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Stabilization of chemostats using feedback linearization and reduction of dimension.

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Abstract— **Stabilization via feedback linearization of models of competition between two species of microroganisms for two essential resources based in the chemostat is considered, extending previous work recently done by the authors; see [1], [14] and references therein. We show that though the full four-dimensional system is not stabilizable due to the dynamical properties of the system, it is possible to achieve the goal in modified form by pursuing a process of dimensional reduction prior to feedback linearization that results in replacing the four dimensional analysis with one in three dimensions. This technique has appeared in the literature applied to a similar system in the seminal papers of Hoo and Kantor [7]. However, in that case it appears that the authors thought of the method as a matter of convenience, and apparently did not realize that their original (higher dimensional) system was not stabilizable without utilizing the reduction procedure. (That is, that the reduction process was necessary in order to achieve the stabilization goal.) In this paper we show how the problem and its solution are very similar for both our model and that of Hoo and Kantor. This suggests that the dimensional reduction method could be rather generally applicable.**

I. Introduction

In this paper we apply the mathematical methods of differential geometric nonlinear control theory, especially techniques referred to as *feedback linearization* [10], [13], [18], to a model of two-species competition for two resources in a chemostat (also known as a continuously stirred tank reactor). As in our previous work, we focus on using these control techniques to stabilize the system, using simple proportional derivative controllers on the resulting linear system. Often, the goal is to stabilize to a state where both species coexist at equilibrium. This can be interpreted as a means of circumventing the *competitive exclusion principle*, which in this context states that at most one species can win the competition for resources, and which is regarded as a fundamental tenet of ecological dynamics. For this reason and others, the chemostat is an interesting system on which to apply geometric control techniques. See, *e.g.* [3], [12] and references therein, for other work investigating the use of control theory to achieve multi-species coexistence in chemostats.

With two resources available, it is important to consider how the resources, once consumed, are combined to promote growth. Such considerations have led to a spectrum of resource types [4], [11], [16], [17]. We confine our attention to perfectly complementary (a.k.a. essential) resources [11], [16]: those that fulfill different growth needs, and so must be taken together by the consumer. The dilution rate and the input concentration of one nutrient are taken as input controls.

Hoo and Kantor apply differential geometric control to a single input, single output (SISO) system with a single species and substrate [6], and then extend to a multiple input, multiple output (MIMO) system with two species competing for a single growth-limiting resource where the growth of one species is inhibited by the addition of an external agent [7]. (The latter work is also discussed in [13].) The dilution rate is used as a control input for the SISO case, while a combination of dilution rate and the introduction rate of the inhibitor are needed to achieve their control objectives in the MIMO case. Henson and Seborg [5] examine a model of the growth of a single species on one growth-limiting resource, and use the dilution and input resource concentration as controls. The control goal is to optimize a metabolic product. More recent work [1], [14] has focused on the effects of such biological details as resource type or control signal choices on the mathematical structure of the feedback-linearized system.

The main observation of this work is that the full four-dimensional system is not stabilizable due to the dynamical properties of the system. The reason is that the accessibility condition is violated asymptotically in time regardless of how the controls are chosen. We also show that the four-dimensional system of Hoo and Kantor [7] cannot be feedback linearized for similar reasons. However those authors successfully stabilized the system using feedback linearization by reducing the dimension of the system from four to three. We stress that feedback linearization of the four-dimensional system was not considered in Hoo and Kantor. By reducing our model (with two essential resources) to dimension three, we can also achieve stabilization, at least locally and for an open set of values in the parameters. We notice then that this state space dimension reduction is a necessity in the stabilization process and not a convenience as Hoo and Kantor's presentation may suggest.

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II. THE MODEL

The chemostat is designed to provide a controlled environment in which to study the growth and interaction of microorganisms under nutrient limitation [15]. It can be thought to consist of three vessels: a feed vessel, a culture vessel, and a collecting receptacle. The feed vessel contains adequate quantities of all required nutrients with the exception of those under investigation; these are assumed to be growth limiting. The contents of the feed vessel are supplied to the culture vessel at a rate D, while the medium in the continuously-stirred culture vessel is removed to the collecting receptacle at the same rate. Thus, constant volume is maintained in the culture vessel, and nutrients, microorganisms, and byproducts are removed in proportion to their concentrations. To simplify notation, we will assume that the flow rates have been scaled by the volume of the culture vessel.

We consider a model of competition between two species for two resources. As mentioned in the introduction, we confine ourselves to the case of perfectly complementary (essential) resources as defined by Léon and Tumpson [11] . These fulfill different needs for growth, and so must be taken together by the consumer. With $S(t)$ and $R(t)$ representing the concentrations of resources S and R in the culture vessel at time t, and $x_i(t)$ representing the biomass of the ith population of microorganisms in the culture vessel at time t, $i = 1$, 2, the dynamical system may be written

$$\begin{aligned}
\dot{S}(t) &= (S^0 - S(t))D - \sum_{i=1}^{2} \frac{x_i(t)}{Y_{S_i}} \mathcal{G}_i(S(t), R(t)), \\
\dot{R}(t) &= (R^0 - R(t))D - \sum_{i=1}^{2} \frac{x_i(t)}{Y_{R_i}} \mathcal{G}_i(S(t), R(t)), \\
\dot{x}_1(t) &= x_1(t)(-D + \mathcal{G}_1(S(t), R(t))), \qquad \text{(II.1)} \\
\dot{x}_2(t) &= x_2(t)(-D + \mathcal{G}_2(S(t), R(t))), \\
& S(0),\ R(0) \geq 0,\ x_1(0),\ x_2(0) > 0.
\end{aligned}$$

Here,

$$\mathcal{G}_i(S, R) = \min\{p_i(S), q_i(R)\} \qquad \text{(II.2)}$$

is the rate of conversion of nutrient to biomass of population i per unit of population i as a function of the concentrations of resources S and R in the culture vessel. The function $p_i(S)$ denotes the rate of conversion of nutrient S to biomass of population i per unit of population i when resource S alone is limiting. The function $q_i(R)$ is similarly defined. It is generally assumed that $p_i, q_i : \mathbf{R}_+ \to \mathbf{R}_+$ are C^1 with

$$\begin{aligned}
p_i(0) = 0, \quad p_i'(S) > 0,\ S > 0, \\
q_i(0) = 0, \quad q_i'(R) > 0,\ R > 0,
\end{aligned}$$

for $i = 1$, 2. We will also assume that

$$\lim_{S\to\infty} p_i(S) = m_{S_i} < \infty \text{ and } \lim_{R\to\infty} q_i(R) = m_{R_i} < \infty.$$

A prototype is the Michaelis-Menten functional response to a single resource, given by

$$p_i(S) = \frac{m_{S_i} S}{K_{S_i} + S} \text{ and } q_i(R) = \frac{m_{R_i} R}{K_{R_i} + R}. \qquad \text{(II.3)}$$

Details of the derivation of model (II.1) can be found in [1], where the formulation of [2], [8], [9] is followed.

III. FEEDBACK LINEARIZATION: BACKGROUND

For completeness we present a brief discussion of the generalities of feedback linearization. Feedback linearization implements a combination of a coordinate transformation and a nonlinear feedback control law to convert the system under study into a linear system in Brunovsky normal form [10], [13], [18]. We stress that this is an exact equivalence of systems, not an approximation of a nonlinear system by a linear system. The natural setting in which to apply geometric control is that of an affine control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{j=1}^{M} u_j \mathbf{g}_j(\mathbf{x}), \qquad \text{(III.4)}$$

where M is the number of inputs and the u_j are the controls. Denote by

$$\begin{aligned}
\Delta_0 &= \text{span}\{\mathbf{g}_1, \ldots, \mathbf{g}_M\} \\
\Delta_1 &= \text{span}\{\mathbf{g}_1, \ldots, \mathbf{g}_M, ad_{\mathbf{f}}\mathbf{g}_1, \ldots, ad_{\mathbf{f}}\mathbf{g}_M\} \\
&\cdots \\
\Delta_i &= \text{span}\{ad_{\mathbf{f}}^k \mathbf{g}_j : \quad 0 \leq k \leq i, \quad 1 \leq j \leq M\}, \\
& \quad i = 0, 1, \ldots, n-1.
\end{aligned}$$

Here, $ad_*(*)$ denotes the adjoint action of the Lie algebra of vector fields on itself, *i.e.*, $ad_{\mathbf{f}}\mathbf{g}$ is the Lie derivative of the vector field $\mathbf{g}$ with respect to the vector field $\mathbf{f}$.

A fundamental result that characterizes feedback linearizability for a given nonlinear system is the following.

Theorem 1: [10] Suppose the matrix $g(\mathbf{x}^0) = \big(\mathbf{g}_1(\mathbf{x}^0), \ldots, \mathbf{g}_M(\mathbf{x}^0)\big)$ has rank M. Then (III.4) is feedback linearizable if and only if

1) for each $0 \leq i \leq n-1$, the distribution Δ_i has constant dimension near $\mathbf{x}^0$;
2) the distribution Δ_{n-1} has dimension n;
3) for each $0 \leq i \leq n-2$, the distribution Δ_i is involutive.

In Section IV condition (2) is shown to fail asymptotically for the full four-dimensional system (II.1). (It is also shown to fail for the original four-dimensional system considered by Hoo and Kantor [7].) We then use the asymptotic behavior of system (II.1) to reduce the dimension, and construct the linearizing controllers using the idea of relative degree as described and used in [14] to successfully feedback linearize the resultant three-dimensional system.

IV. FEEDBACK LINEARIZATION: IMPLEMENTATION

System (II.1) can be written as an affine control system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{2} u_i \mathbf{g}_i(\mathbf{x}) \qquad \text{(IV.5)}$$

where $\mathbf{x} = (S, R, x_1, x_2)^\top$, the control vector fields are $\mathbf{g}_1(\mathbf{x}) = (-S, -(R - R^0), -x_1, -x_2)^\top$, $\mathbf{g}_2(\mathbf{x}) =$

$(1,0,0,0)^\top$, and the drift is given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -\sum_{i=1}^{2} \frac{x_i}{Y_{S_i}} \mathcal{G}_i \\ -\sum_{i=1}^{2} \frac{x_i}{Y_{R_i}} \mathcal{G}_i \\ x_1 \mathcal{G}_1 \\ x_2 \mathcal{G}_2 \end{pmatrix}.$$

Standard calculations provide the following result:

Lemma 2: System (II.1) is feedback linearizable if

$$\begin{aligned} & det(\mathbf{g}_1, \mathbf{g}_2, ad_{\mathbf{f}}\mathbf{g}_1, ad_{\mathbf{f}}\mathbf{g}_2) \\ & = (R - R^0)^2 x_1 x_2 (\frac{\partial \mathcal{G}_1}{\partial S}\frac{\partial \mathcal{G}_2}{\partial R} - \frac{\partial \mathcal{G}_1}{\partial R}\frac{\partial \mathcal{G}_2}{\partial S}) \\ & + (R - R^0) x_1 x_2 \Big[(\sum_{i=1}^{2} \frac{x_i}{Y_{R_i}} \frac{\partial \mathcal{G}_i}{\partial S})(\frac{\partial \mathcal{G}_2}{\partial R} - \frac{\partial \mathcal{G}_1}{\partial R}) \\ & + (\sum_{i=1}^{2} \frac{x_i}{Y_{R_i}} \frac{\partial \mathcal{G}_i}{\partial R})(\frac{\partial \mathcal{G}_1}{\partial S} - \frac{\partial \mathcal{G}_2}{\partial S}) \Big] \neq 0. \end{aligned} \tag{IV.6}$$

The curves $p_i(S) = q_i(R),\ i = 1, 2$ divide the SR-plane into three regions given by region 1: $p_i(S) > q_i(R), i = 1, 2$; region 2: $p_1(S) > q_1(R), p_2(S) < q_2(R)$; and region 3: $p_i(S) < q_i(R), i = 1, 2$. Note that region two can be unbounded or bounded, as depicted in figure 1, depending on the type of uptake functions and the parameters therein.

Lemma 3: In regions 1 and 3, system (II.1) is not feedback linearizable.

Proof: In region 1, $\min\{p_i(S), q_i(R)\} = q_i(R)$, so that $\frac{\partial \mathcal{G}_i}{\partial S} = 0$, for $i = 1, 2$. Similarly in region 3, $\min\{p_i(S), q_i(R)\} = p_i(S)$, so that $\frac{\partial \mathcal{G}_i}{\partial R} = 0$, for $i = 1, 2$. In both cases it follows from (IV.6) that

$$det(\mathbf{g}_1, \mathbf{g}_2, ad_{\mathbf{f}}\mathbf{g}_1, ad_{\mathbf{f}}\mathbf{g}_2) = 0$$

and thus the system is not feedback linearizable. ■

We now consider region 2. There the determinant in condition (IV.6) reduces to:

$$\begin{aligned} & det(\mathbf{g}_1, \mathbf{g}_2, ad_{\mathbf{f}}\mathbf{g}_1, ad_{\mathbf{f}}\mathbf{g}_2) \\ & = -(R - R^0) x_1 x_2 \frac{\partial \mathcal{G}_1}{\partial R}\frac{\partial \mathcal{G}_2}{\partial S} \left(R - R^0 + \frac{x_1}{Y_{R_1}} + \frac{x_2}{Y_{R_2}} \right). \end{aligned}$$

Feedback linearization holds in region 2 provided that R does not take on either value R^0 or $R^0 - \frac{x_1}{Y_{R_1}} - \frac{x_2}{Y_{R_2}}$.

Remark 1: The previous results motivate the following observation, which while easy to see, may be underappreciated. In particular, these results show that the controllers obtained via feedback linearization are, in general, not global, *c.f.*, the title of [7]. This fact is true independent of the dimensional reduction analysis that is the main topic of this paper. In fact, there can be reasons other than those seen above for the controllers obtained to fail to be global; namely, the coordinate transformation generated by the procedure may not be a global diffeomorphism. It is appropriate to mention this here because one of the implicit points we make is that the task of the implementation of controllers to achieve specific control goals using geometric methods is by no means complete when the standard feedback linearization procedure has been carried out. Moreover, it is interesting to note that the failure of the controller to be global as shown above is due to properties of the model that have real biological significance. Finally we note that since the system is not feedback linearizable in regions 1 and 3, some additional control apparatus would be needed to control the system to region 2 where feedback linearization can be implemented in order to achieve global results. This question will be considered in future work.

Having determined regions where feedback linearization can be done in both the unbounded and the bounded configuration, we examine conditions under which the system can be stabilized there.

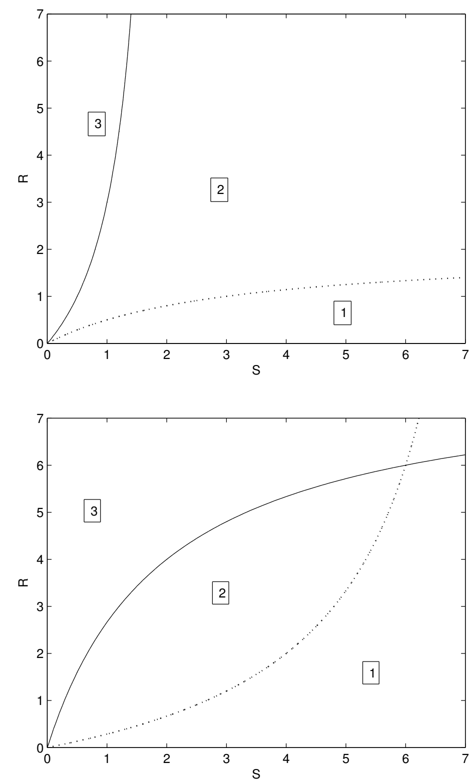

Fig. 1. Top: unbounded region configuration; Bottom: bounded region configuration

Lemma 4: In the case of essential resources, the system in dimension four cannot be stabilized using feedback linearization when the dilution rate D and the input nutrient concentration S^0 are taken as controls.

Lemma 5: For any solution $(S(t), R(t), x_1(t), x_2(t))$ of system (II.1), we have

$$\lim_{t\to\infty} \left(R(t) - R^0 + \frac{x_1(t)}{Y_{R_1}} + \frac{x_2(t)}{Y_{R_2}} \right) = 0.$$

Proof: Setting $\theta = R - R^0 + \frac{x_1}{Y_{R_1}} + \frac{x_2}{Y_{R_2}}$, it follows from system (II.1) that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -D\theta.$$

Since the dilution rate D is regarded as a control, we may assume it is a function of time and that it satisfies $\epsilon \leq D(t)$

for some $\epsilon > 0$. This simply amounts to assuming that the circulation pump is never turned off, nor run in reverse during operation of the chemostat. This seems a natural and not overly restictive assumption; indeed, allowing $D(t) \leq 0$ would be unrealistic. Furthermore, the nonegativity of the dilution rate is something that can be arranged by the operator. If the dilution rate and an input nutrient concentration are to be used as controls, it's clear that the ability to implement the controller requires that $D > 0$, and in fact the dynamical system (II.1) is not valid unless D is strictly positive.

Using the fact that $\dot{\theta} \leq -D(t)\theta$ and Gronwall's lemma we get

$$\theta(t) \leq \theta(0) \exp\big(- \int_0^t D(s) ds \big).$$

Since $\epsilon \leq D(t)$, it is then clear that

$$\theta(t) \leq \theta(0)\, e^{-\epsilon t},$$

hence

$$\lim_{t \to \infty} \theta(t) = 0.$$

Thus $R - R^0 + \frac{x_1}{Y_{R_1}} + \frac{x_2}{Y_{R_2}} = 0$ asymptotically, as stated. ■

We next provide a proof of lemma 4.

Proof: It follows from lemma 3 that system (II.1) is not feedback linearizable in either region 1 nor 3. On the other hand, since $R - R^0 + \frac{x_1}{Y_{R_1}} + \frac{x_2}{Y_{R_2}} = 0$ asymptotically, it is readily seen that

$$\det(\mathbf{g}_1, \mathbf{g}_2, ad_{\mathbf{f}}\mathbf{g}_1, ad_{\mathbf{f}}\mathbf{g}_2) = 0$$

asymptotically along any solution of (II.1). Thus, the accessibility condition must fail as t gets large. ■

The implication here is that feedback linearization-based controllers applied to system (II.1) will ultimately fail. It is important to note that this failure occurs despite the fact that the conditions for feedback linearization (*i.e.*, the accessibility conditions and the determination of functions with appropriate relative degree) are formally met. Indeed, our analysis has shown that under general assumptions, feedback linearization of the four-dimensional chemostat with essential resources must fail due to the dynamics of the system. A solution to this problem can be found in the early paper of Hoo and Kantor [7]. The system analyzed there is

$$\begin{aligned} \dot{x}_1(t) &= [\mu_1(S) - D]x_1, \\ \dot{x}_2(t) &= [\mu_2(S, I) - D]x_2, \\ \dot{I}(t) &= -px_1 I + D(I_f - I), \\ \dot{S}(t) &= -\mu_1(S)\frac{x_1}{Y_1} - \mu_2(S, I)\frac{x_2}{Y_2} + D(S_f - S). \end{aligned} \qquad \text{(IV.7)}$$

We next consider implementation of feedback linearization on (IV.7), and stress that the following analysis was not done in [7]. On one hand we obtain that

Lemma 6: System (IV.7) is feedback linearizable if

$$x_1 x_2 (S_f - S)\frac{\partial \mu_1}{\partial S}\frac{\partial \mu_2}{\partial I}\left(S_f - S - \frac{x_1}{Y_1} - \frac{x_2}{Y_2}\right) \neq 0. \qquad \text{(IV.8)}$$

On the other hand, similarly to the case in system (II.1), we have

Lemma 7: In system (IV.7),

$$\lim_{t \to \infty}\left(S(t) - S_f + \frac{x_1(t)}{Y_{R_1}} + \frac{x_2(t)}{Y_{R_2}}\right) = 0.$$

This fact was used in [7] to implement the dimensional reduction. However, the authors of that work did not state, and seemed not to realize, that this condition implies system (IV.7) fails to be feedback linearizable for dynamical reasons.

We illustrate the failure of feedback linearization on the system (IV.7) numerically in Figure 2. It is easy to see that system (IV.7) has a relative degree vector $(2, 2)$ for the functions $h_1 = \ln \frac{x_1}{x_2}$ and $h_2 = \ln \frac{(S - S_f)^2}{x_1 x_2}$, and we use these functions to generate the linearizing transformation. We show results for arbitrarily chosen values of the parameters and stabilization goals; the behavior shown in Figure 2 is insensitive to the choices. We note that the controller seems to work initially, but eventually (when the determinant of the accessibility matrix becomes sufficiently small), the signals begin to diverge. This is precisely the manner in which feedback linearization fails for system (II.1).

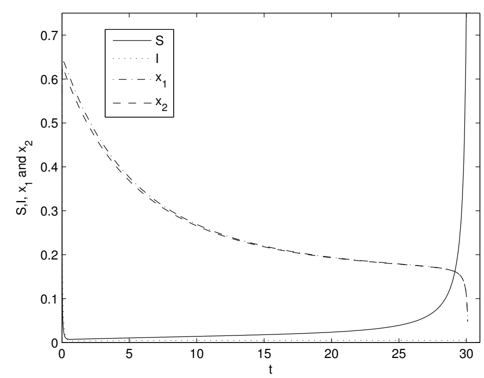

Fig. 2. The figure shows the results of attempting to stabilize Hoo and Kantor's four dimensional model.

Without noting this behavior or condition (IV.8), Hoo and Kantor reduced the dimension of their system (from four to three) using Lemma 7 and successfully stabilized the resultant three-dimensional system. It would appear that this computation was done for convenience rather than out of necessity. Nontheless, since Hoo and Kantor were successful in stabilizing the system using state space dimension reduction, and since the failure of the four-dimensional system with essential resources occurs for very similar reasons, this suggests that we might try to work with the three-dimensional system resulting from reducing the state space dimension of our system. That is, we are led to investigate the system

$$\begin{aligned} \dot{S}(t) &= (S^0 - S(t))D - \sum_{i=1}^{2} \frac{x_i(t)}{Y_{S_i}}\mathcal{G}_i(S(t), R(t)), \\ \dot{x}_1(t) &= x_1(t)(-D + \mathcal{G}_1(S(t), R(t))), \\ \dot{x}_2(t) &= x_2(t)(-D + \mathcal{G}_2(S(t), R(t))), \end{aligned} \qquad \text{(IV.9)}$$

$$S(0),\ R(0),\ x_1(0),\ x_2(0) \geq 0,$$

where $\mathcal{G}_i(S,R)$ is as in (II.2) and it is understood that

$$R = R^0 + \frac{x_1}{Y_{R_1}} + \frac{x_2}{Y_{R_2}}.$$

We begin with an accessibility analysis of (IV.9).

Lemma 8: System (IV.9) is accessible if $x_i \neq 0, i = 1,2$, and

$$\frac{\partial}{\partial S}\left(\mathcal{G}_1(S,R) - \mathcal{G}_2(S,R)\right) \neq 0.$$

Proof: The model can be written as an affine control system $\dot{\mathbf{x}} = \mathbf{f} + u_1\mathbf{g}_1 + u_2\mathbf{g}_2$ where the control inputs are chosen to be $u_1 = S^0D$ and $u_2 = D$, the control vector fields are $\mathbf{g}_1 = (1,0,0)^\top$ and $\mathbf{g}_2 = (-S,-x_1,-x_2)^\top$, and the drift is given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix}\mathbf{f}_1\\ \mathbf{f}_2\\ \mathbf{f}_3\end{pmatrix} = \begin{pmatrix} -\sum_{i=1}^2 \frac{x_i}{Y_{S_i}}\mathcal{G}_i(S,R)\\ x_1\mathcal{G}_1(S,R)\\ x_2\mathcal{G}_2(S,R)\end{pmatrix}.$$

The accessibility matrix, made up of column vectors $\mathbf{g}_1$, $\mathbf{g}_2$ and $ad_{\mathbf{f}}\mathbf{g}_1$, is

$$A = \begin{pmatrix} 1 & -S & -\sum_{i=1}^2 \frac{x_i}{Y_{S_i}}\frac{\partial}{\partial S}\mathcal{G}_i(S,R)\\ 0 & -x_1 & -x_1\frac{\partial}{\partial S}\mathcal{G}_1(S,R)\\ 0 & -x_2 & -x_2\frac{\partial}{\partial S}\mathcal{G}_2(S,R)\end{pmatrix}.$$

And the system is accessible if $detA \neq 0$, *i.e.*

$$x_1x_2\frac{\partial}{\partial S}\left(\mathcal{G}_1(S,R) - \mathcal{G}_2(S,R)\right) \neq 0.$$

■

We next implement feedback linearization. We seek a function φ that satisfies $d\varphi \neq 0$ and $d\varphi\cdot\mathbf{g}_1 = d\varphi\cdot\mathbf{g}_2 = 0$. A possible solution to the above equations is

$$\begin{aligned}\varphi(S,x_1,x_2) &= \varphi(x_1,x_2)\\ &= \ln(x_1/x_2)\\ &=: z_{1,1}.\end{aligned}$$

and the time derivative of φ is

$$\begin{aligned}\dot{z}_{1,1} &= L_{\mathbf{f}}z_{1,1}\\ &= \mathcal{G}_1(S,R) - \mathcal{G}_2(S,R)\\ &=: z_{1,2}.\end{aligned}$$

The problem is solvable if the system has a relative degree vector (r_1,r_2) satisfying $r_1 + r_2 = 3$, the dimension of the state space. Since the relative degree component with respect to φ is $r_1 = 2$, we must then have $r_2 = 1$ which means that the only requirement for $z_{2,1}$ is that it be independent of $z_{1,1}$ and $z_{1,2}$. For instance we can choose $z_{2,1} = x_1$. In the $(z_{1,1}, z_{1,2}, z_{2,1})$-coordinate system, our model may then be expressed in the form

$$\begin{aligned}\dot{z}_{1,1} &= z_{1,2},\\ \dot{z}_{1,2} &= L_{\mathbf{f}+u_1\mathbf{g}_1+u_2\mathbf{g}_2}z_{1,2} \equiv v_1,\\ \dot{z}_{2,1} &= L_{\mathbf{f}+u_1\mathbf{g}_1+u_2\mathbf{g}_2}z_{2,1} \equiv v_2,\end{aligned} \tag{IV.10}$$

where v_1 and v_2 are new control inputs. The system

$$\begin{aligned}L_{\mathbf{f}}z_{1,2} + u_1L_{\mathbf{g}_1}z_{1,2} + u_2L_{\mathbf{g}_2}z_{1,2} &= v_1,\\ L_{\mathbf{f}}z_{2,1} + u_1L_{\mathbf{g}_1}z_{2,1} + u_2L_{\mathbf{g}_2}z_{2,1} &= v_2,\end{aligned}$$

can be written as

$$A\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}v_1 - L_{\mathbf{f}}z_{1,2}\\ v_2 - L_{\mathbf{f}}z_{2,1}\end{pmatrix},$$

where A is the matrix

$$A = \begin{pmatrix}L_{\mathbf{g}_1}z_{1,2} & L_{\mathbf{g}_2}z_{1,2}\\ L_{\mathbf{g}_1}z_{2,1} & L_{\mathbf{g}_2}z_{2,1}\end{pmatrix}.$$

Solving for u_1 and u_2 gives

$$u_1 = \frac{1}{detA}\left(L_{\mathbf{g}_2}z_{2,1}(v_1 - L_{\mathbf{f}}z_{1,2}) - L_{\mathbf{g}_2}z_{1,2}(v_2 - L_{\mathbf{f}}z_{2,1})\right)$$

and

$$u_2 = \frac{1}{detA}\left(L_{\mathbf{g}_1}z_{1,2}(v_2 - L_{\mathbf{f}}z_{2,1}) - L_{\mathbf{g}_1}z_{2,1}(v_1 - L_{\mathbf{f}}z_{1,2})\right)$$

Since

$$\det A = -x_1\left(\frac{\partial G_1}{\partial S} - \frac{\partial G_2}{\partial S}\right),$$

the feedback linearization procedure will fail if

$$\left(\frac{\partial G_1}{\partial S} - \frac{\partial G_2}{\partial S}\right)|_{S=\hat{S}} = 0,$$

or

$$p_1'(\hat{S}) - p_2'(\hat{S}) = 0.$$

Note that this is the same condition encountered in the three-dimensional model of [14]. If we choose the uptake functions (II.3), there is a point $\hat{S}$ where the feedback linearization equations are singular and the singular locus is a good approximation to the basin boundary between the physical and unphysical equilibria. Stabilization of the reduced system (in dimension 3) is illustrated in Figure 3. The parameters used are: $m_{S_1} = m_{R_2} = 5$, $m_{S_2} = m_{R_1} = 6$, $k_{S_1} = k_{R_2} = 0.25$, $k_{S_2} = k_{R_1} = 0.5$. With these parameters, the singular value for S is $\hat{S} = 0.20521309....$ For the initial condition $S = 0.2053$, $x_1 = x_2 = 0.1$, the system stabilizes to a coexistence equilibrium $S = 0.05$, $x_1 = 0.5$ and $x_2 = 0.45$, as is shown in Figure 3 (Top). However, for the slightly different set of initial conditions $S = 0.2052$, $x_1 = x_2 = 0.1$ we notice that stabilization fails since the substrate concentration takes on unphysical values, as is shown in Figure 3 (Bottom). Thus, as in [14], we find that initial conditions on either side of the singular value are asymptotic to different equilibria, with trajectories asymptotic to the trivial equilibrium taking on negative values of some state variables.

V. Discussion

In this work we have considered stabilization via feedback linearization of two-species competition models based in the chemostat. Competition is for two resources, and the dilution rate and input concentration of one nutrient are taken as input controls. We confine our attention to essential resources (those that fulfill different requisite needs for growth, and so must be taken together by the consumer) [11], [16].

This case shares some similarities with the model provided in [7]. The main observation is that feedback linearization fails. The reason for that lies in the fact that due to the dynamics of the system, the accessibility condition is violated

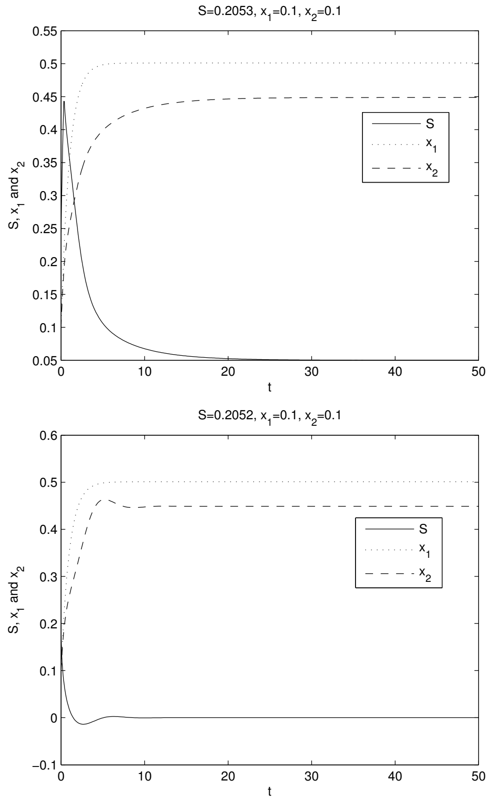

Fig. 3. Top: Stabilization is achieved for the initial conditions $S = 0.2053$, $x_1 = x_2 = 0.1$. The value of $S = 0.2053$ is slightly greater than the singular value $\hat{S} = 0.20521309....$ Resource S is stabilized at the value 0.05 as anticipated by the goal setting computations.
Bottom: Stabilization fails for the initial conditions $S = 0.2052$, $x_1 = x_2 = 0.1$. The value of $S = 0.2052$ is slightly less than the singular value $\hat{S} = 0.20521309....$ The system is not properly stabilized since resource S takes on negative (nonphysical) values.

asymptotically in time regardless of how the controls are chosen. It was also shown that the four-dimensional system of Hoo and Kantor cannot be feedback linearized for similar reasons. However those authors successfully stabilized the system using feedback linearization by reducing the dimension of the system from four to three as suggested by the vanishing of the quantity $S - S_f + \frac{x_1}{Y_1} + \frac{x_2}{Y_2}$. We stress that feedback linearization of the four-dimensional system was not considered in Hoo and Kantor. By reducing our model with two essential resources to dimension three, we can also achieve stabilization, at least locally and for an open set of values of the parameters. We notice then that this state space dimension reduction is a necessity in the stabilization process and not a convenience as Hoo and Kantor's presentation may suggest.

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