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Networked Control of Spatially Distributed Processes with Sensor-Controller Communication Constraints

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Abstract— **This work presents a methodology for the design of a model-based networked control system for spatially distributed processes described by linear parabolic partial differential equations (PDEs) with measurement sensors that transmit their data to the controller/actuators over a bandwidth-limited communication network. The central design objective is to minimize the transfer of information from the sensors to the controller without sacrificing closed-loop stability. To accomplish this, a finite-dimensional model that captures the dominant dynamic modes of the PDE is embedded in the controller to provide it with an estimate of those modes when measurements are not transmitted through the network, and the model state is then updated using the actual measurements provided by the sensors at discrete time instances. Bringing together tools from switched systems, infinite-dimensional systems and singular perturbations, a precise characterization of the minimum stabilizing sensor-controller communication frequency is obtained under both state and output feedback control. The stability criteria are used to determine the optimal sensor and actuator configurations that maximize the networked closed-loop system's robustness to communication suspensions. The proposed methodology is illustrated using a simulation example.**

I. Introduction

With the significant growth in computing and networking abilities in recent times, as well as the rapid advances in actuator and sensor technologies, there has been an increased reliance in the process industry on sensor and control systems that are accessed over communication networks rather than dedicated links. The defining feature of networked control systems is that the control system components (sensors, controller, actuators) are connected using a shared communication network (which could be wired or wireless) over which information, such as the plant output and control input, are exchanged. The communication channel is typically shared by multiple feedback control loops. Compared with dedicated, point-to-point cables, communication networks have many advantages, such as reduced installation and maintenance time and costs, flexibility and ease of diagnosis and reconfiguration, as well as enhanced fault-tolerance and supervisory control. Yet, control over networks also poses a number of fundamental issues that need to be resolved before process operation can take full advantage of their potential. These issues, which stem from intrinsic limitations on the information transmission and processing capabilities of the communication medium, challenge many of the assumptions in traditional process control theory dealing with the study of dynamical systems linked through ideal channels, and have emerged as topics of significant research interest to the control community (e.g., see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10] for some results and references in this area).

The development of a control strategy that enforces the desired closed-loop objectives with minimal communication requirements between the control system components is an appealing goal since it reduces reliance on the communication medium and helps reduce network resource utilization. This is an important consideration particularly when the communication medium is a (potentially unreliable) wireless sensor network where conserving network resources is key to prolonging the service life of the network. Beyond saving on communication costs, the study of this problem provides an assessment of the robustness of a given control system and allows the designers to identify the fundamental limits on the tolerance of a given networked control system to communication suspension. This can be a major consideration in deciding a priori whether the desired control objectives can be met with a certain kind of network.

Despite the substantial and growing body of work on control over communication networks, the overwhelming majority of research studies in this area have focused on lumped parameter systems modeled by ordinary differential, or difference, equations. Many important engineering applications, however, are characterized by spatial variations, owing to the underlying physical phenomena such as diffusion, convection, and phase-dispersion, and are naturally modeled by partial differential equations (PDEs). Typical examples of these systems include transport-reaction processes and fluid flow systems. Unlike spatially homogeneous processes, the control problem arising in the context of spatially-distributed processes often involves the regulation of spatially distributed variables (such as temperature and concentration spatial profiles) using spatially-distributed control actuators and measurement sensors. While the study of distributed parameter systems in process control has been an active area of research (e.g., see [11], [12], [13], [14], [15], [16], [17] and the references therein), the design and implementation of networked control systems for spatially distributed processes remain open problems that need to be investigated and addressed.

Motivated by these considerations, we present in this work a methodology for the design of networked control

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systems for a class of spatially distributed processes modeled by linear parabolic PDEs with measurement sensors that transmit their data to the controller/actuators over a bandwidth-limited communication network. Our objective is to enforce closed-loop stability while keeping the sensor-controller communication to a minimum in order to reduce network resource utilization. The key ideas are to embed a finite-dimensional model that captures the dominant dynamic modes of the PDE in the controller to provide it with an estimate of those modes when measurements are not transmitted through the network, and to update the model state using the actual measurements provided by the sensors at discrete time instances. The networked closed-loop system is analyzed by bringing together tools from switched systems, infinite-dimensional systems and singular perturbations leading to a precise characterization of the minimum sensor-controller communication frequency.

The rest of the paper is organized as follows. Following some mathematical preliminaries, modal decomposition techniques are used in Section II to obtain a finite-dimensional system that captures the dominant dynamic characteristics of the PDE. This system is then used in Sections III and IV, respectively, to design finite-dimensional state and output feedback networked control architectures and obtain explicit characterizations of the minimum allowable sensor-controller communication frequency in terms of model uncertainty, and sensor and actuator locations. Finally, the proposed methodology is used to stabilize an unstable steady-state of a diffusion-reaction process under sensor-controller communication constraints.

II. Preliminaries

A. Class of Systems

We consider spatially-distributed processes modeled by linear parabolic PDEs of the form:

$$\frac{\partial \bar{x}(z,t)}{\partial t} = \alpha\frac{\partial^2 \bar{x}(z,t)}{\partial z^2} + \beta\bar{x}(z,t) + \omega\sum_{i=1}^{m} b_i(z)u_i(t) \quad (1)$$

$$y_j(t) = \int_0^{\pi} q_j(z)\bar{x}(z,t)dz,\ j = 1,\cdots,n \quad (2)$$

subject to the boundary and initial conditions:

$$\bar{x}(0,t) = \bar{x}(\pi,t) = 0,\ \bar{x}(z,0) = \bar{x}_0(z) \quad (3)$$

where $\bar{x}(z,t) \in \mathbb{R}$ denotes the process state variable, $z \in [0,\pi] \subset \mathbb{R}$ is the spatial coordinate, $t \in [0,\infty)$ is the time, u_i denotes the i-th manipulated input (control actuator), $b_i(\cdot)$ is a function that describes how the control action is distributed in $[0,\pi]$, $y_j(t) \in \mathbb{R}$ is a measured output, $q_j(\cdot)$ is a function that describes how the measured output is distributed in $[0,\pi]$, the parameters $\alpha > 0, \beta, \omega$ are constants, and $\bar{x}_0(z)$ is a smooth function of z.

Throughout the paper, the notations $\|\cdot\|$ and $\|\cdot\|_2$ will be used to denote the L_2 norms associated with a finite-dimensional and infinite-dimensional Hilbert spaces, respectively. Furthermore, a bounded linear operator $\mathcal{M}$ is said to be power-stable if there exists positive real numbers β and γ such that $\|\mathcal{M}^k\| \le \beta e^{-\gamma k}$, for any non-negative integer k. The spectral radius of a bounded linear operator $\mathcal{M}$ is defined as $r(\mathcal{M}) = \lim_{k\to\infty}\|\mathcal{M}^k\|^{1/k} \le \|\mathcal{M}\|$. From these definitions, it can be verified that $\mathcal{M}$ is power-stable if and only if $r(\mathcal{M}) < 1$. Finally, the notation $x(t_k^-)$ will be used to denote the limit $\lim_{t\to t_k^-} x(t)$.

For a precise characterization of the class of PDEs considered in this work, we formulate the PDE of Eqs.1-3 as an infinite-dimensional system in the state space $\mathcal{H} = L_2(0,\pi)$, with inner product $\langle\omega_1,\omega_2\rangle = \int_0^{\pi}\omega_1(z)\omega_2(z)dz$, and norm $\|\omega_1\|_2 = \langle\omega_1,\omega_1\rangle^{\frac{1}{2}}$, where ω_1,ω_2 are two elements of $L_2(0,\pi)$. Defining the input and output operators as $\mathcal{B}u = \omega\sum_{i=1}^{m} b_i(\cdot)u_i$, $\mathcal{Q}x = [\langle q_1,x\rangle\ \langle q_2,x\rangle\cdots\langle q_n,x\rangle]'$, the system of Eqs.1-3 can be written in the following form:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u,\ \ x(0) = x_0 \quad (4)$$

$$y = \mathcal{Q}x \quad (5)$$

where $x(t)$ is the state function defined on an appropriate Hilbert space, $\mathcal{A}$ is the differential operator, $u = [u_1\ u_2\ \cdots\ u_m]'$ and $x_0 = \bar{x}_0(z)$. For $\mathcal{A}$, the solution of the eigenvalue problem ($\mathcal{A}\phi_j = \lambda_j\phi_j,\ j = 1,\ldots,\infty$), where λ_j denotes an eigenvalue and ϕ_j denotes an eigenfunction, yields real and ordered eigenvalues. Also, for a given α and β, only a finite number of unstable eigenvalues exist, and the distance between two consecutive eigenvalues (i.e., λ_j and λ_{j+1}) increases as j increases. Furthermore, for parabolic PDEs, the spectrum of $\mathcal{A}$ can be partitioned, where $\sigma_1(\mathcal{A}) = \{\lambda_1,\cdots,\lambda_m\}$ contains the first m (with m finite) "slow" eigenvalues and $\sigma_2(\mathcal{A}) = \{\lambda_{m+1},\lambda_{m+2},\cdots\}$ contains the remaining "fast" stable eigenvalues where $|\lambda_m|/|\lambda_{m+1}| = O(\epsilon)$ and $\epsilon < 1$ is a small positive number characteristic of the large separation between the slow and fast eigenvalues of $\mathcal{A}$. This implies that the dominant dynamics of the PDE can be described by a finite-dimensional system, and motivates the use of modal decomposition to derive a finite-dimensional system that captures the dominant (slow) dynamics of the PDE.

B. Modal Decomposition

Let $\mathcal{H}_s$, $\mathcal{H}_f$ be modal subspaces of $\mathcal{A}$, defined as $\mathcal{H}_s = \text{span}\{\phi_1,\ldots,\phi_m\}$ and $\mathcal{H}_f = \text{span}\{\phi_{m+1},\phi_{m+2},\ldots\}$. Defining the orthogonal projection operators, $\mathcal{P}_s$ and $\mathcal{P}_f$, such that $x_s = \mathcal{P}_s x$, $x_f = \mathcal{P}_f x$, the state of the system of Eq.4 can be decomposed as $x = x_s + x_f$. Applying $\mathcal{P}_s$ and $\mathcal{P}_f$ and using the decomposition of x, the system of Eqs.4-5 can be decomposed as:

$$\dot{x}_s = \mathcal{A}_s x_s + \mathcal{B}_s u,\ \ x_s(0) = \mathcal{P}_s x_0 \quad (6)$$

$$\dot{x}_f = \mathcal{A}_f x_f + \mathcal{B}_f u,\ \ x_f(0) = \mathcal{P}_f x_0 \quad (7)$$

$$y = \mathcal{Q}x_s + \mathcal{Q}x_f \quad (8)$$

where $\mathcal{A}_s = \mathcal{P}_s\mathcal{A}$ is an $m\times m$ diagonal matrix of the form $\mathcal{A}_s = \text{diag}\{\lambda_j\}$, $\mathcal{B}_s = \mathcal{P}_s\mathcal{B}$, $\mathcal{A}_f = \mathcal{P}_f\mathcal{A}$ is an unbounded differential operator which is exponentially stable (following from the fact that $\lambda_{m+1} < 0$ and the selection of $\mathcal{H}_s$ and $\mathcal{H}_f$), $\mathcal{B}_f = \mathcal{P}_f\mathcal{B}$. In what follows, the x_s- and x_f-subsystems will be referred to as the slow and fast subsystems, respectively.

III. Analysis and Design of Networked Control System Under State Feedback

A. Finite-dimensional Networked Control Architecture

1) Feedback controller synthesis: To realize the desired networked control structure, the first step is to synthesize a stabilizing feedback controller of the form $u(x_s) = \mathcal{K}x_s$, where $\mathcal{K}$ is the feedback gain chosen to exponentially stabilize the origin of the non-networked closed-loop slow subsystem. It is worth noting that, due to the stability of $\mathcal{A}_f$, a controller that exponentially stabilizes the finite-dimensional slow subsystem also exponentially stabilizes the infinite-dimensional closed-loop system.

2) Reducing sensor-controller communication over the network: To minimize the transfer of information between the sensors and the controller without sacrificing closed-loop stability, a dynamic model of the slow subsystem is embedded in the controller to provide it with an estimate of the evolution of the slow states when measurements of those states are not transmitted over the network. This allows the sensors to send their data at discrete times, since the model can provide an approximation of the slow dynamics. Feedback from the sensors to the controller is performed by updating the state of the model using the actual slow state provided by the sensors at discrete time instances. Under this communication logic, the networked control law is implemented as follows:

$$\begin{array}{rcl} u(t) & = & \mathcal{K}\widehat{x}_s(t) \\ \dot{\widehat{x}}_s(t) & = & \widehat{\mathcal{A}}_s\widehat{x}_s(t) + \widehat{\mathcal{B}}_s(z)u(t),\ t \in (t_k, t_{k+1}) \\ \widehat{x}_s(t_k) & = & x_s(t_k),\ k = 0, 1, 2, \cdots \end{array} \tag{9}$$

where $\widehat{x}_s$ is an estimate of x_s, $\widehat{\mathcal{A}}_s$ and $\widehat{\mathcal{B}}_s$ are bounded operators that model the dynamics of the slow subsystem. Note from Eq.9 that a choice of $\widehat{\mathcal{A}}_s = \mathcal{O}$, $\widehat{\mathcal{B}}_s = \mathcal{O}$, corresponds to the special case where in between consecutive transmission times, the model acts as a zero-order hold by keeping the last available measurement from the sensors until the next one is available from the network.

3) Characterizing the minimum allowable communication frequency: A key parameter in the analysis of the control law of Eq.9 is the update period $h := t_{k+1} - t_k$, which determines the frequency at which the controller receives measurements from the sensors through the network to update the model state. To simplify the analysis, we focus on the case when the update period is constant and the same for all the sensors, i.e., we require that all sensors communicate their measurements concurrently every h seconds. To characterize the maximum allowable update period between the sensors and the controller, we define the model estimation error as $e_s(t) = x_s(t) - \widehat{x}_s(t)$, where $e_s \in \mathcal{H}_s$ represents the difference between the slow state of the system of Eq.6 and the state of its model given in Eq.9. Defining the augmented state $\xi = [x_s\ e_s]'$ which is an element of the extended state space $\mathcal{H}_s^e = \mathcal{H}_s \times \mathcal{H}_s$, it can be shown that the augmented slow subsystem can be formulated as a switched system and written in the following operator-matrix form for clarity:

$$\begin{array}{rcl} \dot{\xi}(t) & = & \mathcal{F}\xi(t),\ \ t \in (t_k, t_{k+1}) \\ e_s(t_k) & = & 0,\ \ k = 0, 1, 2, \cdots, \end{array} \tag{10}$$

where

$$\mathcal{F} = \begin{bmatrix} \mathcal{A}_s + \mathcal{B}_s(z)\mathcal{K} & -\mathcal{B}_s(z)\mathcal{K} \\ \widetilde{\mathcal{A}}_s + \widetilde{\mathcal{B}}_s(z)\mathcal{K} & \widehat{\mathcal{A}}_s - \widetilde{\mathcal{B}}_s(z)\mathcal{K} \end{bmatrix} \tag{11}$$

is a bounded linear operator with a dense domain $D(\mathcal{F}) := D(\mathcal{A}_s) \times D(\mathcal{A}_s) \to \mathcal{H}_s^e$, $\widetilde{\mathcal{A}}_s = \mathcal{A}_s - \widehat{\mathcal{A}}_s$, $\widetilde{\mathcal{B}}_s = \mathcal{B}_s - \widehat{\mathcal{B}}_s$ represent the modeling errors. Note that while the slow state x_s evolves continuously in time, the error e_s is reset to zero at each transmission instance since the state of the model is updated every h seconds.

In order to derive conditions for closed-loop stability, we need to express the closed-loop response as a function of the update period. To this end, it can be shown that the system described by Eq.10 with initial condition $\xi(t_0) = [x_s(t_0)\ 0]' = \xi_0$ has the following solution for $t \in [t_k, t_{k+1})$:

$$\xi(t) = \mathcal{T}_{\mathcal{F}}(t - t_k)\left(\mathcal{I}_s\mathcal{T}_{\mathcal{F}}(h)\mathcal{I}_s\right)^k \xi_0 \tag{12}$$

with $t_{k+1} - t_k = h$, where $\mathcal{T}_{\mathcal{F}}(t) : \mathcal{H}_s^e \to \mathcal{H}_s^e$ is a C_0-semigroup generated by $\mathcal{F}$ on $\mathcal{H}_s^e$, $\mathcal{I}_s = \mathrm{diag}\,[\mathcal{I}\ \mathcal{O}]$, $\mathcal{I}$ is the identity operator. The following proposition provides a necessary and sufficient condition for stability of the networked finite-dimensional closed-loop system. The proof is conceptually similar to the one presented in [3] – except that it involves operators defined over functional spaces – and is omitted for brevity.

Proposition 1: *Consider the networked closed-loop system of Eqs.6-9 and the augmented system of Eqs.10-11 whose solution is given by Eq.12. Let $\mathcal{M}(h) = \mathcal{I}_s\mathcal{T}_{\mathcal{F}}(h)\mathcal{I}_s$. Then the zero solution, $\xi = [x_s\ e_s]' = [0\ 0]'$, is exponentially stable if and only if $r(\mathcal{M}(h)) < 1$.*

Remark 1: By examining the structure of $\mathcal{F}$ in Eq.11, it can be seen that the minimum stabilizing communication frequency is dependent on the degree of mismatch between the dynamics of the slow subsystem and the model used to describe it. Given bounds on the size of the uncertainty, the stability criteria of Proposition 1 can therefore be used to determine the range of stabilizing update periods, h. Alternatively, for a fixed h, the maximum tolerable process-model mismatch can be determined. Note that since $\mathcal{M}$ is defined over a finite-dimensional space, its spectral radius can be determined by computing the eigenvalues of $\mathcal{M}$.

Remark 2: Due to the dependence of $\mathcal{F}$ on $\mathcal{B}_s(z)$ (which is parameterized by the control actuator locations), the maximum allowable update period is also dependent on the choice of the control actuator locations. The result of Proposition 1 can therefore be used to identify the actuator locations that are more robust to communication suspension (i.e., the ones that require measurement updates less frequently than others) for a given feedback gain.

B. Analysis of the Networked Infinite-dimensional System

Theorem 1 below establishes that the characterization of the maximum allowable update period obtained based on the finite-dimensional system is exactly preserved when implementing the networked control law on the infinite-dimensional system. As shown in the proof that follows, this is due to the fact that the networked closed-loop slow subsystem is decoupled from the stable fast subsystem which is driven by a bounded and converging input.

Theorem 1: *Consider the infinite-dimensional system of Eqs.6-7 subject to the control and update laws of Eq.9. Then, the zero solution is exponentially stable if and only if* $r(\mathcal{M}(h)) < 1$*, where* $\mathcal{M}(h) = \mathcal{I}_s\mathcal{T}_{\mathcal{F}}(h)\mathcal{I}_s$.

Proof: Substituting the control law of Eq.9 into the system of Eqs.6-7, it can be shown that the infinite-dimensional networked closed-loop system takes the form:

$$\begin{aligned} \dot{\xi}(t) &= \mathcal{F}\xi(t), \;\; t \in (t_k, t_{k+1}) \\ \dot{x}_f(t) &= \mathcal{A}_f x_f(t) + \mathcal{B}_f\mathcal{K}x_s(t) - \mathcal{B}_f\mathcal{K}e(t) \\ e_s(t_k) &= 0, \;\; k = 0, 1, 2, \cdots \end{aligned} \tag{13}$$

Sufficiency: From the properties $\mathcal{A}_f$ and the fact that $\mathcal{B}_f$ is a bounded operator, it follows [18] that there exists a C_0-semigroup $\mathcal{T}_f$ such that the x_f-subsystem admits the following mild solution:

$$x_f(t) = \mathcal{T}_f(t)x_f(0) + \int_0^t \mathcal{T}_f(t-\tau)\mathcal{B}_f\mathcal{K}\left(x_s(t) - e(t)\right)d\tau \tag{14}$$

Furthermore, since $\mathcal{A}_f$ is a stable operator, the spectrum of $\mathcal{A}_f$ satisfies $\sup\{Re\ \sigma(\mathcal{A}_f)\} < -\gamma_f$ for some $\gamma_f > |\beta - \alpha(m+1)^2|$; and thus, $\mathcal{T}_f$ satisfies $\|\mathcal{T}_f(t)\|_2 \le M_0 e^{-\gamma_f t}$; $t \ge 0$ [18]. Based on this, by taking the 2-norm in space of both sides of Eq.14, the following bound can be obtained:

$$\|x_f(t)\|_2 \le K_0 e^{-\gamma_f t}\|x_f(0)\|_2 + M_1\left|\int_0^t e^{-\gamma_f(t-\tau)}\|\xi(\tau)\|d\tau\right| \tag{15}$$

where $K_0 \ge M_0$, $M_1 = 2M_0\|\mathcal{B}_f\mathcal{K}\|_2$ and we have used the fact that $\|x_s\| \le \|\xi\|$ and $\|e_s\| \le \|\xi\|$. From the result of Proposition 1, we have $\|\,\xi(\tau)\,\| \le \varphi e^{-\gamma_s\tau}\|\,\xi(0)\,\|$ which when substituted into Eq.15 yields:

$$\|x_f(t)\|_2 \le K_0 e^{-\gamma_f t}\|x_f(0)\|_2 + M_2(e^{-\gamma_s t} + e^{-\gamma_f t})\|\xi(0)\|$$

where $M_2 = M_1\varphi/|(\gamma_f - \gamma_s)|$. Finally, defining $\gamma = \min\{\gamma_f, \gamma_s\} > 0$ and $K_f = 4 \cdot \max\{K_0, M_2, \varphi\} > 0$, we arrive at the following bound:

$$\left\|\begin{bmatrix} \xi(t) \\ x_f(t) \end{bmatrix}\right\|_2 \le K_f e^{-\gamma t}\left\|\begin{bmatrix} \xi(0) \\ x_f(0) \end{bmatrix}\right\|_2 \tag{16}$$

which implies that the origin is exponentially stable.

Necessity: Using a contradiction argument, we assume that the zero solution is exponentially stable but that $r(\mathcal{M}(h)) > 1$. However, if $r(\mathcal{M}(h)) > 1$ the slow subsystem will be unstable (from Proposition 1) and $x_s(t)$ will grow unbounded with time. From Eq.14, this also means that we cannot ensure $x_f(t)$ will converge to zero for a general initial condition. This implies that the zero solution of the infinite-dimensional system cannot be exponentially stable, and thus we have a contradiction. This completes the proof.

IV. Output Feedback Implementation of the Networked Control Strategy

In this section, we consider the case where measurements of the state variable are available only at a finite number of locations in the spatial domain. The main idea is to include a finite-dimensional state observer that uses the available output measurements to generate estimates of x_s and to use these estimates to update the model state. The networked output feedback controller is then implemented as follows:

$$\begin{aligned} u(t) &= \mathcal{K}\widehat{x}_s(t), \;\; t \in [t_k, t_{k+1}) \\ \dot{\widehat{x}}_s(t) &= \widehat{\mathcal{A}}_s\widehat{x}_s(t) + \widehat{\mathcal{B}}_s(z)u(t) \\ \dot{\bar{x}}_s(t) &= (\widehat{\mathcal{A}}_s - \mathcal{L}\mathcal{Q})\bar{x}_s(t) + \widehat{\mathcal{B}}_s u(t) \; + \; \mathcal{L}y(t) \\ \widehat{x}_s(t_k) &= \bar{x}_s(t_k), \;\; k = 0, 1, 2, \cdots \end{aligned} \tag{17}$$

where $\bar{x}_s$ is the observer-generated estimate of the slow state, $\mathcal{L}$ is the observer gain and $t_{k+1} - t_k = h$ is the update period. As noted in [3], implementing this set-up requires that the observer be collocated with the sensor and that it has a copy of the model and controller, and has knowledge of h in order to acquire $u(t)$, which is at the other side of the communication link. In this way $u(t)$ is simultaneously and continuously generated at both ends of the feedback path with the only requirement that the observer makes sure that the model has been updated to ensure that both the controller and the observer are synchronized.

Defining the error variable $e_o = \bar{x}_s - \widehat{x}_s$, where $e_o \in \mathcal{H}_s$ is the difference between the observer's estimate of the slow state and the model state, and introducing the augmented state $\xi_o = [x_s\ \bar{x}_s\ e_o]'$, which is an element of the extended state space $\mathcal{H}_o^e = \mathcal{H}_s \times \mathcal{H}_s \times \mathcal{H}_s$, it can be shown that the augmented slow subsystem can be formulated as a switched system of the form:

$$\begin{aligned} \dot{\xi}_o(t) &= \mathcal{F}_o\xi_o(t) + \mathcal{G}_o x_f, \;\; t \in (t_k, t_{k+1}) \\ e_o(t_k) &= 0, \;\; k = 0, 1, 2, \cdots, \end{aligned} \tag{18}$$

where the operators $\mathcal{F}_o$ and $\mathcal{G}_o$ are given below in matrix form for clarity:

$$\mathcal{F}_o = \begin{bmatrix} \mathcal{A}_s & \mathcal{B}_s\mathcal{K} & -\mathcal{B}_s\mathcal{K} \\ \mathcal{L}\mathcal{Q} & \mathcal{C} & -\widehat{\mathcal{B}}_s\mathcal{K} \\ \mathcal{L}\mathcal{Q} & -\mathcal{L}\mathcal{Q} & \widehat{\mathcal{A}}_s \end{bmatrix}, \mathcal{G}_o = \begin{bmatrix} 0 \\ \mathcal{L}\mathcal{Q} \\ \mathcal{L}\mathcal{Q} \end{bmatrix} \tag{19}$$

and $\mathcal{C} = \widehat{\mathcal{A}}_s + \widehat{\mathcal{B}}_s\mathcal{K} - \mathcal{L}\mathcal{Q}$. The following theorem provides a stability condition for the networked infinite-dimensional closed-loop system that ties the update period with the separation between the slow and fast eigenvalues of $\mathcal{A}$, $\epsilon = |\lambda_m|/|\lambda_{m+1}|$. In the statement of this theorem, the notation $\mathcal{T}_{\mathcal{F}_o}(t) : \mathcal{H}_o^e \rightarrow \mathcal{H}_o^e$ refers to a C_0-semigroup generated by $\mathcal{F}_o$ on $\mathcal{H}_o^e$, and $\mathcal{I}_o = \mathrm{diag}\,[\mathcal{I}\ \mathcal{I}\ \mathcal{O}]$, where $\mathcal{I}$ is the identity operator. The proof of the theorem can be obtained using singular perturbation arguments [12] and is omitted for brevity.

Theorem 2: *Consider the infinite-dimensional system of Eqs.6-8 subject to the control and update laws of Eq.17. Then, if* $r(\mathcal{M}_o(h)) < 1$*, where* $\mathcal{M}_o(h) = \mathcal{I}_o\mathcal{T}_{\mathcal{F}_o}(h)\mathcal{I}_o$*, there exists* $\epsilon^* > 0$ *such that if* $\epsilon \in (0, \epsilon^*]$*, the zero solution of the closed-loop system is exponentially stable.*

Remark 3: According to the result of Theorem 2, an update period that stabilizes the approximate finite-dimensional system of Eqs.18-19 with $x_f = 0$ continues to stabilize the infinite-dimensional system provided that the separation between the slow and fast eigenvalues is sufficiently large. Therefore, unlike the state feedback result of Theorem 1, a restriction must be placed on the separation between the slow and fast eigenvalues to ensure closed-loop stability. This restriction, which requires that a sufficient number of slow states and measurements be included in the controller design, is needed to ensure that the error introduced by updating the model state using $\bar{x}_s$ rather than x_s is sufficiently small. Notice also that, unlike the state feedback case, the evolution of the augmented slow subsystem of Eq. 18 is dependent on x_f (which is a consequence of using y instead of x_s in the controller).

Remark 4: Beyond generalizing the results in [3] to distributed parameter systems modeled by linear PDEs, the results of Theorems 1 and 2 provide an explicit characterization of the dependence of the maximum allowable update period on the choice of the sensor and actuator spatial locations. This dependence is a consequence of the parametrization of the operators $\mathcal{F}$ and $\mathcal{F}_o$ by the actuator and sensor locations. The results can therefore be used to identify optimal sensor and actuator locations that are most robust to communication suspension.

V. Simulation study: Application to a Diffusion-Reaction Process

In this section, we illustrate through computer simulations how the networked control systems described earlier can be used to stabilize the open-loop unstable zero solution of a linearized diffusion-reaction process of the form of Eq.1 subject to the boundary and initial conditions of Eq.3, where $\alpha = 1$, $\beta = 1.66$, and $\omega = 2$. We consider the first eigenvalue as the dominant one and use standard Galerkins method to derive an ODE that describes the temporal evolution of the amplitude of the first eigenmode: $\dot{a}_1 = \lambda_1 a_1 + g(z_a)u$, where $\bar{x}(z,t) = \sum_{i=1}^{\infty} a_i(t)\phi_i(z)$, $g(z_a) = 2\langle\phi_1(z), b(z)\rangle$, and a single point actuator (with finite support) is used for stabilization, i.e., $b(z) = 1/(2\mu)$ for $z \in [z_a - \mu, z_a + \mu]$, where μ is a sufficiently small number, and $b(z) = 0$ elsewhere. The ODE is used to design the networked controllers which are then implemented on a 30-th order Galerkin discretization of the PDE.

The state feedback results are presented first. In this case, and following the methodology outlined in Section III, the networked closed-loop system takes the form $\dot{\xi}(t) = \Lambda\xi(t), t \in [t_k, t_{k+1})$, where $\xi := [a_1\ e_1]^T$, $e_1 = a_1 - \widehat{a}_1$, $\widehat{a}_1$ is an estimate of a_1 generated by a model of the form $\dot{\widehat{a}}_1 = \widehat{\lambda}_1\widehat{a}_1 + \widehat{g}(z_a)u$, $\widehat{\lambda}_1$ and $\widehat{g}(z_a)$ are estimates of λ_1 and $g(z_a)$, respectively, and

$$\Lambda = \begin{bmatrix} \lambda_1 + g(z_a)k & -g(z_a)k \\ \widetilde{\lambda}_1 + \widetilde{g}(z_a)k & \widehat{\lambda}_1 - \widetilde{g}(z_a)k \end{bmatrix}$$

where $\widetilde{\lambda}_1 = \lambda_1 - \widehat{\lambda}_1$, $\widetilde{g}(z_a) = g(z_a) - \widehat{g}(z_a)$ and k is the feedback control gain. For any initial condition $\xi_0 = [a_1(0)\ 0]^T$, the system admits the following response $\xi(t) = e^{\Lambda(t-t_k)}M^k(h)\xi_0$, for $t \in [t_k, t_{k+1})$, where $M(h) = I_s e^{\Lambda h} I_s$ and $I_s = \text{diag}\,[1\ 0]$, and is therefore exponentially stable if and only if the eigenvalues of the matrix M are strictly inside the unit circle. The above expressions show that the eigenvalues of M depend on the mismatch between the model and the plant, the control gain, the actuator location, and the update period. In the remainder of this section, we will investigate the interplays between these parameters. Since closed-loop stability requires all eigenvalues of M to lie within the unit circle, it is sufficient to consider only the maximum eigenvalue magnitude which we denote by λ_{max}.

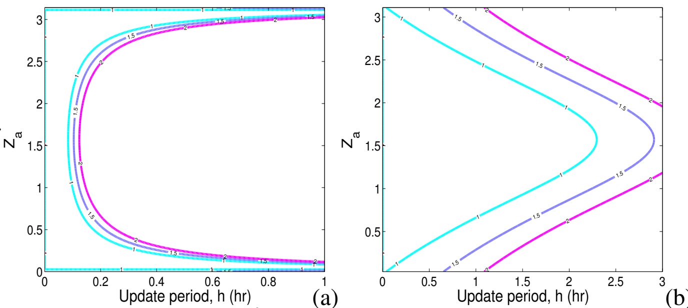

Fig. 1. Dependence of $\lambda_{\max}$ on the update period and actuator position under (a) a zero-order hold scheme and a fixed feedback gain, and (b) a model-based scheme and a location-dependent feedback gain.

Fig.1(a) is a contour plot showing the dependence of λ_{max} on both the position of the actuator, z_a, and the update period, h, when a constant feedback gain of $k = -15$ is used for the state feedback controller and a zero-order hold model is used for the update (i.e., the controller just holds the last value of a_1 received from the network until the next measurement is transmitted). The area enclosed by the unit contour line represents the stability region of the plant. It can be seen that (1) the set of stabilizing actuator locations increases as the update period decreases and (2) the maximum stabilizing update period shrinks as the actuator is moved closer to the middle. This result can be explained by the fact when all actuator locations share the same feedback gain, the non-networked closed-loop response $\dot{a}_1 = (\lambda_1 + g(z_a)k)a_1$ is fastest at the middle location (where the first eigenfunction has a maximum) and therefore more frequent updates are needed at this location. The predictions of Fig.1(a) are further confirmed by the simulation results in Fig.2 which show that the closed-loop system is stable (right) for ($h = 0.1, z_a = 3$) and unstable (left) for ($h = 0.1, z_a = 2$) when the networked control system is implemented on the PDE.

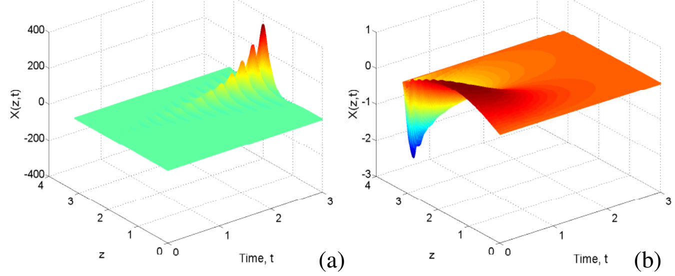

Fig. 2. Closed-loop state profiles with zero-order hold for (a) ($h = 0.1, z_a = 2$) and (b) ($h = 0.1, z_a = 3$).

The trend predicted by Fig.1(a) can change if the feedback gain is chosen to vary with the actuator location. An example of this is shown in Fig.1(b) which is a contour plot showing the dependence of λ_{max} on z_a and h when k is designed to place the non-networked closed-loop eigenvalue at -5 and an uncertain model (with $\widehat{\lambda}_1 = 0.5$ and $\widehat{g}(z_a) = 2$) is used to estimate the evolution of a_1 when actual measurements are not received through the network. The area enclosed by the unit contour line represents the stability region of the plant. The plot shows that the maximum allowable update period increases as the actuator is moved closer to the middle. The predictions of Fig.1(b) are confirmed by the simulation results in Fig.3 which show that the closed-loop system is stable (left) for $(h = 0.2, z_a = 2)$ and unstable (right) for $(h = 0.2, z_a = 3)$.

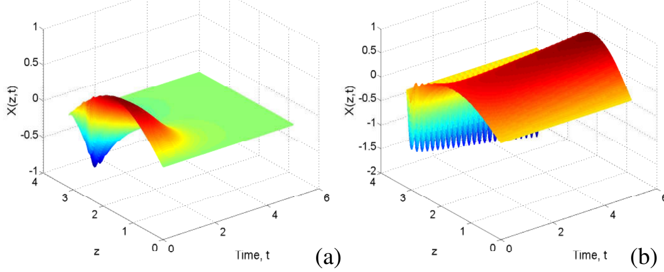

Fig. 3. Closed-loop state profiles with imperfect model for (a) $(h = 0.2, z_a = 2)$ and (b) $(h = 0.2, z_a = 3)$.

To address the output feedback control problem where a_1 is not directly measurable, we use a state observer of the form $\dot{\bar{a}}_1 = (\widehat{\lambda}_1 - LQ_s(z_s))\bar{a}_1 + \widehat{g}(z_a)u + Ly$ to estimate a_1 from the measured output, $y(t) = \langle q(z - z_s), \bar{x}(z,t)\rangle$, provided by a point sensor located at $z = z_s$, where $\bar{a}_1$ is the observer estimate of a_1, $Q_s(z_s) = \langle q(z - z_s), \phi_1(z)\rangle$, $q(z-z_s)$ is the sensor distribution function and the observer gain L is chosen so that $\widehat{\lambda}_1 - LQ_s(z_s) < 0$. Following the analysis presented in Section IV, it can be shown that the networked output feedback control system is exponentially stable if an only if the eigenvalues of the matrix $M_o(h) = I_o e^{\Lambda_o h} I_o$ are inside the unit circle, where:

$$\Lambda_o = \begin{bmatrix} \lambda_1 & g(z_a)k & -g(z_a)k \\ LQ_s(z_s) & \widehat{\lambda}_1 + \hat{g}(z_a)k - LQ_s(z_s) & -\widehat{g}(z_a)k \\ LQ_s(z_s) & -LQ_s(z_s) & \widehat{\lambda}_1 \end{bmatrix}$$

and $I_o = \text{diag}\,[1\ 1\ 0]$. Unlike the state feedback case, closed-loop stability under output feedback is also dependent on the location of the measurement sensor. Fig.4 is a contour plot showing the dependence of λ_{max} on both the location of the sensor, z_s, and the update period, when a zero-order hold scheme is used for the output feedback controller with constant controller and observer gains of $k = -15$ and $L = 100$, respectively, and a fixed actuator location at $z_a = 2$ are used. The area enclosed by the unit contour line is the stability region. The predictions of Fig.4

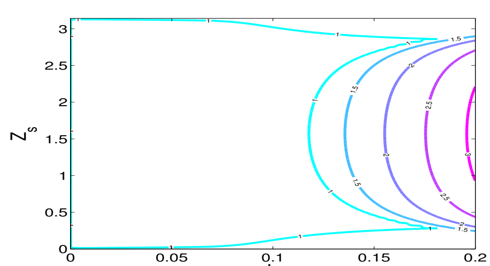

Fig. 4. Dependence of $\lambda_{\max}$ on the update period and sensor position under networked output feedback control structure.

are consistent with the plots in Fig. 5, which show that the closed-loop system is stable (left) for $(z_s = 1.5,\ h = 0.04)$ which lies inside the stability region, and unstable (right) for $(z_s = 1.5,\ h = 0.14)$ which is outside the region.

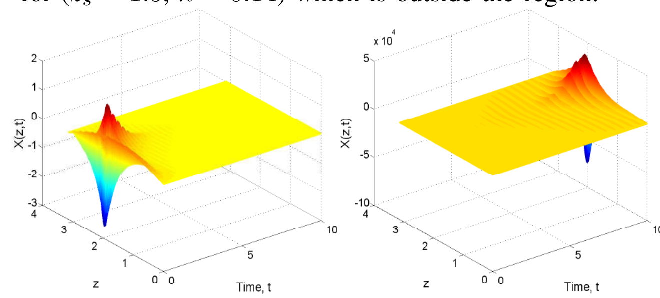

Fig. 5. Closed-loop state profiles under networked output feedback control structure for $(z_s = 1.5,\ h = 0.04)$ (left) and for $(z_s = 1.5,\ h = 0.14)$ (right).

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