

• It is noted that the main condition that the adaptive law needs to satisfy for the overall system stability is that $u\dot{u}/(1 + \omega^T \omega)$ is bounded for all values of its arguments. Analysis of this condition reveals that it is satisfied if the condition (d) of Theorem 1 is satisfied. Since the least squares-based algorithms described in the previous sections satisfy this condition, they can be used to estimate the plant parameters without affecting the system stability.

VI. Higher-order Plants

In this case the plant dynamics is assumed in the form:

$$\dot{x} = Ax + BKu \tag{24}$$

where $x : \mathbb{R}^+ \to \mathbb{R}^n$, $u : \mathbb{R}^+ \to \mathbb{R}^m$, $m \geq n$, and where $A \in \mathbb{R}^{n\times n}$ and $K = \text{diag}[k_1 \ k_2 \ ... \ k_m]$, $k_i \in [\epsilon_i, 1]$, $\epsilon_i << 1$, are uncertain, while $B \in \mathbb{R}^{n\times m}$ is known. It is assumed that x and u are measurable. Let a denote a vector of elements of A, and let $p = [a^T \ k^T] \in \mathcal{S}_p = \{p : (p_i)_{min} \leq p \leq (p_i)_{max}, \ i = 1, 2, ..., n^2 + m\}$.

The objective is to design a control signal $u(t)$ such that $\lim_{t\to\infty}[x(t) - x_m(t)] = 0$, where x_m is an output of the following reference model:

$$\dot{x}_m = A_m x_m + B_m r, \tag{25}$$

where $A_m \in \mathbb{R}^{n\times n}$ is Hurwitz, $B_m \in \mathbb{R}^{n\times n}$, and $r \in \mathbb{R}^n$ denotes a bounded reference input vector.

CEAC Law: The CEAC law is chosen as:

$$u = \hat{K}B^T(B\hat{K}^2 B^T)^{-1}(-\hat{A}x + A_m x + B_m r). \tag{26}$$

Static Adaptive Observer: To derive a static adaptive observer, equation (24) is first filterd to obtain:

$$x = \lambda_F x_F + Ax_F + BKu_F \tag{27}$$

modulo exponentially decaying terms due to initial conditions. This equation is now rewritten as:

$$x = \lambda_F x_F + F(x_F)a + BU_F k \tag{28}$$

where $F(x_F) = \text{diag}([x_F^T \ x_F^T \ ... \ x_F^T])$, and $U_F = \text{diag}[u_{1F} \ u_{2F} \ ... \ u_{mF}]$. The observer is now chosen as:

$$\hat{x} = \lambda_F x_F + \Omega(x_F, u_F)\hat{\theta}, \tag{29}$$

where $\Omega = [F \ BU_F]$, $\theta = [a^T \ k^T]^T$ and $\hat{\theta} = [\hat{a}^T \ \ \hat{k}^T]^T$.

We next note that, in the case when projection is used, the adaptive algorithms from Section IV can be written for the above observer in a unified manner as:

$$\dot{\hat{\theta}} = \text{Proj}_{\hat{\theta}\in\mathcal{S}_\theta}\{-\frac{P\Omega^T e}{1 + \tilde{\omega}^T \tilde{\omega}}\} \tag{30}$$

$$\dot{P} = \begin{cases} \beta P - \delta_1 P \dfrac{\tilde{\omega}\tilde{\omega}^T}{1 + \tilde{\omega}^T \tilde{\omega}} P, \text{ if } \ \|P(t)\| \leq \bar{p} \ \text{ and } \ \beta > 0 \\ 0, \qquad\qquad \text{if } \ \|P(t)\| > \bar{p} \ \text{ and } \ \beta > 0 \\ -\delta_2 P \dfrac{\tilde{\omega}\tilde{\omega}^T}{1 + \tilde{\omega}^T \tilde{\omega}} P \ \text{ and } P(t_R^+) = P_0 = a_0 I, \ \text{if } \beta = 0 \end{cases}$$

where $P(0) = P_0 = P_0^T > 0$, $\|P_0\| \leq \bar{p}$, $\bar{p}$ is an upper bound on the norm of P, and $\delta_i \in \{0, 1\}$. It is now seen that: (i) for $\delta_i = 0$, $\beta = 0$ and $P(0) = \Gamma$, the above algorithm reduces to the gradient algorithm with normalization and projection; (ii) when $\beta > 0$, $\delta_1 = 1$ and $\delta_2 = 0$, it reduces to the least squares with normalization and exponential forgetting; and (iii) when $\beta = 0$, $\delta_1 = 0$ and $\delta_2 = 1$, it reduces to least squares with normalization and covariance resetting. In all cases $P(t)$ is bounded and its upper bound is known. Hence these algorithms have the following common properties:

$$\left.\begin{array}{ll} (a) & \hat{\theta} \text{ is bounded.} \\ (b) & \dfrac{e}{\sqrt{1 + \tilde{\omega}^T \tilde{\omega}}} \in \mathcal{L}^\infty \cap \mathcal{L}^2. \\ (c) & \dot{\hat{\theta}} = \dot{\tilde{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2, \text{ and} \\ (d) & \dfrac{\Omega\dot{\hat{\theta}}}{e} \in \mathcal{L}^\infty \text{ for all values of arguments} \end{array}\right\} \tag{31}$$

Theorem 3: If the adaptive laws satisfy the properties (31), then the plant, controlled by the CEAC will be stable, and $\lim_{t\to\infty}[x(t) - x_m(t)] = 0$.

Proof: The observer equation is differentiated to obtain:

$$\dot{\hat{x}} = -\lambda_F e + \Omega\dot{\hat{\theta}} + \hat{A}x + B\hat{K}u$$

which is derived based on the fact that

$$\begin{aligned} \dot{\Omega}(x_F, u_F)\hat{\theta} &= [\dot{F}(x_F) \quad B\dot{U}_F]\hat{\theta} \\ &= -\lambda_F(\hat{x} - \lambda_F x_F) + [F(x) \quad BU]\hat{\theta} \end{aligned}$$

and where the identity $[F(x) \quad BU]\hat{\theta} = \hat{A}x + B\hat{K}u$ is used. The latter follows from the identities: $Ax \equiv F(x)a$ and $Ku \equiv Uk$. After substituting the control law (26), we have:

$$\dot{\hat{x}} = \Omega\dot{\hat{\theta}} - \lambda_F e + A_m x + B_m r. \tag{32}$$

Subtracting the reference model equation yields:

$$\dot{e}_m = A_m e_m + \Omega\dot{\hat{\theta}} - (\lambda_F I + A_m)e, \tag{33}$$

where $e_m = \hat{x} - x_m$.

Similarly as in the first order case, it can be concluded that, since $\hat{\theta}$ is bounded and $\dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$, the system (7) is a linear time varying system with bounded parameters in which the signals can grow at most exponentially. Hence analysis based on the growth rates of signals can again be used, and the rest of the Theorem can be proved along exactly the same lines as in the case of Theorem 1. ■

Dynamic Adaptive Observer: The observer is chosen as:

$$\dot{\hat{x}} = -\Lambda e + \Omega(x, u)\hat{\theta}, \tag{34}$$

where $\Lambda = \text{diag}[\lambda_1 \ \lambda_2 \ ... \ \lambda_m]$, $\lambda_i > 0$ and $\Omega = [F(x) \quad BU]$.

Let $e = \hat{x} - x$. Upon subtracting (28) from the observer equation, one obtains:

$$\dot{e} = -\Lambda e + \Omega(x, u)\tilde{\theta}, \tag{35}$$

where $\tilde{\theta} = \hat{\theta} - \theta$. The error model is rewritten as:

$$\dot{e} = -\Lambda e + \sum_{i=1}^{n^2+m} \omega_i \tilde{\theta}_i, \tag{36}$$

where ω_i is the ith column of Ω.

Let, for simplicity, $\lambda_1 = \lambda_2 = ... = \lambda_m = \lambda$. In this case the normalization signal $\tilde{\omega}$ is chosen as $\tilde{\omega} = [x^T \ u^T \ \hat{x}/\sqrt{\lambda}]^T$.

Theorem 4: If the adaptive laws for adjusting the parameters of the CEAC (26) satisfy the properties (31) with a normalization $1 + \tilde{\omega}^T\tilde{\omega}$, and if the observer gain is chosen to satisfy $\lambda > c_0$, where c_0 is a calculable bound, then all the signals in the system are bounded and $\lim_{t\to\infty}[x(t) - x_m(t)] = 0$.

Proof: We first note that

$$\frac{e^T\dot{e}}{1 + \tilde{\omega}^T\tilde{\omega}} = \frac{-\lambda e^T e + \sum_{i=1}^{n^2+m} e^T\omega_i\tilde{\theta}_i}{1 + \tilde{\omega}^T\tilde{\omega}}.$$

For adaptive laws (30), using the properties of adaptive algorithms with projection, it can be readily shown that the following holds:

$$\sum_{i=1}^{n^2+m} \frac{\tilde{\theta}_i\dot{\hat{\theta}}_i}{\gamma_i} \le -\sum_{i=1}^{n^2+m} \frac{\tilde{\theta}_i e^T\omega_i}{1 + \tilde{\omega}^T\tilde{\omega}}.$$

Hence:

$$\frac{e^T\dot{e}}{1 + \tilde{\omega}^T\tilde{\omega}} + \sum_{i=1}^{n^2+m} \frac{\tilde{\theta}_i\dot{\hat{\theta}}_i}{\gamma_i} \le 0.$$

We also note that:

$$\tilde{\omega}^T\dot{\tilde{\omega}} = -(x^T(Ax + BKu) + u^T\dot{u} - \hat{x}^T(\hat{x} - x - \Omega\hat{\theta}/\lambda)).$$

Hence the proof reduces to demonstrating that $u^T\dot{u}/(1 + \tilde{\omega}^T\tilde{\omega})$ is bounded for all values of arguments.

We note that the CEAC can be rewritten as: $B\hat{K}u = -\hat{A}x + A_m x + B_m r$. Hence $B\hat{K}\dot{u} = -\Omega(x, u)\dot{\hat{\theta}} + A_m x + B_m r$. It is seen that, if the property (31)(d) is satisfied, it follows that $| - u^T\Omega(x, u)\dot{\hat{\theta}}| \le c_0\lambda\|u\|(\|\hat{x}\| + \|x\|)$.Since elements of $\hat{K}$ are bounded away from zero, it now follows that $u^T\dot{u}$ is bounded for all values of arguments. Hence

$$\left|\frac{\tilde{\omega}^T\dot{\tilde{\omega}}}{1 + \tilde{\omega}^T\tilde{\omega}}\right| \le \bar{c}_1 + \frac{\bar{c}_2}{\lambda}, \forall(x, u, \hat{x}), \forall(\theta, \hat{\theta}) \in \mathcal{S}_\theta.$$

It can now be readily shown that choosing λ such that $\lambda > c_0 = (\bar{c}_1 + \sqrt{\bar{c}_1^2 + 4\bar{c}_2})/2$, guarantees that $e/\sqrt{1 + \tilde{\omega}^T\tilde{\omega}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$, and the rest of the proof follows. ■

VII. Conclusions

In the paper it is shown that the CEAC law, in which parameter estimates are generated using static observers and several adaptive laws with normalization, results in a stable system in which the tracking control objective is achieved. The adaptive laws include gradient algorithms with normalization and projection, and least squares with covariance resetting and exponential forgetting.

This is followed by an analysis of the case when dynamic observers and adaptive laws with normalization are used to generate parameter estimates. It is shown that the normalized adaptive laws can be used provided that the observer is sufficiently fast and that its gain is chosen to be larger than a calculable worst-case bound.

The results from the paper enable the use of several different adaptive algorithms with normalization to generate parameter estimates for the CEAC law.

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