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Adaptive Robust Control of a Class of Nonlinear Systems in Semi-strict Feedback Form with Non-uniformly Detectable Unmeasured Internal States

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Abstract— **This paper proposes a novel control method for a special class of nonlinear systems in semi-strict feedback form. The main characteristics of this class of systems is that the unmeasured internal states are non-uniformly detectable, which means that no observer for these states can be designed to make the observation error exponentially converge to zero. In view of this, a projection-based adaptive robust control law is developed in this paper for this kind of system. This method uses a projection-type adaptation algorithm for the estimation of both the unknown parameters and the internal states. Robustifying feedback term is synthesized to make the system robust to uncertain nonlinearities and disturbances. It is theoretically proved that all the signals are bounded, and the control algorithm is robust to bounded disturbances and uncertain nonlinearities with guaranteed transient performance. Furthermore, the output tracking error converges to zero asymptotically if the system has only parametric uncertainties. The class of system considered here has wide engineering applications, and a practical example - control of mechanical systems with dynamic friction - is used as a case study. Simulation results are obtained to demonstrate the applicability of the proposed control methodology.**

Index Terms— **Nonlinear Control; Dynamic Uncertainties; Adaptive Robust Control; Dynamic Friction**

I. INTRODUCTION

The control of nonlinear systems with various kinds of uncertainties is receiving more and more attention these years. Parametric uncertainties and non-parametric uncertainties (external disturbances) are two major sources of uncertainties. To deal with them, the deterministic robust control (DRC) [8] and the adaptive control (AC) [4] have been developed. The deterministic robust controllers are able to guarantee transient performance and final tracking accuracy in the presence of various kinds of uncertainties. However, some problems like switching or infinite-gain feedback [8] will happen. In contrast, the adaptive controllers [4] are able to achieve asymptotic tracking in the presence of parametric uncertainties without using infinite gain feedback. However, this approach may result in unbounded control signals in the presence of external disturbances. In [9], an adaptive robust control (ARC) algorithm has been proposed, which incorporates the design methods of DRC and AC effectively. The resulting ARC controllers have the advantages of both DRC and AC while overcoming their practical limitations.

Besides parametric uncertainties and uncertain nonlinearities, some systems may be further subjected to dynamic uncertainties. This kind of system has exogenous dynamic systems whose states can not be measured. The control of this kind of system has received more and more attention in recent years because some real systems are of that form, e.g., the dynamic friction in [1], [7] and the eccentric rotor in [2]. In [2], an adaptive controller was designed for a class of extended strict feedback nonlinear systems in which the unmeasured states enter the systems in a linear affine fashion. However, it is unclear how the approach can be made robust to uncertain nonlinearities and disturbances. In [3], Jiang and Praly proposed a modified robust adaptive control procedure for a class of uncertain nonlinear systems subject to dynamic uncertainties. However, since this method does not explicitly use the structural information of the original system, it does not have some desirable properties like asymptotic output tracking in presence of parametric uncertainties only. In [10], [5], an observer based ARC algorithm was proposed. Robustness and asymptotic tracking can both be achieved using this algorithm. However, the original system is assumed to be uniformly detectable. This assumption limits the application of this method because some systems, e.g., the mechanical systems with dynamic friction, does not satisfy the assumed detectability condition.

In this paper, we propose a novel ARC algorithm for the control of a class of nonlinear systems in semi-strict feedback form whose unmeasured internal states are bounded but not uniformly detectable. For this kind of system, no observer can be designed to make the observation error converge to zero. Instead, we design a projection-type adaptation algorithm to give the state estimation. It is theoretically proved that with the proposed control law, the closed-loop system is robust to nonlinear uncertainties and disturbances and has guaranteed transient performances. Furthermore, in the presence of parametric uncertainties only, asymptotic output tracking can be achieved. These two characteristics combine the good merits of DRC and AC. To illustrate its applicability, we take a practical example - the control of linear motor system with dynamic friction - as a case study. This system satisfies the assumptions made in the paper, i.e., the unmeasured internal states are bounded but not uniformly detectable. The simulation results demonstrate the applicability of the proposed method in practical applications.

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II. Problem Formulation

In this paper, we consider the following nonlinear system.

$$\begin{array}{rcl} \dot{\eta} & = & F_\eta(x)\theta + G_\eta(x)\eta + H_\eta(x) + \Delta_\eta(x,\eta,u,t), \\ \dot{x}_i & = & x_{i+1} + \theta^T \varphi_{\theta i}(\bar{x}_i) + h_i(\bar{x}_i) + \Delta_i(x,\eta,u,t), \\ & & \qquad 1 \le i \le l-1 \\ \dot{x}_l & = & u + \theta^T \varphi_{\theta l}(x) + \varphi_{\eta l}^T(x)\eta + h_l(x) + \Delta_l(x,\eta,u,t), \\ y & = & x_1 \end{array} \tag{1}$$

where $x = [x_1, \cdots, x_l]^T \in R^l$ is the vector of measurable states. $\bar{x}_i = [x_1, \cdots, x_i] \in R^i$ is the vector of first i measurable states. u and y are the control input and output, respectively. $\eta \in R^m$ is the vector of unmeasured internal states. $\theta \in R^p$ is the vector of unknown constant parameters. $F_\eta \in R^{m\times p}$, $G_\eta \in R^{m\times m}$, $H_\eta \in R^m$, $\varphi_{\theta i} \in R^p$, $h_i \in R$ and $\varphi_{\eta l} \in R^m$ are matrices, vectors or scalars of known smooth functions. Δ_η and Δ_i represent the lumped unknown nonlinear functions such as disturbances and modeling errors.

Remark 1: In order to simplify the deduction and focus on how to deal with non-uniformly detectable internal states, the internal states here are assumed to appear only in the dynamic equation directly related to input u. However, with some unharmful modifications of the control algorithm, the class of systems which can be handled with the proposed method can be extended to the same one as in [10], i.e., the internal states can appear from l-th to n-th dynamic equation.

Now some practical assumptions are made as follows:

Assumption 1: The extents of parametric uncertainties are known. And the uncertain nonlinearities are bounded by known functions. In other words, parametric uncertainties and uncertain nonlinearities satisfy

$$\begin{array}{rcl} \theta \in \Omega_\theta & \stackrel{\Delta}{=} & \{\theta : \ \theta_{min} \le \theta \le \theta_{max}\}, \\ \Delta_\eta \in \Omega_{\Delta_\eta} & \stackrel{\Delta}{=} & \{\Delta_\eta : \ |\Delta_\eta(x,\eta,u,t)| \le \delta_\eta(x)\}, \\ \Delta_i \in \Omega_{\Delta_i} & \stackrel{\Delta}{=} & \{\Delta_i : \ |\Delta_i(x,\eta,u,t)| \le \delta_i(\bar{x}_i)\}. \end{array} \tag{2}$$

Assumption 2: η is physically bounded with known bounds, i.e., $\eta \in \Omega_\eta$, where Ω_η is a known bounded convex set.

Remark 2: This assumption is different from the uniform detectability assumption made in [10], [5]. In [10], [5], the pair $(\varphi_{\eta l}^T, \ G_\eta)$ is assumed to satisfy the uniform detectability condition, i.e., there exists an $\omega(x) = [\omega_1(x), \cdots, \omega_m(x)]^T$, such that the unperturbed system $\dot{\varepsilon} = A(x)\varepsilon$ is exponentially stable, where $A(x) = G_\eta(x) - \frac{\partial \omega}{\partial x_l}\varphi_{\eta l}^T$. But for some practical systems, e.g., mechanical systems with dynamic friction, this condition can not be satisfied. In this paper, we will deal with the systems where $(\varphi_{\eta l}^T, \ G_\eta)$ may not be uniformly detectable, but the internal states are physically bounded by known bounds.

Assumption 3: There exists a positive definite matrix $\Gamma_\eta \in R^{m\times m}$ such that $\Gamma_\eta^{-1} G_\eta(x) + G_\eta^T(x)\Gamma_\eta^{-1} \le 0, \ \forall \ x \in R^l$.

Besides the above assumption on $G_\eta(x)$, we also make the following mild assumption on how the parametric uncertainties affect the dynamics of unmeasured internal states:

Assumption 4: Let $F_{\eta j}(x)$ be the j-th column of $F_\eta(x)$. Then dynamic systems $\dot{\zeta}_j = F_{\eta j}(x) + G_\eta(x)\zeta_j \ (1 \le j \le p)$ with the input x and state $[\zeta_1, \cdots, \zeta_p]$ are bounded-input-bounded-state stable in the sense that for every $x(t) \in L_\infty^l[0\ \infty)$, the solution $[\zeta_1, \cdots, \zeta_p]$ starting from any initial condition is bounded, i.e., $[\zeta_1(t), \cdots, \zeta_p(t)] \in L_\infty^{m\times p}[0\ \infty)$.

Let $y_d(t)$ be the desired motion trajectory, which is assumed to be known, bounded, with bounded derivatives up to l-th order. The objective is to synthesize a bounded control input u such that the output $y = x_1$ tracks $y_d(t)$ as closely as possible in spite of various model uncertainties and unmeasured states.

III. Discontinuous Projection Based ARC Backstepping Design

A. Parameter Projection

Let $\hat{\theta}$ denote the estimate of θ and $\tilde{\theta}$ the estimation error, i.e. $\tilde{\theta} = \hat{\theta} - \theta$. A discontinuous projection based ARC design will be constructed to solve the tracking control problem for (1). Specifically, under Assumption 1, the parameter estimate $\hat{\theta}$ is updated through a parameter adaptation law with the form

$$\dot{\hat{\theta}} = \mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta \tau_\theta) \tag{3}$$

where Γ_θ is a symmetric positive definite (s.p.d.) diagonal adaptation rate matrix, τ_θ is an adaptation function to be synthesized later. $\mathrm{Proj}_{\hat{\theta}} = [\mathrm{Proj}_{\hat{\theta}_1}(\bullet_1), \cdots, \mathrm{Proj}_{\hat{\theta}_p}(\bullet_p)]^T$ where each projection function is defined as

$$\mathrm{Proj}_{\hat{\theta}_i}(\bullet) = \begin{cases} 0 & \text{if } \hat{\theta}_i \ge \theta_{imax} \text{ and } \bullet > 0 \\ 0 & \text{if } \hat{\theta}_i \le \theta_{imin} \text{ and } \bullet < 0 \\ \bullet & \text{otherwise} \end{cases} \tag{4}$$

It can be shown that for any adaptation function τ_θ, the projection mapping guarantees

$$\begin{array}{ll} \text{P1} & \hat{\theta} \in \Omega_\theta = \{\hat{\theta} : \ \theta_{min} \le \theta \le \theta_{max}\} \\ \text{P2} & \tilde{\theta}^T(\Gamma_\theta^{-1}\mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta\tau_\theta) - \tau_\theta) \le 0 \end{array} \tag{5}$$

B. State Estimation

The estimation of unmeasured states η forms the core part of this paper. In [10], [5], using the detectability condition, the estimation error ε is proved to converge to zero exponentially, thus the effect of ε will 'diminish'. It is impossible, however, to make the estimation error converge to zero without the detectability condition when the pair $(\varphi_{\eta l}^T, \ G_\eta)$ is not assumed to be uniformly detectable. In view of this, we add an adaptation function to the state estimator and apply the projection algorithm. With this approach, although the estimation error may not converge to zero, the output tracking error will do in presence of parametric uncertainties only, as will be proved later in this paper. Furthermore, the boundedness of the estimation signals is guaranteed with the projection algorithm, which will also be used later to synthesize the robustifying feedback term to guarantee transient performance and final tracking accuracy.

Let $\zeta_j \in R^m$ $(0 \le j \le p)$ be the estimated variables, similar to those defined in [10], [5]. Let the estimation law be

$$\begin{aligned}\dot{\zeta}_0 &= \mathrm{Proj}_{\zeta_0}(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) \\ &\triangleq \begin{cases} \left(I - \Gamma_\eta \frac{n_{\zeta_0} n_{\zeta_0}^T}{n_{\zeta_0}^T \Gamma_\eta n_{\zeta_0}}\right)(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) \\ \quad \text{if } \zeta_0 \in \partial\Omega_{\zeta_0} \text{ and } n_{\zeta_0}^T (G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) > 0 \\ G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta, \quad \text{otherwise} \end{cases} \\ \dot{\zeta}_j &= G_\eta \zeta_j + F_{\eta j}, \quad 1 \le j \le p \end{aligned} \tag{6}$$

where $\Gamma_\eta \in R^{m\times m}$ is a positive definite matrix satisfying Assumption 3. τ_η is any function to be synthesized later. Ω_{ζ_0} denotes the time-varying convex set that ζ_0 lies in (sometimes we drop the notation 't' for simplicity), $\partial\Omega_{\zeta_0}$ is its boundary. n_{ζ_0} represents the outward unit normal vector at $\zeta_0 \in \partial\Omega_{\zeta_0}$. Ω_{ζ_0} is derived as follows

$$\Omega_{\zeta_0}(t) = \left\{ a + b : a \in \Omega_\eta,\ |b| \le \sup_{t>0}\Big[\sum_{j=1}^{p} \max(|\theta_{maxj}|, |\theta_{minj}|)|\zeta_j(t)|\Big] \right\}, \tag{7}$$

where $\sup\limits_{t>0}(\bullet)$ function denotes the supremum of all $\bullet(t)$ from beginning to the current time. Since Ω_η is convex, it can be easily checked that Ω_{ζ_0} is also convex.

Now put ζ_j, $1 \le j \le p$ into a matrix $\zeta = [\zeta_1 \ \cdots \ \zeta_p]$. We have $\dot{\zeta} = G_\eta \zeta + F_\eta$. Defining the estimation error to be $\varepsilon = \zeta_0 + \zeta\theta - \eta$, then we have the following lemma

Lemma 1: For any function τ_η,

i) If $\zeta_0(0) \in \Omega_{\zeta_0}(0)$, then $\zeta_0(t) \in \Omega_{\zeta_0}(t)$.

ii)

$$\varepsilon^T \Gamma_\eta^{-1}[\mathrm{Proj}_{\zeta_0}(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) - G_\eta \zeta_0 - H_\eta - \Gamma_\eta \tau_\eta] \le 0. \tag{8}$$

Proof: At any time, if ζ_0 touches the bound, i.e., $\zeta_0 \in \partial\Omega_{\zeta_0}$, then according to (6),

$$\begin{aligned} & n_{\zeta_0}^T \mathrm{Proj}_{\zeta_0}(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) \\ =& \begin{cases} n_{\zeta_0}^T \left(I - \Gamma_\eta \frac{n_{\zeta_0} n_{\zeta_0}^T}{n_{\zeta_0}^T \Gamma_\eta n_{\zeta_0}}\right)(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) \\ \quad \text{if } n_{\zeta_0}^T (G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) > 0 \\ n_{\zeta_0}^T (G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta), \quad \text{otherwise} \end{cases} \\ =& \begin{cases} 0, \text{ if } n_{\zeta_0}^T (G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) > 0 \\ n_{\zeta_0}^T (G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta), \quad \text{otherwise} \end{cases} \\ \le& \ 0 \end{aligned} \tag{9}$$

Thus, the derivative of ζ_0 always points inward or to the tangential direction of current Ω_{ζ_0} at the point ζ_0. From (7), $\Omega_{\zeta_0}(t)$ is monotonically expanding. So we conclude that $\zeta_0(t) \in \Omega_{\zeta_0}(t)$ if $\zeta_0(0) \in \Omega_{\zeta_0}(0)$.

For ii), we see that:

Case 1: If either $\zeta_0 \in \partial\Omega_{\zeta_0}$ or $n_{\zeta_0}^T(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) > 0$ is not true, then $\mathrm{Proj}_{\zeta_0}(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) = G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta$, ii) is obviously true.

Case 2: If $\zeta_0 \in \partial\Omega_{\zeta_0}$ and $n_{\zeta_0}^T(G_\eta \zeta_0 + H_\eta + \Gamma_\eta \tau_\eta) > 0$, then ζ_0 is on the boundary of Ω_{ζ_0}. From (7), $\eta - \zeta\theta \in \Omega_{\zeta_0}$, since Ω_{ζ_0} is convex, $n_{\zeta_0}^T \varepsilon = n_{\zeta_0}^T(\zeta_0 - (\eta - \zeta\theta)) \ge 0$. Then, a simple mathematical deduction leads to ii). ■

Lemma 1 is very important. Although ε may not converge to zero in our design, with the proposed state estimator, we also have Lemma 1 to help us. Later on in the proof of part B of Theorem 1, making use of Lemma 1, we will construct a Lyapunov function different to those used in [10], [5], and prove the asymptotic output tracking in a different way.

C. ARC Controller Design

1) Step $1 \le i \le l-1$: First, we denote $\alpha_0(t) = y_d(t)$. At step i $(1 \le i \le l-1)$, let z_i be the error between the state x_i and the desired control signal α_{i-1}, then $z_i = x_i - \alpha_{i-1}$. Take its derivative

$$\dot{z}_i = x_{i+1} + \theta^T \varphi_{\theta i} + h_i + \Delta_i - \dot{\alpha}_{i-1}, \tag{10}$$

Noting that $\dot{\alpha}_{i-1} = \sum\limits_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}(x_{j+1} + \theta^T\varphi_{\theta j} + h_j + \Delta_j) + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + \frac{\partial\alpha_{i-1}}{\partial t}$, we have $\dot{z}_i = x_{i+1} + \alpha_{ic} + \alpha_{iu}$, where

$$\begin{aligned} \alpha_{ic} &= \hat{\theta}^T(\varphi_{\theta i} - \textstyle\sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_{\theta j}) \\ &\quad + h_i - \textstyle\sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j} h_j - \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j} x_{j+1} - \frac{\partial\alpha_{i-1}}{\partial t} \\ \alpha_{iu} &= -\tilde{\theta}^T(\varphi_{\theta l} - \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_{\theta j}) - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} + \Delta_i - \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}\Delta_j \end{aligned} \tag{11}$$

are compensatible and uncompensatible parts respectively.

We construct a control function α_i for the virtual input x_{i+1} such that x_i tracks its desired control law α_{i-1} synthesized at step $i-1$.

$$\begin{aligned} &\alpha_i(\bar{x}_i, \hat{\theta}, t) = \alpha_{ia} + \alpha_{is},\ \alpha_{ia} = -z_{i-1} - \alpha_{ic},\ \alpha_{is} = \alpha_{is1} + \alpha_{is2}, \\ &\alpha_{is1} = -k_{is} z_i,\ k_{is} \ge g_i + |\tfrac{\partial\alpha_{i-1}}{\partial\hat{\theta}} C_{\theta i}|^2 + |C_{\phi i}\Gamma_\theta \phi_i|^2, \end{aligned} \tag{12}$$

where g_i is a positive constant, $C_{\theta i}$ and $C_{\phi i}$ are positive constant diagonal matrices. Let $z_{i+1} = x_{i+1} - \alpha_i$ denote the input discrepancy. With (12), we have

$$\dot{z}_i + k_{ls} z_i = z_{i+1} - z_{i-1} + \alpha_{is2} - \tilde{\theta}^T \phi_i + \tilde{\Delta}_i - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{13}$$

where $\phi_i = \varphi_{\theta i} - \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}\varphi_{\theta j}$ and $\tilde{\Delta}_i = \Delta_i - \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}\Delta_j$ (let $\tilde{\Delta}_1 = \Delta_1$, $\phi_1 = \varphi_{\theta 1}$). Choosing $V_i = V_{i-1} + \frac{1}{2}z_i^2$, then its time derivative is

$$\dot{V}_i = z_i z_{i+1} + \sum_{j=1}^{i}\left[-k_{js} z_j^2 + z_j(\alpha_{js2} - \tilde{\theta}^T\phi_j + \tilde{\Delta}_j) - \frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} z_j\right] \tag{14}$$

The ARC design can be applied to synthesize a robust control function α_{ls2} satisfying the following two conditions

$$\begin{aligned} i.&\quad z_i(\alpha_{is2} - \tilde{\theta}^T\phi_i + \tilde{\Delta}_i) \le \epsilon_i \\ ii.&\quad z_i \alpha_{is2} \le 0 \end{aligned} \tag{15}$$

where ϵ_i is a positive design parameter.

2) Step l: At the last step (step l), noting that the derivative of $z_l = x_l - \alpha_{l-1}$ is

$$\dot{z}_l = u + \theta^T(\varphi_{\theta l} + \zeta^T \varphi_{\eta l}) + \varphi_{\eta l}^T \zeta_0 + h_l - \dot{\alpha}_{l-1} + \Delta_l - \varphi_{\eta l}^T \varepsilon \tag{16}$$

Noting that $\dot{\alpha}_{l-1} = \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j}(x_{j+1} + \theta^T \varphi_{\theta j} + h_j + \Delta_j) + \frac{\partial \alpha_{l-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial \alpha_{l-1}}{\partial t}$, we have $\dot{z}_l = u + \alpha_{lc} + \alpha_{lu}$, where

$$\begin{array}{rcl} \alpha_{lc} & = & \hat{\theta}^T(\varphi_{\theta l} + \zeta^T \varphi_{\eta l} - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j}\varphi_{\theta j}) + \varphi_{\eta l}^T \zeta_0 \\ & & +h_l - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j} h_j - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j} x_{j+1} - \frac{\partial \alpha_{l-1}}{\partial t} \\ \alpha_{lu} & = & -\tilde{\theta}^T(\varphi_{\theta l} + \zeta^T \varphi_{\eta l} - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j}\varphi_{\theta j}) \\ & & -\frac{\partial \alpha_{l-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \varphi_{\eta l}^T \varepsilon + \Delta_l - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j}\Delta_j \end{array} \quad (17)$$

We construct the control input u such that x_l tracks its desired ARC control law α_{l-1} synthesized at step $l-1$.

$$\begin{array}{l} u(x, \zeta_0, \zeta, \hat{\theta}, t) = \alpha_{la} + \alpha_{ls}, \ \alpha_{la} = -z_{l-1} - \alpha_{lc}, \\ \alpha_{ls} = \alpha_{ls1} + \alpha_{ls2}, \quad \alpha_{ls1} = -k_{ls} z_l, \\ k_{ls} \geq g_l + |\frac{\partial \alpha_{l-1}}{\partial \hat{\theta}} C_{\theta l}|^2 + |C_{\phi l}\Gamma_\theta \phi_l|^2 + c_\theta |\psi_l|^2, \end{array} \quad (18)$$

where g_l and c_θ are positive constants, $\psi_l = \varphi_{\eta l}$, $C_{\theta l}$ and $C_{\phi l}$ are positive constant diagonal matrices to be specified later. Let $z_{l+1} = x_{l+1} - \alpha_l$ denote the input discrepancy. With (18), we have

$$\dot{z}_l + k_{ls} z_l = -z_{l-1} + \alpha_{ls2} - \tilde{\theta}^T \phi_l - \psi_l^T \varepsilon + \tilde{\Delta}_l - \frac{\partial \alpha_{l-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \quad (19)$$

where $\phi_l = \varphi_{\theta l} + \zeta^T \varphi_{\eta l} - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j}\varphi_{\theta j}$ and $\tilde{\Delta}_l = \Delta_l - \sum_{j=1}^{l-1} \frac{\partial \alpha_{l-1}}{\partial x_j}\Delta_j$. Choosing $V_l = V_{l-1} + \frac{1}{2}z_l^2$, then its time derivative is

$$\dot{V}_l = \sum_{j=1}^{l}\left[-k_{js} z_j^2 + z_j(\alpha_{js2} - \tilde{\theta}^T \phi_j - \psi_j^T \varepsilon + \tilde{\Delta}_j) - \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} z_j\right] \quad (20)$$

where $\psi_j^T = 0, \ \forall j < l$. The ARC design can be applied to synthesize a robust control function α_{ls2} satisfying the following two conditions

$$\begin{array}{ll} i. & z_l(\alpha_{ls2} - \tilde{\theta}^T \phi_l - \psi_l^T \varepsilon + \tilde{\Delta}_l) \leq \epsilon_l \\ ii. & z_l \alpha_{ls2} \leq 0 \end{array} \quad (21)$$

Remark 3: One smooth example of α_{ls2} satisfying (21) can be found in the following way. Let h_l be any n-th order continuous function satisfying

$$h_l \quad \geq \quad |\theta_M||\phi_l| + |\psi_l||\Omega_{\zeta_0}| + \tilde{\delta}_l \quad (22)$$

where $\theta_M \stackrel{\Delta}{=} \theta_{max} - \theta_{min}$ and $\tilde{\delta}_l \stackrel{\Delta}{=} \sum_{j=1}^{l-1} |\frac{\partial \alpha_{l-1}}{\partial x_j}|\delta_j + \delta_l$. $|\Omega_{\zeta_0}|$ is the length of the set Ω_{ζ_0}, i.e., the maximum distance between any two points in Ω_{ζ_0}. Then α_{ls2} can be chosen as

$$\alpha_{ls2} = -\frac{1}{4\epsilon_l} h_l^2 z_l \quad (23)$$

It is easy to verify that this choice of α_{ls2} satisfies (21).

D. Main Results

Theorem 1: Let the parameter estimates be updated by the adaptation law (3) in which τ_θ is chosen as $\tau_\theta = \sum_{j=1}^{l} \phi_j z_j$, and τ_η is chosen as $\tau_\eta = \sum_{j=1}^{l} z_j \psi_j$.

Let $c_{\theta ji}$ and $c_{\phi ki}$ be the i-th diagonal elements of the diagonal matrices $C_{\theta j}$ and $C_{\phi k}$ respectively. If the controller parameters $C_{\theta j}$ and $C_{\phi k}$ are chosen such that $c_{\phi ki}^2 \geq \frac{n}{4}\sum_{j=2}^{l} \frac{1}{c_{\theta ji}^2}$, $\forall k, i$, Then, the control law (18) guarantees that

A. In general, all signals are bounded. Furthermore, the positive definite function V_l is bounded above by

$$V_l(t) \leq e^{-\lambda_l t} V_l(0) + \frac{\sum_{j=1}^{l} \epsilon_j}{\lambda_l}(1 - e^{-\lambda_l t}) \quad (24)$$

where $\lambda_l = 2\min\{g_1, \cdots, g_l\}$.

B. If after a finite time t_0, there exist parametric uncertainties only (i.e., $\Delta_\eta = 0$ and $\Delta_i = 0, \forall t \geq t_0$), then, in addition to results in A, zero final output tracking error is also achieved, i.e, $z_1 \longrightarrow 0$ and as $t \longrightarrow \infty$.

Proof: For part A, from (12), (18) and (20), we have

$$\begin{array}{rcl} \dot{V}_l & \leq & \sum_{j=1}^{l}\Big\{(-g_j - |\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j}|^2 - |C_{\phi j}\Gamma_\theta \phi_j|^2 - c_\theta |\psi_j|^2) z_j^2 \\ & & +z_j(\alpha_{js2} - \tilde{\theta}^T \phi_j - \psi_j^T \varepsilon + \tilde{\Delta}_j) - z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}}\Big\} \end{array} \quad (25)$$

By completion of square

$$-\sum_{j=2}^{l} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \leq \sum_{j=2}^{l}(|\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j}|^2 z_j^2 + \frac{1}{4}|C_{\theta j}^{-1}\dot{\hat{\theta}}|^2) \quad (26)$$

Noting that $C_{\theta j}^{-1}$ and Γ_θ are diagonal matrices, from (3) and (4), we have

$$\begin{array}{l} \sum_{j=2}^{l} |C_{\theta j}^{-1}\dot{\hat{\theta}}|^2 \ = \ \sum_{j=2}^{l} |C_{\theta j}^{-1}\mathrm{Proj}_{\hat{\theta}}(\Gamma_\theta \tau)|^2 \ \leq \ \sum_{j=2}^{l} |C_{\theta j}^{-1}\Gamma_\theta \tau|^2 \\ \leq \ \sum_{j=2}^{l}(\sum_{k=1}^{l} |C_{\theta j}^{-1}\Gamma_\theta \phi_k z_k|)^2 \ \leq \ l\sum_{j=2}^{l}(\sum_{k=1}^{l} |C_{\theta j}^{-1}\Gamma_\theta \phi_k|^2 z_k^2) \end{array} \quad (27)$$

Thus, if $C_{\theta j}$ and $C_{\phi k}$ satisfy the conditions in the theorem, we have

$$-\sum_{j=2}^{l} z_j \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} \leq \sum_{j=2}^{l} |\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} C_{\theta j}|^2 z_j^2 + \sum_{k=1}^{l} |C_{\phi k}^{-1}\Gamma_\theta \phi_k|^2 z_k^2 \quad (28)$$

From (25) and the properties of each α_{js2}, we have

$$\dot{V}_l \leq \sum_{j=1}^{l}(-g_j z_j^2 + \epsilon_j) \leq -\lambda_l V_l + \sum_{j=1}^{l} \epsilon_j \quad (29)$$

which leads to (24). The boundedness of z_j is thus proved. Using the standard arguments in the backstepping designs [4], it can be proved that all internal signals in the first $l-1$ steps are globally uniformly bounded. Furthermore, since $x_l = z_l + \alpha_{l-1}$, x_l is also bounded. Thus, $x = [x_1, \cdots, x_l]^T$ are bounded. From the bounded-input-bounded-state Assumption 4, Lemma 1 and the bounded internal state Assumption 2, ζ_0, ζ and η are all bounded. Recursively using the fact that $x_i = z_i + \alpha_{i-1}$, it is obvious that α_i and x_i are bounded. Thus the boundedness of u is apparent. This proves the part A.

For part B, when $\Delta_\eta = 0$ and $\Delta_i = 0$, from (25) and (28), noting the condition ii of (21) and (15), we have

$$\dot{V}_l \leq \sum_{j=1}^{l}\left(-g_j z_j^2 - c_\theta |\psi_j|^2 z_j^2 - z_j \tilde{\theta}^T \phi_j - z_j \psi_j^T \varepsilon\right) \quad (30)$$

Define a new p.s.d function V_a as $V_a = V_l + \frac{1}{2}\tilde{\theta}^T \Gamma_\theta^{-1}\tilde{\theta} +$

$\frac{1}{2}\varepsilon^T\Gamma_\eta^{-1}\varepsilon$. Then after a series of derivations,

$$\begin{aligned}\dot{V}_a \leq & \sum_{j=1}^{l}\left(-g_j z_j^2 - c_\theta|\psi_j|^2 z_j^2\right) + \tilde{\theta}^T(\Gamma^{-1}\text{Proj}_{\hat{\theta}}(\Gamma\tau_\theta) - \tau_\theta) \\ & +\varepsilon^T\Gamma_\eta^{-1}[\text{Proj}_{\zeta_0}(G_\eta\zeta_0 + H_\eta + \Gamma_\eta\tau_\eta) \\ & -G_\eta\zeta_0 - H_\eta - \Gamma_\eta\tau_\eta] + \varepsilon^T\Gamma_\eta^{-1}G_\eta\varepsilon\end{aligned} \quad (31)$$

Since $\varepsilon^T\Gamma_\eta^{-1}G_\eta\varepsilon = \frac{1}{2}\varepsilon^T(\Gamma_\eta^{-1}G_\eta(x) + G_\eta^T(x)\Gamma_\eta^{-1})\varepsilon$, from Assumption 3, $\varepsilon^T\Gamma_\eta^{-1}G_\eta\varepsilon \leq 0$. Using (5) and (8), we have $\dot{V}_a \leq \sum_{j=1}^{l} -g_j z_j^2$, from which $z_j \in L_2[0, \infty)$. It is also easy to check that $\dot{z}_j$ is bounded. Hence, by the Barbalat's lemma, $z \to 0$ as $t \to 0$, which proves part B of Theorem 1. ■

IV. PRACTICAL DESIGN EXAMPLE AND SIMULATION RESULTS

A. Systems with Dynamic friction

Nowadays, the control of mechanical systems with dynamic friction has become increasingly popular. A kind of friction model called LuGre model [1] has seen wide application. Now we consider a linear motor driven stage with dynamic friction existing between the contact surfaces. With the LuGre model proposed in [1], [7],

$$\dot{z} = \dot{x} - \frac{|\dot{x}|}{g(\dot{x})}z + \Delta_z \quad (32)$$

$$g(\dot{x}) = \alpha_0 + \alpha_1 e^{-(\dot{x}/v_s)^2} \quad (33)$$

$$m\ddot{x} = Ku - \sigma_0 z - \sigma_1 h(\dot{x})\dot{z} - \alpha_2\dot{x} + \Delta_x \quad (34)$$

where m is the mass, u is the input voltage, K is the gain from voltage to the force, z represents the unmeasurable internal friction state, σ_0, σ_1, α_2 are unknown friction force parameters that can be physically explained as the stiffness, the damping coefficient of bristles, and viscous friction coefficient. x, $\dot{x}$ are the position and velocity respectively. $g(\dot{x})$ describes the Stribeck effect: $\sigma_0\alpha_0$ and $\sigma_0(\alpha_0+\alpha_1)$ represent the levels of the Coulomb friction and stiction force respectively, and v_s is the Stribeck velocity. Δ_z and Δ_x are the modeling errors and disturbances. Let $y_d(t)$ be the desired motion trajectory, which is assumed to be known, bounded, with bounded derivatives up to the second order. We want to design a control law u, such that the output x can track $y_d(t)$ as close as possible, in spite of various uncertainties.

Let us denote $\eta = [\frac{\sigma_0 z}{K} \ \ \frac{\sigma_1 z}{K}]^T$ to be the unmeasured internal states, $\theta = [\frac{\sigma_0}{K} \ \frac{\sigma_1}{K} \ \frac{\alpha_2}{K} \ \bar{\Delta}]^T$ and $\theta_m = \frac{m}{K}$ be the unknown parameters with known bounds, where $\bar{\Delta}$ is the constant portion of disturbances, and $\bar{\Delta} = (\sigma_1 h(\dot{x})\Delta_z + \Delta_x)/K - \tilde{\Delta}$. $\tilde{\Delta}$ is the time-varying portion of the disturbances. Denote $x = [x_1 \ x_2]^T = [x \ \dot{x}]^T$. Then the system can be represented by

$$\begin{aligned}\dot{\eta} &= F_\eta\theta + G_\eta\eta + \Delta_\eta, \\ \dot{x}_1 &= x_2 \\ \theta_m\dot{x}_2 &= u + \theta^T\varphi_\theta + \varphi_\eta^T\eta + \Delta_x \\ y &= x_1\end{aligned} \quad (35)$$

where

$$F_\eta = \begin{bmatrix} x_2 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \end{bmatrix}, \quad G_\eta = \begin{bmatrix} -\frac{|x_2|}{g(x_2)} & 0 \\ 0 & -\frac{|x_2|}{g(x_2)} \end{bmatrix}$$
$$\varphi_\theta = [0 \ -h(x_2)x_2 \ -x_2 \ 1]^T, \quad \varphi_\eta = [-1 \ \frac{h(x_2)|x_2|}{g(x_2)}]^T,$$
$$\Delta_\eta = [\sigma_0\Delta_z \ \sigma_1\Delta_z]^T, \quad \Delta_x = \tilde{\Delta} \quad (36)$$

This system is of the form (1), except that the unknown parameter θ_m appears in front of $\dot{x}_2$. In this case, we only need to make a small modification to the proposed algorithm, as shown later. Now, we will show that this system satisfies all the assumptions made in section II and the internal states are non-uniformly detectable.

Since the physical meanings of all unknown parameters in (35) are known, it is safe to assume that the unknown parameters, uncertain nonlinearities and disturbances are bounded by known bounds. Thus Assumption 1 is satisfied. It can also be seen that the pair (φ_η^T, G_η) is not uniformly detectable. Because at $x = [x_1 \ 0]^T$, we have $G_\eta = \mathbf{0}^{2\times2}$. Then for any $\omega(x) \in R^2$, the matrix $A(x) = G_\eta(x) - \frac{\partial\omega}{\partial x_2}\varphi_\eta^T = \begin{bmatrix} \frac{\partial\omega_1}{\partial x_2} & 0 \\ \frac{\partial\omega_2}{\partial x_2} & 0 \end{bmatrix}$ will always have zeros in the second column. Thus, techniques in [10], [5] do not apply here. However, since z represents the deflection of bristles between the contact surfaces, z is physically bounded [1]. Then $\eta = [\frac{\sigma_0 z}{K} \ \ \frac{\sigma_1 z}{K}]^T$ is also bounded. So Assumption 2 is satisfied.

Since $G_\eta(x)$ is a diagonal negative semi-definite matrix for all x, for any diagonal matrix $\Gamma_\eta > 0$, we have $\Gamma_\eta^{-1}G_\eta(x) + G_\eta^T(x)\Gamma_\eta^{-1} = 2\Gamma_\eta^{-1}G_\eta(x) \leq 0, \ \forall \ x \in R^2$. So Assumption 3 is satisfied. It can also be easily checked that Assumption 4 is satisfied. Since all assumptions required for the system are satisfied, we can use the technique proposed in this paper to design a control law.

B. Control Law Design

Letting $\alpha_0 = y_d(t)$, and $z_1 = x_1 - \alpha_0$ be the tracking error, then, $\dot{z}_1 = \dot{x}_1 - \dot{\alpha}_0 = x_2 - \dot{\alpha}_0$. Selecting $\alpha_1 = -k_1 z_1 + \dot{\alpha}_0$ to be the desired x_2, and defining $z_2 = x_2 - \alpha_1$, then

$$\theta_m\dot{z}_2 = u + \theta^T(\varphi_\theta + \zeta\varphi_\eta) + \varphi_\eta^T\zeta_0 - \varphi_\eta^T\varepsilon + \Delta_x + k_1\theta_m\dot{z}_1 - \theta_m\ddot{\alpha}_0 \quad (37)$$

Then the state estimator and the parameter adaptation laws are chosen as

$$\begin{aligned}&\dot{\zeta}_0 = \text{Proj}_{\zeta_0}(G_\eta\zeta_0 + \Gamma_\eta\varphi_\eta z_2), \ \ \dot{\zeta} = G_\eta\zeta + F_\eta \\ &\dot{\hat{\theta}} = \text{Proj}_{\hat{\theta}}(\Gamma_\theta(\varphi_\theta + \zeta^T\varphi_\eta)z_2), \ \ \dot{\hat{\theta}}_m = \text{Proj}_{\hat{\theta}_m}[\gamma_{\theta_m}(k_1\dot{z}_1 - \ddot{\alpha}_0)z_2].\end{aligned} \quad (38)$$

In this case, Ω_η is a square set, i.e., $\Omega_\eta = [\eta_{min1}, \eta_{max1}] \times [\eta_{min2}, \eta_{max2}]$. From the special structure of F_η and G_η, it is obvious that only ζ_{11} and ζ_{22} take effect. With this fact, $\Omega_{\zeta_0} = [\zeta_{0min1}, \zeta_{0max1}] \times [\zeta_{0min2}, \zeta_{0max2}]$, where

$$\begin{aligned}\zeta_{0maxi} &= \eta_{maxi} + \sup_{t>0}[\max(|\theta_{maxi}|, |\theta_{mini}|)|\zeta_{ii}(t)|] \\ \zeta_{0mini} &= \eta_{mini} - \sup_{t>0}[\max(|\theta_{maxi}|, |\theta_{mini}|)|\zeta_{ii}(t)|]\end{aligned} \quad (39)$$

for $i = 1, 2$. α_{2s1} is chosen as $\alpha_{2s1} = k_2 z_2$. For α_{2s2}, we use the form given by (23): $\alpha_{2s2} = -\frac{1}{4\epsilon}h^2 z_2$. h is chosen to be the right side of (22), which is a continuous function with respect to φ_θ, φ_η, ζ. The control law is thus given by

$$u = -k_2 z_2 + \alpha_{2s2} - \hat{\theta}^T(\varphi_\theta + \zeta^T \varphi_\eta) - \varphi_\eta^T \zeta_0 - \hat{\theta}_m(k_1 \dot{z}_1 + \ddot{\alpha}_0) \tag{40}$$

C. Simulation Results

For simulation, we choose the system parameters to be the same as that used in [6], i.e., $\theta_m = 0.12$, $\theta = [7000\ 1176\ 0.166\ 0]^T$, $g(x_2) = \frac{0.1236+0.0861e^{-|x_2/0.0022|}}{7000}$, $h(x_2) = \frac{0.00013}{0.00013+|x_2|}$. The internal state z is within ± 0.00005. In addition, we set the uncertain nonlinearity term $\Delta_x = 0.1\sin(\frac{2\pi}{0.02}x_1) + 0.1cos(\frac{2\pi}{0.02}x_1) + 0.02sin(\frac{4\pi}{0.02}x_1) + 0.02cos(\frac{4\pi}{0.02}x_1)$ to simulate the effect of cogging forces on the linear motor, and $\Delta_\eta = n(t)|x_2|$ to represent the modeling error of the internal state dynamics where $n(t)$ is a uniformly distributed random number between -0.2 and 0.2.

The controller parameters are chosen as $\hat{\theta}_m(0) = 0.1$, $\hat{\theta}(0) = [6000\ \ 1100\ \ 0.2\ \ 0]^T$, $\theta_{mmax} = 0.2$, $\theta_{mmin} = 0.08$, $\theta_{max} = [10000\ \ 1500\ \ 0.5\ \ 0.5]^T$, $\theta_{max} = [4000\ \ 500\ \ 0\ \ -0.5]^T$, $k_1 = 50$, $k_2 = 10$, $\epsilon = 0.008$, $\gamma_{\theta_m} = 5 \times 10^8$, $\Gamma_\theta = \text{diag}\{2.5 \times 10^{11}\ \ 2.5 \times 10^9\ \ 10\ \ 500\}$, $\Gamma_\eta = \text{diag}\{1500\ \ 200\}$, $\eta_{max} = [z_{max} \cdot \theta_{1max}\ \ \ z_{max} \cdot \theta_{2max}]^T = [0.5\ \ 0.075]^T$, $\eta_{min} = [z_{min} \cdot \theta_{1max}\ \ z_{min} \cdot \theta_{2max}]^T = [-0.5\ \ -0.075]^T$, $\zeta_0(0) = [0\ \ 0]^T$ and $\zeta(0) = \mathbf{0^{2\times 4}}$.

The design trajectory is chosen as a sinusoidal signal, with the amplitude of 0.002 and the frequency of $1Hz$. We set the initial value of the unknown internal state z to be 0.00003. The tracking error and control input with disturbances added to the system are plotted in Fig. 1. As can be seen from the plots, the tracking error converges very fast after the first few cycles, showing a good transient performance and final tracking accuracy. Furthermore, the input signal is bounded, and the tracking error is less than 0.05% magnitude of the desired trajectory in spite of large disturbances, showing the robustness and good capability of disturbance rejection of the proposed ARC algorithm.

Then, we remove all the disturbances and modeling errors, and use the same trajectory, same initial internal state value and same controller parameters. The tracking error and control input are plotted in Fig. 2. As can be seen, in the presence of parameters uncertainties only, asymptotic output tracking is achieved. These results demonstrate the applicability of the proposed method in practical design cases.

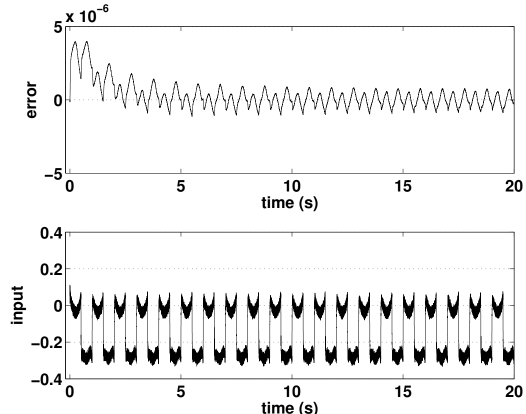

Fig. 1. Tracking error and control input, system with disturbances

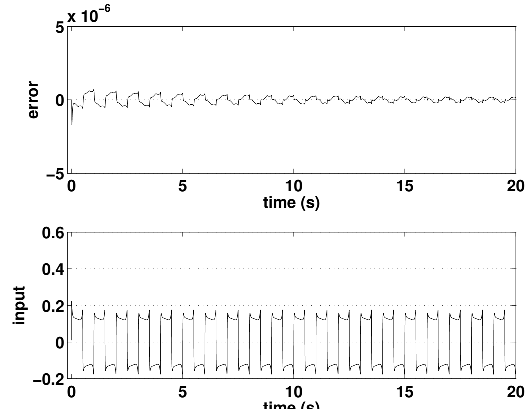

Fig. 2. Tracking error and control input, system without disturbances

V. CONCLUSION

In this paper, a discontinuous projection based adaptive robust control algorithm has been designed for a class of nonlinear systems in semi-strict feedback form with bounded but non-uniformly detectable internal states. Specifically, discontinuous projection algorithm has been used to give the estimation of both the internal states and the unknown parameters. This algorithm has been theoretically proved to be robust to disturbances and uncertain nonlinearities with guaranteed transient performance while having asymptotic output tracking performance in the presence of parametric uncertainties only. A practical example - control of mechanical systems with dynamic friction - is used for case study. The simulation results demonstrate the applicability of the proposed control methodology.

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