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Sliding Mode Boundary Control of Unstable Parabolic PDE Systems with Parameter Variations and Matched Disturbances

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***Abstract*—This paper considers the stabilization problem of a one-dimensional unstable heat conduction system subject to parametric variations and boundary uncertainties. This system is modeled as a parabolic partial differential equation (PDE) and is only powered from one boundary with a Dirichlet type of actuator. By taking the Volterra integral transformation, we obtain a nominal PDE with asymptotic stability characteristics in the new coordinates when an appropriate boundary control input is applied. The associated Lyapunov function can then be used for designing an infinite-dimensional sliding surface, on which the system exhibits exponential stability, invariant of the bounded matched disturbance, and is robust against certain types of parameter variations. A continuous variable structure boundary control law is employed to attain the sliding mode on the sliding surface. The proposed method can be extended to other parabolic PDE systems such as diffusion-advection system. Simulation results are demonstrated and compared with the other outstanding back-stepping control schemes.**

***Index Terms*—Boundary control; chattering, distributed-parameter systems; sliding surface; Lyapunov methods.**

I. INTRODUCTION

Many physical phenomena governed by PDEs, such as heat conduction [1], wave propagation [2], and beam flexure [3], are inevitably subject to certain degrees of modeling uncertainties or exogenous disturbances in the interior domain or at the boundary. These so-called distributed parameter systems are often controlled through the entire domain, mobile object [4], or merely instrumented with boundary actuators. Among them, the boundary control mechanism demonstrates more facilitation in controller implementation, although the degree of freedom in design is much more limited. Boundary control for PDE systems has become an important research area and has been well investigated in recent years [5,6]. However, even in the simplest case of heat conduction systems, the problems with uncertainties still appear to be formidable. Some discontinuous control strategies have been successfully applied to the truncated finite-dimensional model in the context of distributed control [7]–[9]. In this paper, it is of our interest to construct a simple discontinuous surface and to develop sliding mode boundary control laws for infinite-dimensional systems without model truncations. We will adopt the Lyapunov approach to sliding surface design [10] into this infinite-dimensional system version. A continuous sliding mode boundary control law is then designed to achieve system stabilization in spite of some bounded system parameter variations and boundary exogenous disturbances.

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The variable structure control (VSC) methodology has been applied to infinite-dimensional systems in the distributed control mechanism [11]–[13] as well as in the boundary control mechanism [14,15]. By utilizing the semi-group operator theory, Utkin [12] presented the sliding mode discontinuous distributed control scheme for the heat process under the compact commutability conditions with the assumptions of open-loop stability and full state accessability. Drakunov *et al*. [15] proposed a sliding mode controller for the boundary control problem of the stable heat equation with boundary disturbances. An integral transformation was employed to reformulate the problem into a first-order PDE. A sliding manifold as a function of the distributed states is presented. In this paper, we follow a similar idea and present a novel discontinuous sliding surface for the boundary control problem of an unstable parabolic PDE system via the Lyapunov method.

We first consider the boundary control problem of a one-dimensional thermal unstable system in heat conduction with constant coefficients and only subject to the matched boundary disturbance. In Section IV, we will further consider the boundary control problem of the heat equation with parametric variations. Let $U(x,t)$ be the temperature distribution of the rod of length l with respect to some desired (nominal) value. The governing equation is a linear second order parabolic PDE

$$U_t(x,t) = \alpha U_{xx}(x,t) + \beta U(x,t) \tag{1}$$

for $x \in [0,l]$, $t > 0$ and the subscripts denote the derivatives. The $U(x,t)$ is the temperature distribution of the rod with length l . The constant $\alpha > 0$ corresponds to square of the thermal diffusivity, and the destabilizing reaction parameter $\beta \in \mathbb{R}$, are arbitrary constants. The homogeneous boundary condition at $x = 0$ is

$$U(0,t) = 0 \tag{2}$$

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The control input is applied at opposite end $x = l$, and is corrupted by an exogenous disturbance $d(t) \in C^1([0, \infty)$. We consider the Dirichlet boundary actuator

$$U(l, t) = Q(t) + d(t), \tag{3}$$

where $Q(t)$ is the control input and the disturbance $d(t)$ is assumed to be bounded. For $d(t) = Q(t) = 0$, system (1) can have an arbitrary large number of unstable eigenvalues for large β/α [1]. The objective is to regulate this unstable and uncertain infinite-dimensional system to zero distribution, so $\lim_{t\to\infty} U(x, t) \equiv 0$, for $x \in (0, l)$, for by sliding mode methodology.

The contribution of this paper is to construct an easy to implement discontinuous sliding surface for boundary control of the parabolic PDE systems from the viewpoint of Lyapunov method.

II. Construction of Sliding surface

A. Coordination transformation

We first use the Volterra integral transformation to convert the original problem into new coordinates [16]–[18]

$$\omega(x, t) = U(x, t) - \int_0^x k(x, y)U(y, t)dy \tag{4}$$

where $k(x, y)$ is a function of two variables and is often referred to as the kernel or nucleus of the integral equation. The system (1)-(3) is then mapped into a new coordinate, but with uncertainties,

$$\omega_t(x, t) = \alpha\omega_{xx}(x, t) - c\omega(x, t) \tag{5}$$

$$\omega(0, t) = 0 \tag{6}$$

$$\omega(l, t) = Q(t) + d_\omega(t) \tag{7}$$

where $c > 0$ is a free parameter for setting the desired rate of stability. d_ω is regarded as the new boundary disturbance matched to the control input from the viewpoint of ω-coordinate. It is given by

$$d_\omega(t) = d(t) - \int_0^l k(l, y)U(y, t)dy \tag{8}$$

Substituting (1)-(3) into (5)-(7), using the relationship (4) and also introducing the notation $\frac{d}{dx}k(x, x) = k_x(x, x) + k_y(x, x)$, the kernel function $k(x, y)$ should satisfy

$$k_{xx}(x, y) - k_{yy}(x, y) = \lambda k(x, y), \qquad (x, y) \in \mathcal{T}, \tag{9}$$

$$k(x, 0) = 0 \tag{10}$$

$$k(x, x) = -\frac{\lambda}{2}x \tag{11}$$

where $\mathcal{T} = \{x, y : 0 < y < x < l)$ and $\lambda = (\beta + c)/\alpha$.

Existence and uniqueness of this integral operator (4) can be tracked back to 1980s such as in [19,20]. For this hyperbolic PDE system (9)-(11), similarly with wave equation solved using D'Alembert's formula, it could be reformulated in variables of $x+y$ and $x-y$, and transformed into an integral equations [17]–[19]. Through the method of successive approximation, the analytical solution to this simple case for heat equation with constant coefficient has been solved in [17,18] as

$$k(x, y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}} \tag{12}$$

where I_i is a modified Bessel function of order i. The well-posed property and inverse transformation of (4) are also investigated in [18]. Here, we focus on the problems of sliding manifold design.

B. Constructing sliding surface via Lyapunov's direct method

Select the Lyapunov function as

$$V(t) = \frac{1}{2}\int_0^l \omega^2(x, t)dx > 0 \tag{13}$$

Substituting the PDE (5) and the boundary conditions (6)-(7) into its time derivative, we have

$$\dot{V}(t) = \alpha\omega_x(l, t)\omega(l, t) - \alpha\int_0^l \omega_x^2(x)dx - c\int_0^l \omega^2(x)dx \tag{14}$$

Choose the switching surface for Dirichlet actuation be

$$S(t) = \omega_x(l, t) = 0 \tag{15}$$

Then, on the sliding surface (15), it yields

$$\dot{V}(t) = -\alpha\int_0^l \omega_x^2(x, t)dx - c\int_0^l \omega^2(x, t)dx < 0$$

This sliding surface (15) does not provide an extra degree of freedom for selecting the desired eigenvalues in the sliding mode dynamics. However, in the ω-coordinate (5), the parameter furnishes as a design parameter c for choosing the rate of convergence. We have the following lemma.

Lemma 1: The system (5)-(7) on the sliding surface (15) is exponentially stable in $L_2(0, l)$ norm, with a decay rate $2c + \frac{\alpha}{2l^2}$.

Proof: According to [21], the Poincaré inequality can be modified as

$$\int_0^l \omega^2(x, t)dx \le 2l\omega^2(0, t) + 4l^2\int_0^l \omega_x^2(x, t)dx \tag{16}$$

With the boundary condition (6) and (16), we get

$$\dot{V}(t) \le -(c + \frac{\alpha}{4l^2})\int_0^l \omega^2(x, t)dx = -\frac{(\alpha + 4cl^2)}{2l^2}V(t) < 0$$

Therefore, it is $V(t) \le V(0)e^{-\frac{(\alpha+4cl^2)}{2l^2}t}$.

On the sliding surface (15), the influence of the control Q and the matched boundary disturbance $d(t)$ are completely excluded. Thus, this PDE system on sliding surface is exponentially stable. ■

In [21], point-wise asymptotic stability of the equilibrium $\omega \equiv 0$ for $x \in [0, l]$ can be further checked via the Agmon inequality. The more positive the parameter c is or the shorter length l is, the L_2 norm of ω has more rapid decay.

Hereafter, the control objective is to design the boundary control law Q to drive the system state towards and restrain it on the sliding surface $S(t) = 0$. In the original coordinate system (1)

$$S(t) = U_x(l,t) - k(l,l)U(l,t) - \int_0^l k_x(l,y)U(y,t)dy. \quad (17)$$

C. Stability in sliding mode

Lemma 1 assures exponential stability of the system in coordinate (5)-(7) on the sliding surface (15). In particular, the explicit sliding mode dynamics can be obtained as

$$\omega(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\frac{(c+\lambda_n^2)}{\alpha}t}\sin(\lambda_n x)\int_0^l \omega(\zeta,0)\sin(\lambda_n\zeta)d\zeta \quad (18)$$

where $\lambda_n = (2n+1)\pi/2l$, $\omega(x,0) = u_0(x) - \int_0^x k(x,y)u_0(y)dy$, and $u_0(x) = U(x,0)$ are the initial condition of the ω-system and the U-system, respectively. To express the corresponding sliding mode dynamics of (1)-(3) on the surface (17) in the original coordinate, we use the inverse transformation [17]–[20]

$$U(x,t) = \omega(x,t) + \int_0^x L(x,y)\omega(y,t)dy \quad (19)$$

The kernel function $L(x,y)$ takes the form similar to $k(x,y)$

$$L(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} \quad (20)$$

where J_1 is a standard Bessel function of the first order. Thus, the explicit solution of the original system (1) on sliding mode (17) becomes

$$U(x,t) = \tfrac{2}{l}\textstyle\sum_{n=1}^{\infty} e^{-\frac{(c+\lambda_n^2)t}{\alpha}}\left(\left[\sin(\lambda_n x) - \int_0^x \lambda y \cdot\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}\sin(\lambda_n y)dy\right]\cdot\int_0^l \left[u_0(x) + \int_0^x \lambda y \times \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}u_0(y)dy\right]\sin(\lambda_n x)dx\right) \quad (21)$$

Due to boundedness of J_1 and the reasonable initial data $u_0(x)$, the term within the brackets is the bounded function of x. Therefore, $U(x,t)$ has exponential stability.

Theorem 1: The system (1)-(3) with sliding mode on the surface (17) is exponentially stable.

Note that this proposed sliding surface function (17) requires full states accessibility in general. However, if $\beta \leq 0$, system (1) will be open-loop stable without control. In this case, the kernel function is no longer needed for stabilization purpose. The sliding surface can be assigned as (17) with a zero kernel function; that is $k(x,y) = 0$, such that $S(t) = U_x(l,t)$. And a simple point observation suffices. An example will be demonstrated in Section V.

III. Design of Sliding mode boundary controllers

In this section two sliding mode controllers are presented in different coordinates. The first one is based on the transformed ω-coordinates, and the latter is constructed from the original U-coordinates. Both of them are continuous.

A. Controller design in mapped coordinate

Take the time derivative of the sliding surface variable $S(t)$ of (15) and substitute into the PDE (5) to render,

$$\dot{S}(t) = \omega_{xt}(l,t) = \alpha\omega_{xxx}(l,t) - c\omega_x(l,t) \quad (22)$$

in which the boundary control Q does not appear. Integrating both sides of (5) in terms of x from 0 to l, and then taking time derivative of t , we obtain

$$\begin{aligned}\int_0^l \omega_{tt}(x,t)dx &= \alpha\left[\omega_{xt}(l,t) - \omega_{xt}(0,t)\right] - \int_0^l c\omega_t(x,t)dx \\ &= \alpha\dot{S}(t) - \alpha\omega_{xt}(0,t) - \int_0^l c\omega_t(x,t)dx\end{aligned}$$

Since the above result is irrelevant to the spatial variable x, it could be rewritten as

$$\int_0^l \left[\omega_{tt}(x,t) + \frac{\alpha}{l}\dot{S}(t) + \frac{\alpha}{l}\omega_{xt}(0,t) + c\omega_t(x,t)\right]dx = 0 \quad (23)$$

For a physical heat conduction system, the above integrand is bounded on the interval $[0,l]$, i.e.,

$$|f_s(x,t)| = \left|\omega_{tt}(x,t) - \frac{\alpha}{l}\dot{S}(t) - \frac{\alpha}{l}\omega_{xt}(0) + c\omega_t(x,t)\right| < \infty$$

Let $f_s(l,t) = d_0(t)$ for $x = l$, where $d_0(t)$ is an unknown but bounded function of time. We have

$$\dot{S}(t) = \frac{l}{\alpha}\omega_{tt}(l,t) + \frac{cl}{\alpha}\dot{Q}(t) + \frac{cl}{\alpha}\dot{d}_\omega(t) + \omega_{xt}(0,t) - \frac{l}{\alpha}d_0(t) \quad (24)$$

It indicates that the relative order from $Q(t)$ to $S(t)$ is zero. We propose the sliding mode boundary control law as

$$Q(t) = -K\int_0^t \text{sign}(S(\tau))d\tau \quad (25)$$

where $K > \left|\frac{\alpha}{cl}\frac{l}{\alpha}\omega_{tt}(x,t) + \frac{cl}{\alpha}\dot{d}_\omega(t) + \omega_{xt}(0,t) - \frac{l}{\alpha}d_0(t)\right|$ with $S(t)$ is selected as (15). The system will reach the switching surface $S(t) = 0$ in a finite time and restrained on it. To see this, represent the time derivative of sliding surface in (24) as

$$\dot{S}(t) = \frac{cl}{\alpha}\dot{Q}(t) + g(t) \quad (26)$$

where $g(t) = \frac{l}{\alpha}\omega_{tt}(x,t) + \frac{cl}{\alpha}\dot{d}_\omega(t) + \omega_{xt}(0,t) - \frac{l}{\alpha}d_0(t)$, which is completely unknown but is assumed bounded. Select the Lyapunov candidate function for $S(t)$ as

$$V_s(t) = \frac{1}{2}S^T(t)S(t) \quad (27)$$

Taking the time derivative of $V_s(t)$ and substituting (26) yields

$$\dot{V}_s(t) = S(t)\dot{S}(t) = \frac{cl}{\alpha}S(t)\dot{Q}(t) + S(t)g(t)$$
$$< -\|S\|(-\frac{cl}{\alpha}\dot{Q}(t) - \|g(t)\|)$$

Substituting the derivative of proposed controller (25) yields $\dot{V}_s(t) < -\sigma\|S\|^2$, with $\sigma > 0$. Thus, the sliding surface reachability condition is satisfied. Once the system is constrained within the sliding surface $S(t) = 0$, for $t > t_s$, where t_s is the time that the sliding mode is attained, an ideal sliding motion takes place. The system will exponentially converge to the origin, $\lim_{t\to\infty} U(x,t) \equiv 0$.

B. Controller design in original coordinate

Here, we present the sliding mode boundary control law design in the original coordinate using the same format as (25).

$$Q(t) = -K_m \int_0^t \text{sign}(S(\tau))d\tau \tag{28}$$

where $S(\tau)$ is selected as (17). Taking the time derivative of in (17) yields

$$\dot{S}(t) = U_{xt}(l,t) + \frac{\lambda l}{2}U_t(l,t) - \int_0^l k_x(l,y)U_t(y,t)dy \tag{29}$$
$$= \alpha U_{xxx}(l,t) + \beta U_x(l,t) + \frac{\lambda l}{2}(\dot{Q}(t) + \dot{d}(t))$$
$$- \int_0^l k_x(l,y)[\alpha U_{yy}(y,t) + \beta U(y,t)]dy$$
$$= \frac{\lambda l}{2}\dot{Q}(t) + g_2(t)$$

where $g_2(t) = \frac{\lambda l}{2}\dot{d}(t) + \alpha[U_{xxx}(l,t) - \int_0^l k_x(l,y) U_{yy}(y,t)dy] + \beta[U_x(l,t) - \int_0^l k_x(l,y)U(y,t)dy]$ is a lumped uncertainties signal, which is completely unknown but is assumed bounded, that is $\|g_2(t)\| \le \delta_2(t)$. With $K_m > \frac{2}{\lambda l}\delta_2(t)$ and $\delta_2(t)$ being bounded, the motion of the system will reach the sliding mode $S(t) = 0$ in a finite time. To show this, select the same Lyapunov candidate function as in (27). The time derivative of $V_s(t)$ can be obtained as

$$\dot{V}_s(t) = S(t)\dot{S}(t) = \frac{\lambda l}{2}S(t)\dot{Q}(t) + S(t)g_2(t)$$
$$< -\|S\|(-\frac{\lambda l}{2}\dot{Q}(t) - \|g_2(t)\|)$$

Substituting the derivative of the proposed controller (28) yields $\dot{V}_s(t) = S(t)\dot{S}(t) < -\sigma\|S\|^2$. The reaching condition is then satisfied, so the system will converge to the equilibrium manifold as $U(x,t) = 0$ as $t \to \infty$.

It is seen that for the control laws (25) and (28) the resultant system behaviors in ω-coordinates and the original U-coordinates are equivalent.

IV. PDE Systems with Mismatched Parametric Uncertainties and Matched Disturbances

Consider when the parabolic PDE is subject to not only the boundary disturbance but also parameter variations. The system model (1) is reformulated as

$$\begin{aligned} U_t(x,t) &= (\alpha + \Delta\alpha)U_{xx}(x,t) + (\beta + \Delta\beta)U(x,t) \\ &= \alpha U_{xx}(x,t) + \beta U(x,t) + f(x,t) \end{aligned} \tag{30}$$

where $f(x,t) = \Delta\alpha U_{xx}(x,t) + \Delta\beta U(x,t) \in C^1([0,l] \times [0,\infty)$ denotes the lumped effect of system parameter variations. Assume it is bounded. Using the transformation (4) renders

$$\omega_t(x,t) = \alpha\omega_{xx}(x,t) - c\omega(x,t) + f_\omega(x,t) \tag{31}$$

with $f_\omega(\cdot)$ is the effect of uncertainties in the ω-coordinate

$$f_\omega(x,t) = f(x,t) - \int_0^x k(x,y)f(y,t)dy \tag{32}$$

By using the relationship (32) and the properties of kernel function $k(x,y)$ in (9)-(11), the term $f_\omega(\cdot)$ could be further represented as

$$\begin{aligned} f_\omega(x) &= \Delta\alpha U_{xx} + \Delta\beta U \\ &\quad - \textstyle\int_0^x k(x,y)\left[\Delta\alpha U_{yy}(y) + \Delta\beta U(y)\right]dy \\ &= \Delta\alpha[\omega_{xx} + 2\tfrac{d}{dx}k(x,x)U - \tfrac{1}{\alpha}f(0,t)] + \Delta\alpha \\ &\quad \times \textstyle\int_0^x \left[k_{xx}(x,y) - k_{yy}(x,y)\right]U(y)dy + \Delta\beta\omega \\ &= \Delta\alpha\left[\omega_{xx}(x,t) - \lambda\omega(x,t)\right] + \Delta\beta\omega(x,t) \end{aligned} \tag{33}$$

with $f(0,t) = \frac{\alpha\Delta\beta}{(\alpha+\Delta\alpha)}U(0,t) = 0$, which is obtained from (1) and (2).

Theorem 2: The system (30) with both parameter variations (33) and boundary disturbance (3) is exponentially stable on the sliding surface (15) if

$$c > max\{0, c_0 - \frac{(\alpha + \Delta\alpha)}{4l^2}\} \tag{34}$$

where $c_0 = \frac{\alpha\Delta\beta - \beta\Delta\alpha}{\alpha+\Delta\alpha}$.

Proof: Extending from Lemma 1 and using the same Lyapunov function in (13), it yields

$$\dot{V}(t) = \int_0^l \omega(x,t)\left[\alpha\omega_{xx}(x,t) - c\omega(x,t) + f_\omega(x,t)\right]dx$$
$$= (\alpha + \Delta\alpha)(Q + d_\omega)\omega_x(l) - (\alpha + \Delta\alpha)\int_0^l \omega_x^2(x,t)dx$$
$$- (c - \Delta\beta + \lambda\Delta\alpha)\int_0^l \omega^2(x,t)dx$$

When the system on the sliding mode (15), we can further use the λ value in (9) to simplify it as

$$\dot{V}(t) \le -(\alpha + \Delta\alpha)\int_0^l \omega_x^2(x,t)dx - (c - c_0)\int_0^l \omega^2(x,t)dx$$

Using Poincare inequality (16), the above equation can be represented as

$$\dot{V}(t) \le -\frac{1}{2}(c - c_0 + \frac{(\alpha + \Delta\alpha)}{4l^2})V(t) < 0$$

when the condition (34) satisfied. ■

V. Extension to other benchmark problems

The results in the previous sections can be extended to some other benchmark problems such as unstable heat equation with the Neumann boundary actuator and diffusion-advection systems.

A. Neumann boundary actuators

Consider the unstable heat equation (1)-(2) with Neumann actuator

$$U_x(l,t) = Q(t) + d(t) \tag{35}$$

Utilizing the same transform (4), the transformed PDE system and the zero end conditio are identical to (5) and (6). The boundary condition in the control input end is represented as

$$\omega_x(l,t) = Q(t) + d_\omega(t)$$

where $d_\omega(t) = d(t) - k(l,l)U(l,t) - \int_0^l k_x(l,y)U(y,t)$ is the new disturbance matched to the control input. Choose the switching surface as

$$S(t) = \omega(l,t) \tag{36}$$

Then, this system has the exponential stability with a decay rate $2c + \alpha/2l^2$ on the sliding surface (36), and the corresponding sliding mode boundary control is similar to (28), that is

$$Q(t) = -K_n \int_0^t \text{sign}(S(\tau))d\tau \tag{37}$$

with $S(t)$ is selected as (36) and $K_n > \frac{\alpha}{c}\delta_n(t)$, where

$$|g_n(t)| = \left| -\frac{1}{c}\omega_{tt}(x,t) + \frac{\alpha}{cl}\dot{d}(t) \right| \le \delta_n(t)$$

B. Diffusion-advection systems

Consider the boundary control of diffusion-advection systems, whose system dynamics is modeled by

$$\begin{cases} \omega_t(x,t) = \varepsilon\omega_{xx}(x,t) + \kappa\omega_x(x,t) \\ \omega(0,t) = 0 \\ \omega_x(l,t) + q_3\omega(l,t) = Q(t) + d(t) \end{cases} \tag{38}$$

for $x \in [0,l]$, $\varepsilon > 0$, $q_3 \in \mathbb{R}_+$, and κ is an arbitrary constant. The boundary condition at $x = l$ is Robin type, which can be reduced into Dirichlet $q_3 = +\infty$ as or Neumann type as $q_3 = 0$. Following the previous results, we can select the sliding surface as

$$S(t) = \omega(l,t) \tag{39}$$

The system (38) actuated with Robin boundary control with sliding mode on the surface (39) is exponentially stable in $L_2(0,l)$ norm, with a decay rate $\varepsilon/4l^2$. The sliding mode boundary control still can apply for the Robin case $q_3 \neq \{0,+\infty\}$ as following

$$\dot{Q}(t) = -K_d\text{sign}(S(t)) \tag{40}$$

The other actuator cases with $q_3 = \{0,+\infty\}$ can be straightly extended.

VI. Simulation results

In this section, the simulation studies are conducted to verify the feasibility of the proposed controller (28) to the boundary control problem of unstable parabolic PDE systems (30). The performance is compared with other benchmark backstepping controllers proposed in [17,18] as

$$Q(t) = Q_s = \int_0^l k(l,y)U(y,t)dy \tag{41}$$

Only the results of Dirichlet boundary actuator is provided because the behavior of the closed loop system for the Neumann actuator is completely comparable.

Let us consider system (30) and (28) with $l = 1m$, $\beta = 17$, $\alpha = 1$ and with initial condition $U(x,0) = -0.01e^{6.7x}\sin(8\pi x)$, in which one unstable eigenvalues locates in 7.13. Two cases are considered here, the first case is an unstable heat system only with matched disturbance $d(t)$, and the second case is an unstable heat system subject to parameter variations up to 50% and boundary matched disturbance $d(t)$ as well. The uncertainties are assigned as $\Delta\alpha = 0.5$, $\Delta\beta = 8$, and $d(t) = 2 + 0.25\sin(20t)$.

For the case (a), the parameters are setup with $c = 1$, and $K = 10$, and in the controllers (41) and (28), abbreviated as SMC and BC, respectively. Utilizing the finite-difference method for a numerical study, the simulation results of the PDE system with the matched disturbance is illustrated in Fig.1. The proposed sliding mode controller can effectively stabilize the unstable system, and the L_2-norm will converge to zero as $t \to 0.5\,\text{sec}.$, and the smoothness of the controller effort is reasonable. Without the ability to deal with uncertainties, backstepping control (41) can not inhibit the effect of disturbance, while the sliding mode boundary controllers (28) can successfully demonstrate the robustness to the matched disturbance. In case (b), both the parameter variations and the boundary disturbance still have a great influence on the performance of the backstepping controller, but it is substantially slashed into a small region via the proposed sliding mode controllers, as shown in Fig.2. The steady state error in L_2-norm can be further reduced via the extra integral action incorporated. From these simulations, the presented method has revealed the robustness and performance in the boundary control problems of parabolic PDE systems.

VII. Conclusion

This paper contributes to the design of sliding surface and sliding mode boundary controller of a parabolic PDE system using the Lyapunov method. Although this method requires full-state feedback and does not have the advantage of order reduction, the presented control schemes are continuous and completely infinite-dimensional model is utilized. The proposed methodology can be easily extended to other benchmark parabolic PDE systems as long as the solution of kernel function $k(x,y)$ is obtained.

The Lyapunov function (13) plays the similar role as the matrix P in the Riccati equation of a finite-dimensional

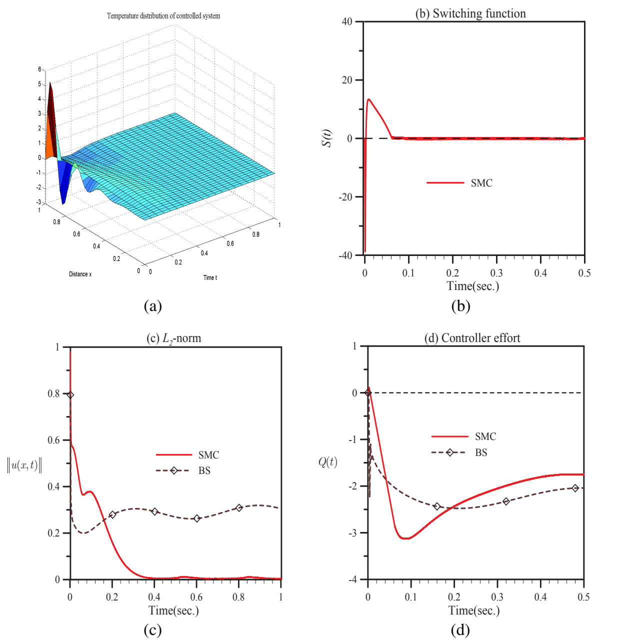

Fig. 1. Closed-loop system responses of Case (a) (solid line: SMC, dashed line: BC). (a) Temperature distribution by SMC.(b) Sliding surface. (c) Comparisons of L_2-norm. (d) The history of applied boundary controllers.

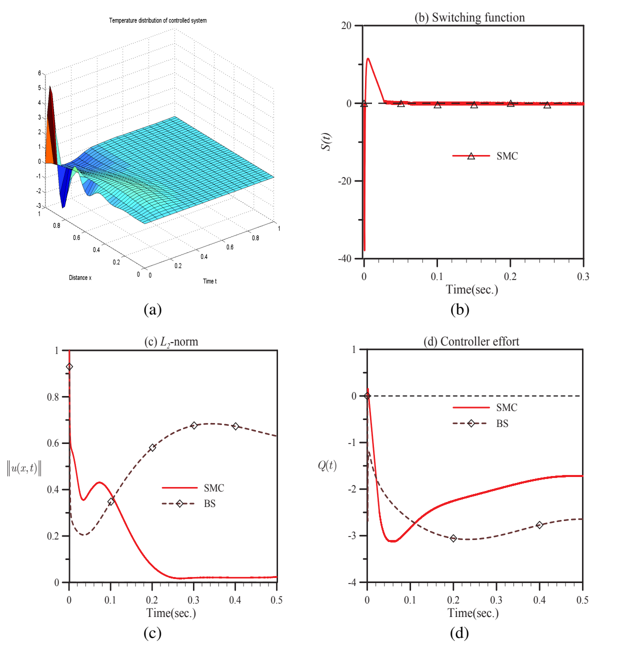

Fig. 2. Closed-loop system responses of case (b) (solid line: SMC, dashed line: BC). (a) Temperature distribution by SMC. (b) Sliding surface. (c) Comparisons of L_2-norm. (d) The history of applied boundary controllers.

system as

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (42)$$

with $R > 0$ and $Q \geq 0$. The versatility of Lyapunov's method may provides a new avenue to deal with other kind PDE problems such as wave, string, beam, etc. Investigation of these problems from the Lyapunov point of view seems to be a promising approach.

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