2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Design of Guaranteed Cost Overlapping Controllers for a Class of Uncertain State-Delay Systems

Josep M. Rossell and Francisco Palacios-Quiñonero

Abstract— **The paper deals with a class of linear continuous-time state-delay systems with norm-bounded uncertainties which are composed by overlapped subsystems. The main goal is to design overlapping guaranteed cost controllers for this class of systems by using the corresponding feasible solution of a linear matrix inequality (LMI) problem. In the overlapping decompositions context the selection of so-called complementary matrices is crucial. The paper presents a procedure to obtain numerical complementary matrices such that a bounded cost of the quadratic performance index is minimized. A simple example is supplied to illustrate the use of the proposed strategy.**

I. INTRODUCTION

Frequently, complex systems share components and can be treated as interconnected systems with overlapped subsystems. For these kind of systems a mathematical framework, the *Inclusion Principle*, has been developed [6], [7], [8], [9], [11]. The inclusion principle gives the conditions under which an initial system, sharing some components, can be expanded to a bigger dimensional space in such a manner that the overlapped subsystems appear now as disjoint. In this virtual new system, decentralized control laws can be designed to be contracted and implemented into the initial one in order to control it.

The initial and expanded system are related by linear transformations. These transformations involve a set of so-called complementary matrices. The influence of the choice of these matrices on properties like stability, controllability or observability has been illustrated in previous works [1], [2], [3], [4].

A generic goal is to design robust controllers which make the resulting closed-loop systems not only asymptotically stable but also guaranteeing an adequate level of performance. In this paper an LMI approach will be used to obtain the control laws. Working with overlapping decompositions we are interested in designing decentralized controllers such that the corresponding gain matrices have a tridiagonal block form, which offer maximal improvement in performance at a minimal cost in information exchange, [12]. The main motivation of this paper is to offer a computational strategy to obtain numerical complementary matrices which are used to design an overlapping controller with a minimum cost bound of the performance index.

This work was supported by the Committee for Science and Technology (CICYT) of Spain under Grant DPI2008-06699-C02-02

Josep M. Rossell and F. Palacios are with the Department of Applied Mathematics III, Universitat Politècnica de Catalunya (UPC), 08240-Manresa, Spain (josep.maria.rossell@upc.edu), (francisco.palacios@upc.edu)

II. BACKGROUND RESULTS

Consider two systems described by the state equations

$$\begin{aligned}\mathbf{S}: \ \dot{x}(t) &= [A+\Delta A(t)]\,x(t) + [B+\Delta B(t)]\,u(t)\\ &\quad + [C+\Delta C(t)]\,x(t-d),\\ x(t) &= \varphi(t), \quad -d \le t \le 0,\end{aligned} \tag{1}$$

$$\begin{aligned}\tilde{\mathbf{S}}: \ \dot{\tilde{x}}(t) &= \left[\tilde{A}+\Delta\tilde{A}(t)\right]\tilde{x}(t) + \left[\tilde{B}+\Delta\tilde{B}(t)\right]u(t)\\ &\quad + \left[\tilde{C}+\Delta\tilde{C}(t)\right]\tilde{x}(t-d),\\ \tilde{x}(t) &= \tilde{\varphi}(t), \quad -d \le t \le 0,\end{aligned} \tag{2}$$

where $x(t)\in\mathbb{R}^{\mathbf{n}}$ and $u(t)\in\mathbb{R}^{\mathbf{m}}$ are the state and the input of $\mathbf{S}$, $\tilde{x}(t)\in\mathbb{R}^{\tilde{\mathbf{n}}}$ and $u(t)\in\mathbb{R}^{\mathbf{m}}$ are the corresponding to $\tilde{\mathbf{S}}$. Let $\varphi(t)$ be a continuous vector valued initial function. Let $x_0{=}x(0)$, $\tilde{x}_0{=}\tilde{x}(0)$ be the initial states of the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$, respectively. Suppose that the dimension of the state vector $x(t)$ of $\mathbf{S}$ is smaller than the vector $\tilde{x}(t)$ of $\tilde{\mathbf{S}}$. The matrices A, B, C, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are constant of appropriate dimensions. $\Delta A(t)$, $\Delta B(t)$, $\Delta C(t)$, $\Delta\tilde{A}(t)$, $\Delta\tilde{B}(t)$ and $\Delta\tilde{C}(t)$ are real-valued matrices of uncertain parameters. Uncertainties are assumed to be norm-bounded as follows:

$$[\Delta A(t)\ \Delta B(t)\ \Delta C(t)] = E\,F(t)\,[E_1\ E_2\ E_3], \tag{3}$$

$$\left[\Delta\tilde{A}(t)\ \Delta\tilde{B}(t)\ \Delta\tilde{C}(t)\right] = \tilde{E}\,\tilde{F}(t)\,\left[\tilde{E}_1\ \tilde{E}_2\ \tilde{E}_3\right], \tag{4}$$

where E, E_1, E_2, E_3, $\tilde{E}$, $\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$ are known constant real matrices. $F(t)$, $\tilde{F}(t)$ are unknown matrix functions with Lebesgue measurable elements such that

$$F^T(t)F(t) \le I, \qquad \tilde{F}^T(t)\tilde{F}(t) \le I. \tag{5}$$

Associated with the systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ we have the following cost functions:

$$J(x_0,u) = \int_0^\infty \left[x^T(t)Q^*x(t) + u^T(t)R^*u(t)\right]\,dt, \tag{6}$$

$$\tilde{J}(\tilde{x}_0,u) = \int_0^\infty \left[\tilde{x}^T(t)\tilde{Q}^*\tilde{x}(t) + u^T(t)\tilde{R}^*u(t)\right]\,dt, \tag{7}$$

respectively. Q^*, $\tilde{Q}^*$ are symmetric positive semidefinite matrices and R^*, $\tilde{R}^*$ are symmetric positive definite matrices.

A. An LMI Approach

Theorem 1: Suppose that there exist constant parameters $\mu{>}0$, $\varepsilon{>}0$, symmetric positive-definite matrices X, S, $Z\in\mathbb{R}^{n\times n}$ and a matrix $Y\in\mathbb{R}^{m\times n}$ such that the following LMI

$$\begin{bmatrix} \Psi & (E_1X+E_2Y)^T & X & Y^T & CS & 0 & X\\ E_1X+E_2Y & -\mu I & 0 & 0 & 0 & 0 & 0\\ X & 0 & -(Q^*)^{-1} & 0 & 0 & 0 & 0\\ Y & 0 & 0 & -(R^*)^{-1} & 0 & 0 & 0\\ SC^T & 0 & 0 & 0 & -S & SE_3^T & 0\\ 0 & 0 & 0 & 0 & E_3S & -\varepsilon I & 0\\ X & 0 & 0 & 0 & 0 & 0 & -S \end{bmatrix} < 0 \tag{8}$$

978-1-4244-4524-0/09/$25.00 ©2009 AACC

is feasible, where $\Psi = AX+BY+(AX+BY)^T+Z+(\mu+\varepsilon)EE^T$. Then, the feedback control law $u(t)=Kx(t)=YX^{-1}x(t)$ is a quadratic guaranteed cost controller for the closed-loop uncertain time-delay system and satisfies

$$J(x_0,u) \leq J^* = \varphi^T(0)X^{-1}\varphi(0) + \int_{-d}^{0} \varphi^T(s)\left[S^{-1}+X^{-1}ZX^{-1}\right]\varphi(s)ds. \tag{9}$$

Proof: A similar proof can be seen in [10]. ■

B. Inclusion Principle

Consider the following transformations:

$$V : \mathbb{R}^{\mathbf{n}} \longrightarrow \mathbb{R}^{\tilde{\mathbf{n}}}, \qquad U : \mathbb{R}^{\tilde{\mathbf{n}}} \longrightarrow \mathbb{R}^{\mathbf{n}}, \tag{10}$$

where V and U are full-rank matrices such that $UV=I$.

Definition 1: (*Inclusion Principle*) A system $\tilde{\mathbf{S}}$ *includes* the system $\mathbf{S}$, denoted by $\tilde{\mathbf{S}} \supset \mathbf{S}$, if there exists a pair of matrices (U,V) satisfying $UV=I$ and such that for any initial state x_0 and any fixed input $u(t)$ of $\mathbf{S}$, the choice $\tilde{x}_0=Vx_0$ of the system $\tilde{\mathbf{S}}$ implies $x(t;x_0,u)=U\tilde{x}(t;Vx_0,u)$ for all t. If $\tilde{\mathbf{S}} \supset \mathbf{S}$, then $\tilde{\mathbf{S}}$ is said to be an *expansion* of $\mathbf{S}$ and $\mathbf{S}$ is a *contraction* of $\tilde{\mathbf{S}}$.

Definition 2: A control law $u(t)=\tilde{K}\tilde{x}(t)$ designed in the system $\tilde{\mathbf{S}}$ is *contractible* to $u(t)=Kx(t)$ of $\mathbf{S}$ if the choice $\tilde{\varphi}(t)=V\varphi(t)$ implies $Kx\big(t;\varphi(t),u(t)\big)=\tilde{K}\tilde{x}\big(t;V\varphi(t),u(t)\big)$ for all t, any initial function $\varphi(t)$ and any fixed input $u(t)$.

C. Complementary Matrices

Suppose that (U,V) is a given pair of matrices. Then, $\tilde{A}$, $\Delta\tilde{A}$, $\tilde{B}$, $\Delta\tilde{B}$, $\tilde{C}$, $\Delta\tilde{C}$, $\tilde{Q}^*$ and $\tilde{R}^*$ can be described as follows:

$$\begin{aligned} \tilde{A} &= VAU+M, & \Delta\tilde{A}(t) &= V\Delta A(t)U, \\ \tilde{B} &= VB+N, & \Delta\tilde{B}(t) &= V\Delta B(t), \\ \tilde{C} &= VCU+M_d, & \Delta\tilde{C}(t) &= V\Delta C(t)U, \\ \tilde{Q}^* &= U^TQ^*U+M_{Q^*}, & \tilde{R}^* &= R^*+N_{R^*}, \end{aligned} \tag{11}$$

where M, N, M_d, M_{Q^*} and N_{R^*} are the so-called *complementary matrices*. For $\tilde{\mathbf{S}}$ to be an expansion of $\mathbf{S}$, a proper selection of M and N is required [6], [7], [8], [9], [11].

In this paper, we assume that the structure of the matrices A, B and C given in (1) have the form

$$A,C = \left[\begin{array}{cc|c} *_{11} & *_{12} & *_{13} \\ \hline *_{21} & *_{22} & *_{23} \\ \hline *_{31} & *_{32} & *_{33} \end{array}\right], \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}, \tag{12}$$

where the submatrices $(*)_{ii}$ and B_{ij} for i=1,2,3, j=1,2 are $n_i \times n_i$ and $n_i \times m_j$ dimensional matrices, respectively. According to (12), a standard selection of the transformation matrix V is given by

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}. \tag{13}$$

Theorem 2: Consider the systems (1) and (2) satisfying (3), (4) and (5). Then $\tilde{\mathbf{S}} \supset \mathbf{S}$ if and only if

$$UM^iV = 0, \quad UM^{i-1}M_dV = 0, \quad UM^{i-1}N = 0 \tag{14}$$

for all $i=1,2,\ldots,\tilde{n}$.

Theorem 3: Consider the systems (1) and (2) satisfying (3), (4) and (5) with the structures given in (12) and (13). Suppose that M_d=0. Then, $\tilde{\mathbf{S}} \supset \mathbf{S}$ if and only if the complementary matrices M and N have the form

$$M = \begin{bmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22}+M_{23}+M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 \\ N_{21} & N_{22} \\ -N_{21} & -N_{22} \\ 0 & 0 \end{bmatrix} \tag{15}$$

and satisfy the conditions

$$\begin{aligned} &\begin{bmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{bmatrix} \left[M_{22}+M_{33}\right]^{i-1} \left[M_{21} \quad M_{22}+M_{23} \quad M_{24}\right] = 0, \\ &\begin{bmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{bmatrix} \left[M_{22}+M_{33}\right]^{i-1} \left[N_{21} \quad N_{22}\right] = 0 \end{aligned} \tag{16}$$

for all $i=1,2,\cdots,\tilde{n}-1$.

Remark 1: By using the transformation V given in (13), Theorem 3 provides the most general structure of the complementary matrices M and N such that $\tilde{\mathbf{S}} \supset \mathbf{S}$.

D. Overlapping Guaranteed Cost Controllers

The objective is to implement an overlapping guaranteed cost control, denoted by $u_D(t)=K_Dx(t)$, in the system (1) but as a contraction of a guaranteed cost control $u_D(t)=\tilde{K}_D\tilde{x}(t)$ designed for the system $\tilde{\mathbf{S}}$, [13]. The gain matrix $\tilde{K}_D$ in the expanded system has the following structure:

$$\tilde{K}_D = \left[\begin{array}{cc|cc} \tilde{K}_{11} & \tilde{K}_{12} & 0 & 0 \\ \hline 0 & 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right] \tag{17}$$

and the contracted gain matrix K_D corresponds to

$$K_D = \tilde{K}_DV = \left[\begin{array}{cc|c} \tilde{K}_{11} & \tilde{K}_{12} & 0 \\ \hline 0 & \tilde{K}_{23} & \tilde{K}_{24} \end{array}\right]. \tag{18}$$

Remark 2: It is possible to use an LMI approach to determine directly a gain matrix K_D with the structure given in (18) for the system $\mathbf{S}$. However, it is necessary to impose structural restrictions on the matrices Y and X in the form

$$Y = \begin{bmatrix} y_{11} & y_{12} & 0 \\ 0 & y_{22} & y_{23} \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{bmatrix} \tag{19}$$

and, consequently, the cost bound J^* can increase considerably.

Remark 3: In (12), if B_{21}=0 and B_{22}=0 the corresponding LMI may be infeasible and in this case the problem can be harder to solve.

III. Computational Procedure

Up to now, we know the structure and the conditions on the complementary matrices M and N given by Theorem 3, but it is necessary to select their numerical values. For this purpose, we consider two stages:

(a) The selection of the initial matrices M_0, N_0 such that $\tilde{\mathbf{S}} \supset \mathbf{S}$.

(b) The implementation of a Matlab-based iterative routine seeking for "optimal" complementary matrices M and N such that the cost bound $\tilde{J}^*$ is minimum.

To solve the stage (a), we can observe that in the literature the complementary matrices M and N are chosen in the following forms (*restrictions and aggregations*) [11]:

1) $M_{12} = \frac{1}{2}A_{12},\ M_{22} = \frac{1}{2}A_{22},\ M_{32} = -\frac{1}{2}A_{22},\ M_{42} = -\frac{1}{2}A_{32}.$ (20)

2) $M_{21} = A_{21}, \quad M_{22} = \frac{1}{2}A_{22}, \quad M_{23} = -\frac{1}{2}A_{22}, \quad M_{24} = -A_{23},$
$N_{21} = B_{21}, \quad N_{22} = -B_{22}.$ (21)

The full computational procedure corresponding to the previous stage (b) can be summarized as follows:

- **Consider** $(A, B, C, E, E_1, E_2, E_3, Q^*, R^*)$ which defines a system **S** together with a cost function $J(x_0, u)$.
- **Select** initial complementary matrices M_0 and N_0 with the structures given in (20) or (21).
- **Select** the matrices M_{R^*}, N_{R^*} to construct an expanded cost function $\tilde{J}(\tilde{x}_0, u)$.
- **Define a function** $\tilde{J}^*(M, N)$ to calculate the bounded cost from the matrices M and N.
- **Minimize** $\tilde{J}^*(M, N)$ starting from the initial matrices M_0, N_0.
  At the end of this process, the optimal complementary matrices M_{opt} and N_{opt} together with the minimum cost bound $\tilde{J}^*_{opt}$ for the expanded system $\tilde{\mathbf{S}}$ are obtained. For these complementary matrices the corresponding gain matrix $\tilde{K}_D$ is calculated.

IV. Example

Consider the system **S** given in (1) with an associated cost function (6) defined by the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E_1 = E_3 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{bmatrix},$$
$$Q^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \varphi(t) = \begin{bmatrix} 0.1 \\ t \\ 0.1 \end{bmatrix}, \quad d = 1. \tag{22}$$

By choosing the complementary matrices

$$M_{Q^*} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5 & 0 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N_{R^*} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{23}$$

we obtain $\tilde{Q}^*$=I_4 and $\tilde{R}^*$=I_2. In order to simplify the problem, the matrix M_d given in (11) is selected as M_d=0. However, any M_d matrix satisfying (15)-(16) can be chosen. In this example, we select the initial complementary matrices M_0 and N_0 in the form

$$M_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad N_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{24}$$

according to (21). For the complementary matrices M_0 and N_0 the initial cost bound is $\tilde{J}^*_0$=2.55. Following the proposed procedure, we obtain

$$M_{opt} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2216 & -1.2500 & 0.7500 & -0.8697 \\ -0.2216 & 1.2500 & -0.7500 & 0.8697 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{25}$$

$$N_{opt} = \begin{bmatrix} 0 & 0 \\ 0.7500 & -0.0263 \\ -0.7500 & 0.0263 \\ 0 & 0 \end{bmatrix}. \tag{26}$$

By using the optimal complementary matrices M_{opt} and N_{opt} given in (25) and (26), the minimum cost bound for the decentralized expanded system $\tilde{\mathbf{S}}$ results to be $\tilde{J}^*_{opt}$=1.41. We can observe that the difference between $\tilde{J}^*_0$ and $\tilde{J}^*_{opt}$ is very significative, almost a 45% of reduction. The corresponding contracted tridiagonal gain matrix is

$$K_D = \tilde{K}_D V = \begin{bmatrix} -9.2617 & -0.2827 & 0 \\ 0 & -2.2021 & -11.5887 \end{bmatrix}, \tag{27}$$

which can be implemented into the initial system **S** in order to control it. All computations have been performed using Matlab LMI Control Toolbox and Optimization Toolbox [5].

V. Conclusion

This paper has dealt with guaranteed cost control for a class of linear continuous-time state-delay uncertain systems which are decomposed into overlapped subsystems. A design strategy to obtain a tridiagonal guaranteed cost controller has been presented. A procedure for the numerical computation of complementary matrices such that a bounded cost is minimized has been given. A simple illustrative example has been offered.

References

[1] L. Bakule, J. Rodellar, and J.M. Rossell. Generalized selection of complementary matrices in the inclusion principle. *IEEE Transactions on Automatic Control*, 45(6):1237–1243, 2000.

[2] L. Bakule, J. Rodellar, and J.M. Rossell. Structure of expansion-contraction matrices in the inclusion principle for dynamic systems. *SIAM Journal on Matrix Analysis and Applications*, 21(4):1136–1155, 2000.

[3] L. Bakule, J. Rodellar, and J.M. Rossell. Controllability-observability of expanded composite systems. *Linear Algebra and its Applications*, 332-334:381–400, 2001.

[4] L. Bakule, J. Rodellar, J.M. Rossell, and P. Rubió. Preservation of controllability-observability in expanded systems. *IEEE Transactions on Automatic Control*, 46(7):1155–1162, 2001.

[5] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *The LMI Control Toolbox*. The MathWorks, Inc., 1995.

[6] M. Ikeda and D.D. Šiljak. Overlapping decompositions, expansions and contractions of dynamic systems. *Large Scale Systems*, 1(1):29–38, 1980.

[7] M. Ikeda and D.D. Šiljak. Overlapping decentralized control with input, state, and output inclusion. *Control-Theory and Advanced Technology*, 2(2):155–172, 1986.

[8] M. Ikeda, D.D. Šiljak, and D.E. White. Decentralized control with overlapping information sets. *Journal of Optimization Theory and Applications*, 34(2):279–310, 1981.

[9] M. Ikeda, D.D. Šiljak, and D.E. White. An inclusion principle for dynamic systems. *IEEE Transactions on Automatic Control*, 29(3):244–249, 1984.

[10] H. Mukaidani. An LMI approach to guaranteed cost control for uncertain delay systems. *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, 50(6):795–800, 2003.

[11] D.D. Šiljak. *Decentralized Control of Complex Systems*. Academic Press, New York, USA, 1991.

[12] D.D. Šiljak and A.I. Zečević. Control of large-scale systems: Beyond decentralized feedback. *Annual Reviews in Control*, 29:169–179, 2005.

[13] A.I. Zečević and D.D. Šiljak. A new approach to control design with overlapping information structure constraints. *Automatica*, 41:265–272, 2005.