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Asymptotic Stability of Constant Time Headway Driving Strategy with Multiple Driver Reaction Delays

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***Abstract*— This paper addresses the asymptotic stability of a continuous-time deterministic car-following model with respect to different driving strategies. More precisely, the model takes into account a constant time headway driving behavior as well as the presence of multiple decision-making and actuation delays in describing the drivers' actions. The stability analysis of the corresponding derived models is challenging and we will focus on characterizing the stability regions in the delay parameter space. Such a problem depicts some interesting properties in terms of delays that simplify the overall analysis leading to some simple frequency-sweeping based algorithms as well as to various practical interpretations of the results in terms of drivers' behaviors. Illustrative examples complete the presentation.**

***Index Terms*— traffic dynamics, car following, delay, stability.**

I. Introduction and Problem Statement

According to a recent research note published by National Highway Traffic Safety Administration (NHTSA) [35], motor vehicle crashes were the leading cause for death in the U.S. in 2002 for the ages between 3 and 33. This can be seen as one of the many reasons why traffic behavior is a research focus since 1930s [14], with increasing interest in the last decades, primarily due to undesirable impacts on the environment and energy conservation concerns [3]. As a consequence, numerous *mathematical models* have been developed via macroscopic and microscopic approaches [1], [4], [7], [23], [38], [39]. Based on the degree of detail and the physics aimed to be captured, these approaches can incorporate various parameters defining the traffic flow including the consideration of single/multiple lanes, on/off ramps, lane changes, traffic lights and their synchronization and roundabouts [14].

Since the framework/ideas deployed to derive mathematical models are too broad, we will focus on a particular subclass which is widely preferred [1], [14], [20], [21], [34], [39], [41] when studying traffic behavior: Pipes model [22], a deterministic microscopic follow-the-leader type model in which drivers cruise at a constant velocity on a single-lane without changing lanes. This selection is strongly motivated by the earlier work where it was shown that Pipes model effectively predicts the car following behavior of human drivers in the experiments [3]. With this motivation in mind, we attempt to reveal intrinsic features of the traffic flow under the conditions Pipes model is desirably reliable in predicting car following traffic flow. This also enables to develop traffic control strategies by constructing decentralized [34], adaptive [36], non-linear spacing controllers [24], [40], [41], gain scheduling techniques [40] and collision avoidance [6] among automated heavy-duty vehicles. Common objective in the cited references is to analytically investigate how the headway (spacing between consecutive vehicles) dynamics propagates upstream of the traffic flow [3], [25], [37]; to propose analysis tools to reveal how headway dynamics behaves under perturbations [3], [6] and to design appropriate controllers to prevent amplification of such perturbations [24], [34], [36], [40], [41]. These efforts fall within the studies of *car string stability* (CSS), which aim to investigate spacing (headway) error propagation upstream of the traffic flow where in some cases the errors may amplify and in some other cases they may attenuate. For an appropriate definition of CSS and related problems, we refer to [3], [36], [40], to cite only a few.

CSS indicates attenuation of the *periodic* perturbations (excitations) arising in the acceleration, velocity and position errors between consecutive vehicles, and propagating from one vehicle to another, while asymptotic stability (AS) refers to exponential decay of the response of the system states (velocity and position of vehicles) in time against impulsive perturbations. Despite the simplicity of the mathematical models, assessment of AS and CSS may not be trivial tasks, as evident from the cited references. In this paper, we focus on AS since instability precludes the CSS analysis. Due to the presence of human drivers and physiological delays of human drivers [1], [4], [7], [9], the microscopic traffic flow problem becomes a *human-in-the-loop dynamics* in which delays play a major role in determining the AS.

Delay in closed loop dynamics is a well-known source of poor performance (low damping), weak robustness and instability [10], [17]. Furthermore, the presence of multiple delays leads to unexpected behaviors in the parameter-space defined by the delays: stability rays, delays ratio sensitivity, delay interference, bounded and unbounded regions [8], [12]. Such problems, far from trivial, are challenging and one of the main research interests in such cases is to find appropriate algorithms for characterizing them globally. In this context, one of the interesting ideas proposed in the literature is based on *frequency-sweeping* [5], [10], [12], [32], which we also use to assess the AS with respect to delays and drivers'

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aggressiveness. To obtain tractable results, we will follow the lines of [1], [3], [14], [15], [31], [39] and assume that delays τ_k are time invariant. The bottleneck in analyzing AS in this paper is then due to our consideration of *driver heterogeneity* in the traffic, and this will bring about *multiple delays*, each one of which is likely to play a different or counter-intuitive role in AS.

This paper is a continuation of authors' earlier work in [26]–[28], [31], [32]. In the earlier work, authors studied *constant headway spacing* strategy of the drivers. Inspired by [3], we study here the effects of *constant time headway* driving strategy of the drivers (Section II) by accounting for multiple delays without approximating the analysis associated with delays. The main objective is to construct a stability analysis framework that considers the multiple delays, and to develop a practicable approach to reveal AS features of traffic flow with respect to delays and driver aggressiveness coefficients (Section III). To our best knowledge, a complete analytical study in this context has not been pursued in the literature and we form our main objective along this line. Illustrative examples in Section IV complete the work. Research in this direction has impacts on understanding human behavior and its interconnection with AS/CSS, studying adaptive cruise control, automated car following control and semi-active driver assistance systems.

Notation. The notations are standard. Set of real and positive real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively, and $\mathbb{C}$ and $\mathbb{C}_-$ represent the entire complex plane and left half of the complex plane, respectively. The imaginary axis is denoted by $j\mathbb{R}$, where $j = \sqrt{-1}$ and we use s for the Laplace variable.

II. Mathematical Modeling

Mathematical modeling and the pertaining discussions in this section are largely borrowed from [1], [3], [27], however, the model is extended based on our main interest of studying the effects of multiple delays. The justification comes from the fact that the vehicles and the drivers are not identical [14], [26], [27], [39] and this heterogeneity brings in different delays to the flow dynamics. Moreover, mechanical properties of different vehicles add different delays and drivers have different driving habits, physiologies and capabilities [1], [27]. We start with Pipes model [3], [4], [14], [22], [23], [28], which is a single-lane continuous-time deterministic microscopic car following model, in which the vehicles of the chain travel at a constant velocity, at the so-called quasi steady-state, without changing lanes. Although Pipes model is single delayed $\tau_k = \tau_{k+1}$, we consider its multiple delay form

$$\dot{v}_k(t) = \alpha_k(v_{k+1}(t-\tau_{k+1}) - v_k(t-\tau_k)),\ k = 1, \dots, n, \quad (1)$$

where v_k is the velocity of the k^{th} vehicle, see Fig 1, $\tau_k > 0$ is the constant delay, n is the number of vehicles and the weighting $\alpha_k > 0$ can be seen as a measure of driver aggressiveness per unit mass. The above differential equation

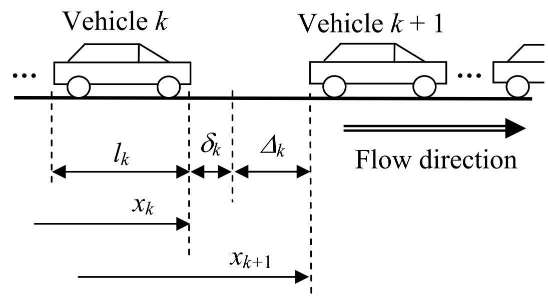

Fig. 1. Platoon of vehicles, inspired from [3].

describes that driver k attempts to vanish the velocity error $v_{k+1}(t) - v_k(t)$ by penalizing it using the gain α_k, but a driver's sensing is not instantaneous, hence the velocity error couples with *delays*.

In [3], it was assumed that delays are identical to each other $\tau_k = \tau$ and it was shown that Pipes model closely predicts the experiments conducted with two manually driven vehicles following each other. The simplicity and the reliability of the model is appealing, however, considering the general case with non-identical delays τ_k complicates the AS analysis.

Constant time headway strategy considers that driver k aims to perform control to maintain a constant time headway. This formulates the headway expression as $\delta_k(t) = x_{k+1}(t) - x_k(t) - l_k - \Delta_k - h_k v_k(t)$, where $h_k > 0$ is the time headway, and l_k and Δ_k do not contribute to AS analysis as they are constants. Considering this driving strategy along with multiple delays, a stability analysis method free of approximations will be developed in main results section, extending our earlier works where we studied different driving strategies [28], [31], [32]. Interestingly, time headway strategy leads to a more complicated AS problem in which some parameters become extremely sensitive against large decision-making gains.

A. Spacing Dynamics with Constant Time Headway

With the knowledge of (1), one has the transfer function

$$G_k(s, \tau_k, \tau_{k+1}) = \frac{\alpha_k e^{-\tau_{k+1}s}}{s + \alpha_k e^{-\tau_k s}}. \quad (2)$$

Let $D_k(s)$, $V_k(s)$ and $A_k(s)$ are the Laplace transforms of time functions $\delta_k(t)$, $v_{k+1}(t) - v_k(t)$ and $\dot{v}_{k+1}(t) - \dot{v}_k(t)$, respectively. From [3], one obtains,

$$\frac{D_k(s)}{D_{k+1}(s)} = \frac{1 - G_k - sh_kG_k}{1 - G_{k+1} - sh_{k+1}G_{k+1}} G_{k+1} = \hat{G}_k(s), \quad (3)$$

$$\frac{V_k(s)}{V_{k+1}(s)} = \frac{A_k(s)}{A_{k+1}(s)} = \frac{1 - G_k}{1 - G_{k+1}} G_{k+1} = \bar{G}_k(s), \quad (4)$$

where some arguments are suppressed for easier reading. Notice that among (3)-(4), studying only Eq.(3) for $h_k \geq 0$ is sufficient since one recovers (4) when $h_k \to 0$ in (3). In the following, we only treat (3) for $h_k > 0$, and extensions to $h_k \to 0$ will be trivial.

B. Preliminaries for AS Analysis

From (3), we have

$$\frac{D_k(s)}{D_{k+1}(s)} = \frac{P(s,\boldsymbol{\tau},\boldsymbol{\alpha})}{f(s,\boldsymbol{\tau},\boldsymbol{\alpha})}, \tag{5}$$

where $f(s,\boldsymbol{\tau},\boldsymbol{\alpha})$ is the characteristic function,

$$\begin{aligned} f(s,\boldsymbol{\tau},\boldsymbol{\alpha}) &= s^2(-1 + h_{k+1}\,\alpha_{k+1}\,e^{-\tau_{k+2}\,s}) \\ &+ s\left(\alpha_{k+1}\,e^{-\tau_{k+2}\,s} - \alpha_{k+1}\,e^{-\tau_{k+1}\,s} - \alpha_k\,e^{-\tau_k\,s}\right. \\ &\qquad\qquad \left. + h_{k+1}\,\alpha_k\alpha_{k+1}\,e^{-(\tau_k+\tau_{k+2})s}\right) \\ &- \alpha_k\alpha_{k+1}\,e^{-\tau_k\,s}\left(e^{-\tau_{k+1}\,s} - e^{-\tau_{k+2}s}\right), \end{aligned} \tag{6}$$

with $\boldsymbol{\tau} = (\tau_k, \tau_{k+1}, \tau_{k+2})$ and $\boldsymbol{\alpha} = (\alpha_k, \alpha_{k+1})$.

In order to analyze AS, one should investigate the location of zeros of $f(s,\boldsymbol{\tau},\boldsymbol{\alpha}) = 0$ on the complex plane $\mathbb{C}$. Several methods have been proposed in the literature for handling such problems in the corresponding parameter space (see, for instance, [16] for some classification and further discussions). In order to make the paper self-contained, a short overview from time delay systems (TDS) literature [11] is provided next regarding the stability analysis of the characteristic function. Characteristic function (6) is quadratic since the highest power of s is two and it is of neutral type in the sense defined by [2], [13].

Since (6) is neutral type, a necessary condition should be checked first before analyzing the AS, that is the stability of the corresponding delay-difference operator having the characteristic function

$$\tilde{f}(s, h_{k+1}, \alpha_{k+1}, \tau_{k+2}) = -1 + h_{k+1}\,\alpha_{k+1}\,e^{-\tau_{k+2}\,s}. \tag{7}$$

If (7) is stable, then the stability of (6) is given only by the point spectrum since the essential spectrum is located on $\mathbb{C}_-$. If this is not the case, the system is always unstable [16]. In the case studies, we will show that the stability condition of (7) imposes an additional constraint on the parameters h_{k+1} and α_{k+1}.

As discussed in [16], the spectral abscissa function (supremum of the real part of the rightmost root) is not necessarily a continuous function in general for the neutral case. However under the assumption of the stability of (7), a loss or acquisition of the exponential stability of the trivial solution of the original system is associated with characteristic roots on the imaginary axis. In other words, similar to the retarded case, the change of stability is given by a root $j\omega$ "crossing" the imaginary axis. Next, in the corresponding delay-parameter space, one needs to make a partition of the space in several regions, where such regions are characterized by two properties: (i) the number of strictly unstable roots of (6) is constant for all the delays located inside the region, and (ii) for each delay-point located on the boundary, there exists at least one characteristic root located on the imaginary axis. The regions corresponding to the case when there are no unstable roots define the stability regions [16]. In the sequel, we present how to find and compute the stability regions. As we see below, the particular form of the characteristic function will allow taking advantage of its structure reducing the stability analysis to a two-delay case.

III. Main Results: Stability Analysis

As mentioned earlier, a frequency sweeping framework is adapted to reveal the stability features of the dynamics. Frequency sweeping enables some convenient geometry arguments leading to a practical stability analysis method which is presented step-by-step. The stability analysis method is a versatile technique which helps analyzing AS of the dynamics for any given driver parameters. Some unexpected and intriguing stability features are pointed out. Using the stability analysis, we particularly investigate (i) how AS is affected in the delay parameter, (ii) how α_k and h_k affect AS, and (iii) how stability robustness is affected by studying independent delays as opposed to assuming all these delays identical. The proofs of theorems are suppressed, but they can be found in [33].

A. Theoretical Development

Recall that detection of imaginary roots, $s = j\omega$, of the characteristic function is the starting point of the stability analysis,

$$f(j\omega,\boldsymbol{\tau},\boldsymbol{\alpha}) = 0, \tag{8}$$

in which we assume that driver aggressiveness coefficients $\boldsymbol{\alpha}$ are known prior to the stability analysis.

In the following, we present two different approaches in analyzing AS on Eq. (8). First approach takes this equation as it is and reveals some properties regarding AS. Second approach reveals some 'delay decoupling' features that exist in this equation. With the availability of decoupling, it will be possible to use the approaches proposed in [12], [32].

1) Approach 1: Replace the exponential functions in (8) with

$$e^{-\tau_m s} = x_m + jy_m, \quad m = k, k+1, k+2, \tag{9}$$

and obtain a new function $g(j\omega, x_m, y_m, \boldsymbol{\alpha})$ where x_m, y_m are real numbers. Exponential terms in (9) define unit circles on (x_m, y_m) planes,

$$\mathcal{C}_m = x_m^2 + y_m^2 - 1 = 0, \quad m = k, k+1, k+2. \tag{10}$$

Theorem 1: Boundaries of stability switching curves (SSC) and ultimately of the stability regions on (τ_{k+1}, τ_{k+2}) plane are independent from the choice of any τ_k.

Theorem 2: The only $s = j\omega$ solution of $g(j\omega, x_m, y_m, \boldsymbol{\alpha}) = 0$ along τ_k axis can be found at $\tilde{\tau}_k = \pi(1 + \mp 4\pi\ell)/(2\alpha_k)$, $\ell = 0, 1, \ldots$.

Notice that Theorem 1 only indicates that the geometry of the potential stability switching curves does not change for different choices of τ_k. On the other hand, choice of τ_k affects AS as we shall show below. What Theorem 2 shows is that there exists only $\tilde{\tau}_k$ points along τ_k axis for which $s = j\omega$ is a solution.

Corrolary 1: Connecting Theorem 1 and Theorem 2, all the points $\tilde{\tau}_k$ are independent of $(\tau_{k+1}, \tau_{k+2}) \in \mathbb{R}^2_+$.

Remark 1: It is easy to check that AS with respect to τ_k holds for only $0 \le \tau_k < \tau_k^*$, where $\tau_k^* = \pi/(2\alpha_k)$. Furthermore, since the characteristic function in (8) represents a neutral type dynamics, for AS it is necessary but not sufficient that the condition $|h_{k+1}\alpha_{k+1}| < 1$ holds. This guarantees the stability of the difference equation (7).

Remark 2: The work in [3] chooses α_{k+1} and h_{k+1} as 0.37 and 1.8, respectively, for human drivers. Although the cited work does not consider multiple delays, these numerical choices can be tested in the inequality conditions obtained within our framework. If the traffic flow dynamics is governed by multiple delays, the condition $|h_{k+1}\alpha_{k+1}| = 0.666 < 1$ is not violated by these realistic numerical choices. Moreover, one calculates the upper bound $\tau_k^* = \pi/(2\alpha_k) = 4.24\, sec.$

Theorem 3: Stability switching curves on $(\tau_{k+1}, \tau_{k+2}) \in \mathbb{R}^2_+$ can be found to be generated by only *two* fundamental (kernel) curves for $\forall \omega \in \mathbb{R}_+$.

It is important to state that the solutions discussed in Theorem 3 belong to the *kernel curves*, [29]. In other words, these solutions are the only generators of infinitely many other solutions that exist on the plane of $\tau_{k+1} - \tau_{k+2}$. Once the kernel curves are identified, it is straightforward to obtain the entire set of stability switching curves.

When $\tau_k = \tau_{k+1} = \tau_{k+2} = 0$, the characteristic function has two roots $s_1 = 0$ and $s_2 = -\alpha_k$. This reveals that the delay system is marginally stable since $s_2 < 0$ and s_1 is an invariant root (it is independent of the choice of the delays), see also [8]. Due to this invariant root, $\omega = 0$ becomes a solution to (6). The following theorem characterizes the stability features associated with $\omega = 0$.

Theorem 4: A characteristic root crosses the origin of $\mathbb{C}$ for any (τ_{k+1}, τ_{k+2}) pair residing on the line equation $L(\tau_{k+1}, \tau_{k+2}) = 1 + \alpha_{k+1}(\tau_{k+2} - \tau_{k+1} - h_{k+1}) = 0$.

Remark 3: Notice that the invariant $s = 0$ root of the dynamics is due to rigid body dynamics of the perturbations. In other words, $s = 0$ defines the static part of the modes of the perturbations. Physically, one may disregard the rigid body dynamics and analyze the behavior of perturbations around a static mode. In this regard, marginal stability feature mentioned above may be seen as AS around the rigid body motion.

Theorem 5: As per the remark above and if $0 \le \tau_k < \pi/(2\alpha_k)$ holds, then the dynamics is AS for any delay $\tau_{k+2} \in \mathbb{R}_+$ along the $\tau_{k+1} = 0$ axis.

Property 1: Assuming difference equation (7) is stable ($|h_{k+1}\alpha_{k+1}| < 1$), the maximum of ω for which $s = j\omega$ is a solution to (8) is upper bounded, [13]. In the following, this conservative upper bound is denoted by $\bar{\omega}$.

2) Approach 2: It is easy to see that the following holds,

$$f(j\omega, \boldsymbol{\tau}, \boldsymbol{\alpha}) = P(j\omega, \tau_k)\, Q(j\omega, \tau_{k+1}, \tau_{k+2}) = 0, \quad (11)$$

$$P(j\omega, \tau_k) = j\omega + \alpha_k e^{-j\omega\tau_k}, \quad (12)$$

$$Q(j\omega, \tau_{k+1}, \tau_{k+2}) = j\omega(h_{k+1}\alpha_{k+1}e^{-j\omega\tau_{k+2}} - 1) + \alpha_{k+1}(e^{-j\omega\tau_{k+2}} - e^{-j\omega\tau_{k+1}}). \quad (13)$$

The manipulation above decouples the effects of τ_k and the pair (τ_{k+1}, τ_{k+2}) to AS. In other words, AS analysis can be divided into two steps; one concerning AS along the axis τ_k and the other in $(\tau_{k+1}, \tau_{k+2}) \in \mathbb{R}^2_+$. Assessing the stability of $P(s, \tau_k) = 0$ is straightforward [18], [19]. Consequently, the objective is now to solve all $(\tau_{k+1}, \tau_{k+2}) \in \mathbb{R}^2_+$ and $\omega \in \mathbb{R}_+$ from Eq. (13) precisely.

Notice that one can write

$$Q(j\omega, \tau_{k+1}, \tau_{k+2}) = Q_k(j\omega) + Q_{k+1}(j\omega)e^{-j\omega\tau_{k+1}} + Q_{k+2}(j\omega)e^{-j\omega\tau_{k+2}},$$

where $Q_k(j\omega) = -j\omega$. For $\omega \neq 0$, define now $a_i = Q_{i+1}/Q_k$, $i = k, k+1$, and Ω as the set of crossing frequencies. Next, the main results can be rewritten as:

Theorem 6: The frequency ω_0 ($\omega_0 > 0$) is a crossing frequency, that is $\omega_0 \in \Omega$, if one of the following conditions is satisfied:

(i) $\omega_0 = |\alpha_k|$;

(ii) The following triangle inequalities hold simultaneously:

$$\left\{ \begin{array}{l} |\, a_k(j\omega_0)\, | + |\, a_{k+1}(j\omega_0)\, | \geq 1 \\ -1 \leq |\, a_k(j\omega_0)\, | - |\, a_{k+1}(j\omega_0)\, | \leq 1. \end{array} \right\}$$

Remark 4: It is important to point out the behavior of the imaginary root crossing for small frequency values. In other words, for $\omega \to 0$, see Theorem 4.

The remaining analysis concerning the classification of the stability crossing curves, as well as the smoothness and the crossing direction characterizations follow closely the ideas presented in [12].

3) Numerical Implementation: An algorithm for deriving the stability crossing curves can be resumed as follows:

- Compute τ_k from Remark 1. If $0 \le \tau_k < \pi/(2\alpha_k)$ holds *and* delay-difference operator (7) is stable, proceed to the following steps. Otherwise the system is unstable.
- Assume first that $\omega \neq 0$ is numerically known and sweep ω within a prescribed range as per Property 1.
- Approach 1: For the given h_{k+1} and $\boldsymbol{\alpha}$ values, solve $g(j\omega, x_m, y_m, \boldsymbol{\alpha}) = 0$ simultaneously with the unit circles $\mathcal{C}_{k+1}$ and $\mathcal{C}_{k+2}$. Common roots x_{k+2}, y_{k+2} can be used along with ω to compute the delay τ_{k+2} from (9). Approach 2: Use ω to compute the triangle inequalities in Theorem 6. If these inequalities are satisfied, one can use the arguments of the vectors $|a_k|$ and $|a_{k+1}|$ to compute the corresponding delays τ_{k+1} and τ_{k+2}.

- Approach 1: Using the common roots x_{k+2}, y_{k+2} from the previous step, obtain the pairs x_{k+1}, y_{k+1} from the common solutions between $g(j\omega, x_m, y_m, \boldsymbol{\alpha}) = 0$ and $\mathcal{C}_{k+1}$. From (9), find τ_{k+1} using x_{k+1}, y_{k+1}, ω.
- Repeat the above steps for a range of ω, $0 < \omega < \bar{\omega}$ and extract all τ_{k+1}, τ_{k+2} pairs. Deploy the trigonometric properties to compute the infinitely many roots using $2\pi\ell/\omega$, $\ell = 0, 1, \dots$ shifting. All these delays lie on the stability switching curves, [29].
- For the given h_{k+1} and $\boldsymbol{\alpha}$ values, find $L(\tau_{k+1}, \tau_{k+2}) = 0$ defined in Theorem 4. Superpose this line on top of the stability switching curves.
- Following from [30], identify asymptotically stable versus unstable regions in the parameter space of $\tau_{k+1} - \tau_{k+2}$.

IV. Illustrative Examples

In this section, we borrow appropriate numerical values from [3]. The cited work suggests that $\alpha_k = 0.37$ and $h_k = 1.8$, $k = 1, \dots, n$. In order to represent the heterogeneity of the drivers, we will use the suggestions as nominal values.

First case analyzes two different driver behaviors $\alpha_k = 0.33$ and $\alpha_{k+1} = 0.40$, but $h_{k+1} = 1.8$ for both drivers. The arising stability map is given in Figure 2, where on this figure the shaded gray regions correspond to AS of the two consecutive spacing dynamics. It is crucial to state that the stability regions are valid only for $0 \leq \tau_k < 4.76$. For $\tau_k \geq 4.76$, the spacing dynamics is unstable for any τ_{k+1}, τ_{k+2}. Also note that AS is a necessary condition but it is not sufficient to claim that traffic flow will be *collision free*. The collision analysis is the following step that needs to be studied once AS is understood. We leave this part of the work to future studies.

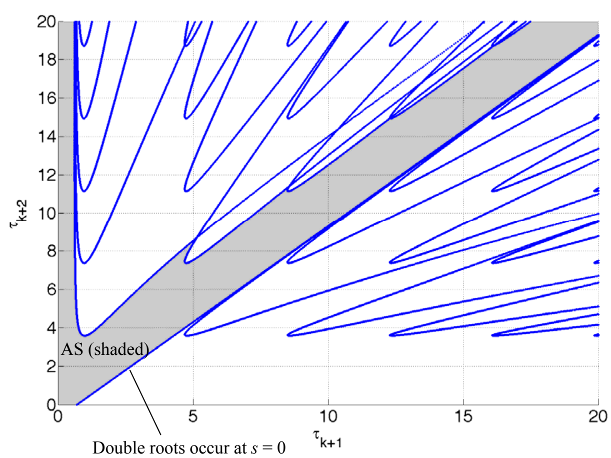

Fig. 2. Asymptotic stability regions (shaded) for $\alpha_k = 0.33$, $\alpha_{k+1} = 0.40$, $h_{k+1} = 1.8$. For stability, $0 \leq \tau_k < 4.76$.

Next, we investigate the AS assuming $\alpha_k = \alpha_{k+1} = 0.37$. Three different scenarios are taken where in each one a different h_{k+1} is used. We choose $h_{k+1} = 1.25$, $h_{k+1} = 1.5$ and $h_{k+1} = 1.8$ and compare the arising stability maps, Figure 3. From practical point of view, $h_{k+1} = 1.8$ is a realistic value for human drivers, whereas $h_{k+1} = 1.2$ is more realistic for automated vehicles. In a sense, Figure 3 compares the effectiveness of human drivers and automated vehicles. We see that stability regions (gray) are narrower and smaller with human drivers, especially around the point $\tau_{k+1} = 2$ and $\tau_{k+2} = 6$.

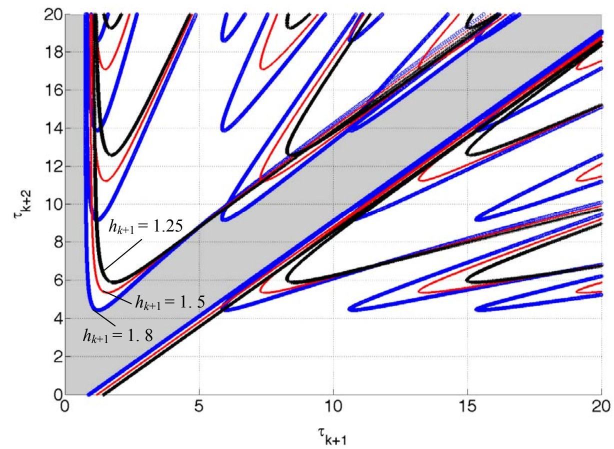

Fig. 3. Effects of constant time headway controller h_{k+1} to stability. $\alpha_k = \alpha_{k+1} = 0.37$, $h_{k+1} = 1.25$ (black curves), $h_{k+1} = 1.5$ (red curves, thin), $h_{k+1} = 1.8$ (blue curves, thicker). For stability, $0 \leq \tau_k < 4.2454$.

We now study how stability is affected if a more aggressive driver follows a less aggressive driver (scenario 1), or vice versa (scenario 2). We take $h_{k+1} = 1.8$ for both drivers in both scenarios. In scenario 1, $\alpha_k = 0.40$ and $\alpha_{k+1} = 0.30$; and in scenario 2, $\alpha_k = 0.30$ and $\alpha_{k+1} = 0.40$. For scenario 1 and scenario 2, stability boundaries are given by black and blue curves, respectively, in Figure 4. Clearly, the black curves leave more space for the stability regions (regions connected to the origin of the plane). This result points out that stability regions of spacing dynamics in delay space become larger in the case when more aggressive drivers (or automated vehicles with higher actuation bandwidth) follow less aggressive drivers (scenario 1). Larger stability regions can also be interpreted as a degree of robustness against uncertainties, perturbations and unmodeled dynamics. Finally, we comment that larger driver aggressiveness coefficients (α_k and α_{k+1}) lead to narrower stability regions. This may be an expected result, however, it surfaces within the context of analytical stability analysis.

V. Conclusions

Spacing (headway) dynamics in a chain of vehicles is studied by departing from a widely studied Pipes model over which we consider the natural presence of drivers' heterogeneity and ultimately multiple delays arising from these drivers due to their 'delayed' reactions to stimuli. A constant time headway driving strategy is considered for all the drivers and the asymptotic stability of the spacing

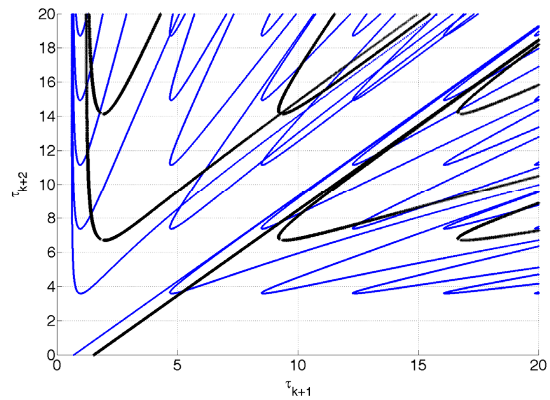

Fig. 4. Effects of a more aggressive driver following a less aggressive driver (Scenario 1: thick black curves and $0 \leq \tau_k < 3.9270$), or vice vera (Scenario 2: thin blue curves and $0 \leq \tau_k < 5.2360$) to asymptotic stability. Constant time headway controller $h_{k+1} = 1.8$ for both drivers.

dynamics is investigated with respect to parameters defining the strategy and drivers' aggressiveness coefficients. A neutral characteristic function with three independent delays governs the stability features which we reveal by developing frequency sweeping and geometric arguments. Case studies demonstrate how stability is affected with respect to constant time headway driving strategy, drivers' aggressiveness and more (less) aggressive drivers following behind less (more) aggressive drivers.

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