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Direct Frequency Response Function Based, Uncertainty Accomodating Optimal Controller Design

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Abstract— **Here we present a new approach to optimal controller design which bridges the gap between system data and its complementary optimal controller. Starting with empirical, open-loop frequency response function (FRF) data, it is shown that the optimal controller can be derived directly without performing system identification. The primary benefit is that we are able to work directly with the measured data and the uncertainties inherent in it. This approach is viewed as advantageous because it has the ability to capture features in the model that a structured uncertainty model could not. Further, we go on to show a method of incorporating the empirical FRF uncertainty into the cost for robustness against plant uncertainty. This method leads to a more precise calculation of H_2 and LQG controllers since it avoids the residual errors associated with performing the traditional intermediary step of system identification, while concurrently accounting for measured system uncertainty.**

I. INTRODUCTION

In recent years, computation of the optimal controller for a given state space model has been relegated to an afterthought. There is still an abundance of research in system identification, but given a perfect state space model, tools at our disposal such as Matlab make it quick and simple to calculate the optimal controller. Therefore, given the ease of model-based control design [4], the current approach may appear unwarranted or unnecessary. However, the major shortcoming of model-based controllers is that they are limited in performance and robustness by the errors present in the nominal plant model. Here, we sidestep this issue by working with empirical data instead of a parameterized system model. Thus the argument at present quintessentially concerns where the borders of optimal control theory should begin, whether it be at the time-domain stage, the frequency response function stage, or the parameterized system model stage. Here we advocate the frequency response function stage and proceed to develop an extension to optimal control theory to handle this perspective, as in [1].

Why are state-space models so ubiquitous in control theory? Is it their convenience? Their conciseness? Have past results led us to believe that they are innately amenable and predisposed toward controller design? Kalman's insights [2] have heretofore fueled nearly a half-century of control theory, but at the expense of its successes, have we lost insight into the physical system by focusing on the error-prone state-space model? After all, when we're doing diagnostics on a controller or a system, isn't the frequency response function (FRF) one of the first things we look at [4],[5]? In fact, it seems transparent on inspection that FRFs are the most fundamental description of a plant when we consider all of the diagnostics we have that revolve around FRFs: Bode plots, Nichols plots, and Nyquist plots. And how many people could look at a 50^{th} order state-space model and have any idea of the system's dynamics? Amidst all of this, it seems hard to imagine why we have relegated the FRF to a diagnostic tool when it appears to be the most intuitive, complete way to represent a linear system.

In the end, even a FRF is an approximation, but in our view, it is the most fundamental model we have to work with. It presents a clear picture of the uncertainties at each frequency, which would be difficult if not impossible to capture in a finite-dimensional state-space model, especially in the presence of the typical nonlinear disturbances that pervade measured data. Thus if we design a controller around a state-space model, we have not only uncertainty in the model but even uncertainty in the uncertainty model, which would seem preposterous to anyone first being introduced to optimal control theory. Therefore dealing directly with the FRF, instead of an intermediate dynamic model, we have a more accurate understanding of the system and its associated uncertainty.

In this paper, we develop a method of tuning an existing controller to more accurately fit the measured FRF, not just the ROM of the system.

II. H_2 CONTROLLER DERIVATION

A. H_2 Cost Setup

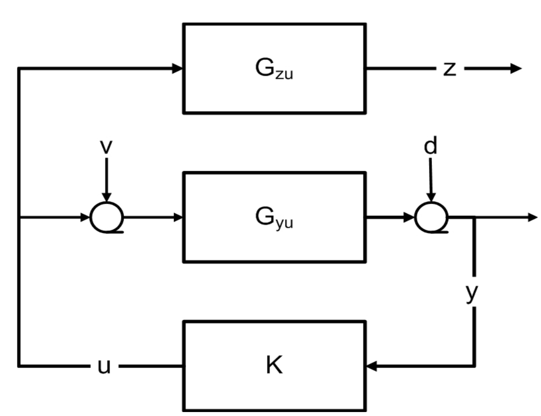

Fig. 1. Block diagram of the control architecture to be considered throughout the paper.

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Consider the the control architecture in Figure 1 with a m-input, p-output plant and consider the following cost, $J_p \in \mathbb{R}^+$, as a weighted combination of the measurement and performance cost:

$$J_p \triangleq \alpha_y J_{py} + \alpha_z J_{pz} \tag{1}$$

where $J_{py} \in \mathbb{R}^+$ is the H_2 norm of the closed-loop transfer function from the input to the measurement, and $J_{pz} \in \mathbb{R}^+$ is the H_2 norm of the closed-loop transfer function from the input to the performance variable. Also, let $G_{yu} \in \mathbb{C}^{p,m}$ represent the open-loop plant transfer function, $G_{zu} \in \mathbb{C}^{p,m}$ the transfer function to the performance variable, and $K \in \mathbb{C}^{m,p}$ the controller transfer function. Then

$$J_{py} \triangleq \frac{1}{\pi} \sum_{k=1}^{len(w)} tr\left[H_y(jw_k)H_y^H(jw_k)\right]\Delta w \tag{2}$$

$$J_{pz} \triangleq \frac{1}{\pi} \sum_{k=1}^{len(w)} tr\left[H_z(jw_k)H_z^H(jw_k)\right]\Delta w \tag{3}$$

$$H_y(jw_k) \triangleq S(jw_k)G_{yu}(jw_k) \triangleq S_k G_{yk} \triangleq H_{yk} \tag{4}$$

$$H_z(jw_k) \triangleq S(jw_k)G_{zu}(jw_k) \triangleq S_k G_{zk} \triangleq H_{zk} \tag{5}$$

$$S(jw_k) \triangleq [I - G_{yu}(jw_k)K(jw_k)]^{-1} \triangleq S_k, \tag{6}$$

where $H_y \in \mathbb{C}^{p,m}$, $H_z \in \mathbb{C}^{p,m}$, and $S \in \mathbb{C}^{p,p}$.

If the frequency response function (FRF) of the plant and performance variable are experimentally determined, then G_{yk} and G_{zk} are known for all relevant $w_k \in \mathbb{R}^+$. Although H_2 controllers minimize the transfer function between input and output, they do not deal with model uncertainty. Thus a stability (or robustness) cost is introduced which accounts for nominal FRF estimation uncertainty. Notice the inverse of the distance in the cost which penalizes the Nyquist curve's proximity to the critical point.

$$J_s \triangleq \frac{1}{\pi} \sum_{k=1}^{len(w)} W(w_k)\frac{1}{d^2(jw_k)}\Delta w \tag{7}$$

$$d(jw_k) \triangleq d_k = \det\left(I - G_{yk}K_k\right), \quad d_k^2 = \| d_k \|^2 \tag{8}$$

$$W(jw_k) \triangleq W_k \triangleq tr\left(\sigma_{nyq,k}\sigma_{nyq,k}^H\right), \tag{9}$$

where $W_k \in \mathbb{R}^+$ is a weighting function which accounts for uncertainty in the Nyquist Domain. Here we use the standard deviation as a metric to account for uncertainty in the closed-loop system, although a different value may be used to obtain a different confidence level in the stability of the closed-loop system and the margins thereby obtained. By uncertainty in the Nyquist domain, it is meant the uncertainty in d. Since this distance is also a function of the controller (8), it can not be measured directly, as in the case of the FRFs. Therefore we approximate the Nyquist uncertainty as:

$$\sigma_{nyq,k}(i,j) \approx \left(\frac{\partial d_k}{\partial G_{yk}(i,j)}\right)\sigma_{G_{yk}}(i,j) \tag{10}$$

where $\sigma_{G_{yk}}(i,j) \in \mathbb{R}$ represents the standard deviation of the FRF at every w_k for the channel between the j^{th} input and the i^{th} output with $\sigma_{nyq,k}(i,j) \in \mathbb{R}$.

$$\frac{\partial d_k}{\partial G_{yk}(i,j)} = d_k tr\left(-S_k\left[\frac{\partial G_{yk}}{\partial G_{yk}(i,j)}\right]K_k\right) \tag{11}$$

$$\frac{\partial d_k}{\partial G_{yk}} = -d_k K_k S_k \tag{12}$$

Here we are presented with a problem in the interpretation of $\sigma_{nyq,k}$ because above what was really meant was uncertainty with respect to each channel at each frequency. However, using the matrix definition of $\frac{\partial d_k}{\partial G_{yk}}$ and $\sigma_{G_{yk}}$, we gather unintended terms. Therefore, we appeal to the Hadamard product (.*):

$$\sigma_{nyq,k} \approx -d_k\left(S_k^T K_k^T\right) .* \sigma_{G_{yk}}, \tag{13}$$

with $\sigma_{nyq,k}, \sigma_{G_{yk}} \in \mathbb{R}^{p,m}$. Plugging back into the stability cost, (7)-(9):

$$\eta = \left(S_k^T K_k^T\right) .* \sigma_{G_{yk}} \tag{14}$$

$$W_k = d_k^2 tr\left(\eta\eta^H\right) \tag{15}$$

$$J_s = \frac{1}{\pi} \sum_{k=1}^{len(w)} tr\left(\eta\eta^H\right)\Delta w \tag{16}$$

Defining a total cost, similar to [1], we now have:

$$J = \alpha J_p + \beta J_s \tag{17}$$

$$\hat{\alpha}_y = \alpha\alpha_y, \quad \hat{\alpha}_z = \alpha\alpha_z \tag{18}$$

$$J = \hat{\alpha}_y J_{py} + \hat{\alpha}_z J_{pz} + \beta J_s \tag{19}$$

B. H_2 *Optimization*

Now that we have formulated the problem, we address the identification of the optimal controller. Notice that the primary difference between the current approach and typical approaches is that nowhere have we assumed a model for the plant. We work strictly with empirical FRF data and its inherent uncertainty. However, the shortcoming of this avenue of controller design is that traditional approaches cannot be used to determine the optimal controller (i.e. Riccati equation) since we are dealing solely with FRF data. Therefore we appeal to the natural solution of deriving gradients of the cost with respect to the controller and optimize using these. The gradients of the measurement, performance, and stability cost with respect to the controller, $\partial J_{py} \in \mathbb{R}^K$, $\partial J_{pz} \in \mathbb{R}^K$, and $\partial J_s \in \mathbb{R}^K$ (where ^{K} denotes the number of elements in the parameterization of the controller), are:

$$\partial J_p = \alpha_y \partial J_{py} + \alpha_z \partial J_{pz} \tag{20}$$

$$\partial J_p = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re\Bigg[tr\Bigg(S_k\{\alpha_y G_{yk}G_{yk}^H + \alpha_z G_{zk}G_{zk}^H\} S_k^H S_k G_{yk}\partial K_k\Bigg)\Bigg]\Delta w \tag{21}$$

$$\partial J_s = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re\Bigg[tr\Bigg(S_k\left[(K_k S_k)^H .* \sigma_{G_{yk}} .* \sigma_{G_{yk}}^H\right] \left[I + K_k S_k G_{yu}\right]\partial K_k\Bigg)\Bigg]\Delta w. \tag{22}$$

Where the total gradient can now be written as:

$$\Gamma =\{\hat{\alpha}_y S_k G_{yk} G_{yk}^H + \hat{\alpha}_z S_k G_{zk} G_{zk}^H\} S_k^H S_k G_{yk} + \beta S_k \left[(K_k S_k)^H .* \sigma_{G_{yk}} .* \sigma_{G_{yk}}^H \right] [I + K_k S_k G_{yu}] \tag{23}$$

$$\partial J = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re \left[tr \left(\Gamma \partial K_k \right) \right] \Delta w \tag{24}$$

with $\Gamma \in \mathbb{C}^{p,m}$ and $\partial J \in \mathbb{R}^K$.

III. LQG Controller Derivation

A. LQG Cost Setup

Usually, LQG controllers, and their associated cost are defined in the time-domain. Here we work with the frequency-domain definition, as presented in [3] with $\underline{Q} \in \mathbb{R}^{p,p}$ the measurement weighting, $R \in \mathbb{R}^{m,m}$ the control weighting, $\underline{V} \in \mathbb{R}^{m,m}$ the covariance of the input noise (noise on top of the input signal), and $W \in \mathbb{R}^{p,p}$ the covariance of the measurement noise:

$$J_p \triangleq \frac{1}{\pi} \sum_{k=1}^{len(w)} tr\big(S_k^H \{\underline{Q} + K_k^H R K_k\} S_k G_{yk} \underline{V} G_{yk}^H + W\{K_k^H G_{yk}^H S_k^H \underline{Q} S_k G_{yk} K_k + S_k^H K_k^H R K_k S_k\}\big) \Delta w. \tag{25}$$

B. LQG Optimization

With the frequency-domain definition of the LQG cost in place, we derive the gradients:

$$\Gamma_P =\{S_k G_{yk} \underline{V} + (S_k G_{yk} K_k + I) W K_k^H\} G_{yk}^H S_k^H \underline{Q} S_k G_{yk} + S_k\{G_{yk} \underline{V} G_{yk}^H + W\} S_k^H K_k^H R (I + K_k S_k G_{yk}) \tag{26}$$

$$\partial J_p = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re \left[tr \left(\Gamma_P \partial K_k \right) \right] \Delta w \tag{27}$$

where $\Gamma_P \in \mathbb{C}^{p,m}$ and $\partial J_p \in \mathbb{R}^K$. The only assumption is that the weighting and covariance matrices are symmetric.

Defining the performance variable as a linear combination of outputs, we can also incorporate a performance variable into the cost (as in the H_2 controller) by modification of the $\underline{Q}$ weighting matrix:

$$z \triangleq C_z y \tag{28}$$

$$\underline{\hat{Q}} \triangleq \hat{\alpha_y} \underline{Q} + \hat{\alpha_z} C_z^H Q_z C_z \tag{29}$$

where Q_z weights the performance variables. The modified cost and gradient incorporating the stability cost as defined previously in (16) and (22) are now:

$$\eta = \left(S_k^T K_k^T \right) .* \sigma_{G_{yk}} \tag{30}$$

$$J \triangleq \frac{1}{\pi} \sum_{k=1}^{len(w)} tr\big(S_k^H \{\underline{\hat{Q}} + K_k^H R K_k\} S_k G_{yk} \underline{V} G_{yk}^H + W\{K_k^H G_{yk}^H S_k^H \underline{\hat{Q}} S_k G_{yk} K_k + S_k^H K_k^H R K_k S_k\} + \beta \eta \eta^H \big) \Delta w \tag{31}$$

$$\Gamma =\{S_k G_{yk} \underline{V} + (S_k G_{yk} K_k + I) W K_k^H\} G_{yk}^H S_k^H \underline{\hat{Q}} S_k G_{yk} + S_k\{G_{yk} \underline{V} G_{yk}^H + W\} S_k^H K_k^H R (I + K_k S_k G_{yk}) + \beta S_k \left[(K_k S_k)^H .* \sigma_{G_{yk}} .* \sigma_{G_{yk}}^H \right] [I + K_k S_k G_{yu}] \tag{32}$$

$$\partial J = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re \left[tr \left(\Gamma \partial K_k \right) \right] \Delta w \tag{33}$$

IV. Parameterization

Up to now, we have not selected a parametrization for the controller. Here we choose the general state-space form:

$$K_k = C_c \left(j w_k I - A_c \right)^{-1} B_c + D_c, \tag{34}$$

where it is at the discretion of the designer whether or not to allow direct feed-through terms ($D_c \neq 0$).

Then defining the gradients for the costs presented previously with respect to the controller:

$$\phi_k = (j w_k I - A_c)^{-1} \tag{35}$$

$$\partial K_k = C_c \phi_k \partial A_c \phi_k B_c + C_c \phi_k \partial B_c + \partial C_c \phi_k B_c + \partial D_c \tag{36}$$

$$\frac{\partial J}{\partial A_c(i,j)} = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re \left[tr \left(\phi_k B_c \Gamma C_c \phi_k \frac{\partial A_c}{\partial A_c(i,j)} \right) \right] \Delta w \tag{37}$$

$$\frac{\partial J}{\partial B_c(i,j)} = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re \left[tr \left(\Gamma C_c \phi_k \frac{\partial B_c}{\partial B_c(i,j)} \right) \right] \Delta w \tag{38}$$

$$\frac{\partial J}{\partial C_c(i,j)} = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re \left[tr \left(\phi_k B_c \Gamma \frac{\partial C_c}{\partial C_c(i,j)} \right) \right] \Delta w \tag{39}$$

$$\frac{\partial J}{\partial D_c(i,j)} = \frac{2}{\pi} \sum_{k=1}^{len(w)} Re \left[tr \left(\Gamma \frac{\partial D_c}{\partial D_c(i,j)} \right) \right] \Delta w \tag{40}$$

where $\phi_k \in \mathbb{C}^{n,n}$, $[\partial J / \partial A_c(i,j)] \in \mathbb{R}$, $[\partial J / \partial B_c(i,j)] \in \mathbb{R}$, $[\partial J / \partial C_c(i,j)] \in \mathbb{R}$, $[\partial J / \partial D_c(i,j)] \in \mathbb{R}$, and Γ is given in either (23) or (32).

V. Examples

A. Example 1

We consider a randomly generated 20^{th} order unstable linear plant. First, an ideal 20^{th} order LQG controller is designed for the nominal plant. Then the FRF for that plant is perturbed by at most 10%, see Figure 2. We then use the nominal controller to initialize the optimization of (25) using the gradients as defined in (26)-(27), all with respect to the perturbed FRF (Figure 2). We use $\hat{\alpha}_y = 1$, $\hat{\alpha}_z = 0$, and $\beta = 0$ (the general LQG problem) to obtain a 20^{th} controller with a lower cost, $J = 465.7446$, than the cost evaluated using the nominal LQG controller, $J = 576.0398$. From Figure 3, we can see that the optimized controller suppressed low frequency peaks, especially along the first input, better than the LQG controller.

B. Example 2

For a second example, we incorporate the uncertainty robustness ($\beta \neq 0$ in (31)). Again we start with a completely arbitrary, randomly generated 20^{th} order stable linear plant. This time we do not perturb the baseline FRF, but instead incorporate a frequency by frequency uncertainty model. We then obtain a controller by optimizing (31) with $\hat{\alpha}_y = 1$, $\hat{\alpha}_z = 0$, and $\beta = 1$. Of course the cost is lower this time for the optimized controller, $J = 0.95094$, since the LQG controller, with a cost, $J = 1.8872$, does not have the same robustness considerations.In Figure 4, we see that the optimized controller has pushed the MIMO Nyquist plot away from the critical point, while maintaining comparable performance, see Figure 5. Note that the reason the closed loop FRFs appear high is that the system turned out to be very lightly damped (all of the poles had a real part between -1 and 0), yet the costs remained relatively low since the modes of the system were nearly DC and thus did not encompass a large bandwidth.

VI. CONCLUSIONS

A. Conclusions

We presented a new method for optimal controller derivation which allows one to forgo the system identification process and thus to work directly with empirical FRF data. We have shown that this approach is more optimal than traditional approaches since the residual errors accrued in the system identification process are not passed on to the final closed-loop system. Several variations were developed and demonstrated with two examples. Also, we have shown the effect of incorporating plant uncertainty. The primary advantage offered is that we deal directly with the empirical FRF and the uncertainty inherent in it.

B. Future Work

As of yet unexplored are parameterizations and optimization routines complementary to the current approach. Although several were incorporated into the current scheme, there is still research to be done in this arena. With research devoted to these areas, the authors believe the strengths of the proposed approach will lead to improved methods for optimal controller design and tuning.

VII. ACKNOWLEDGMENTS

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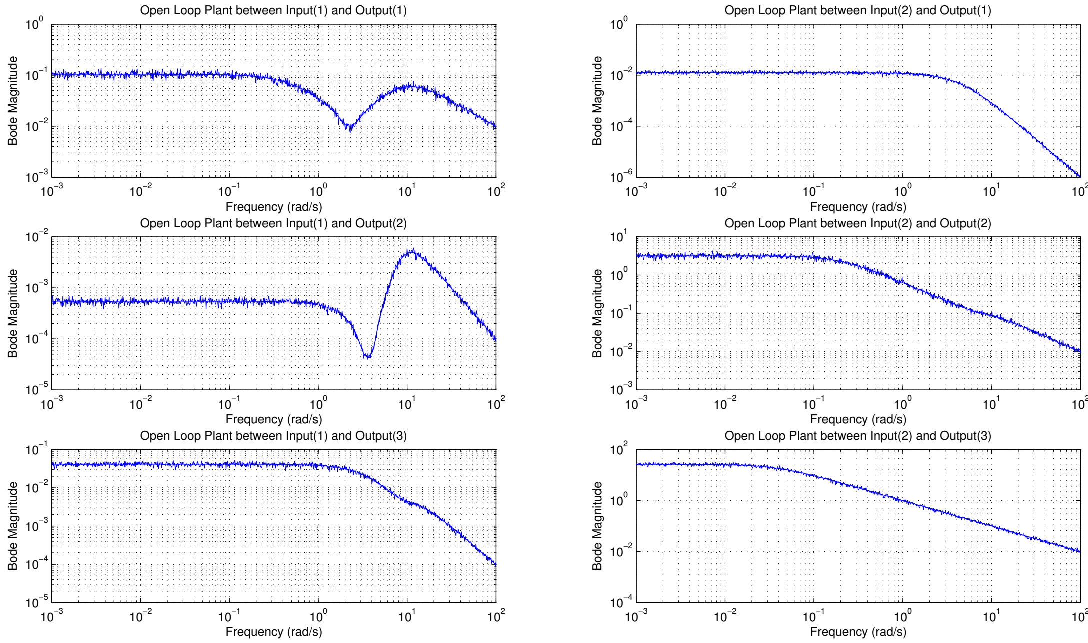

Fig. 2. Example 1. Open-Loop Bode magnitude plot of the plant overlayed with up to 10% noise.

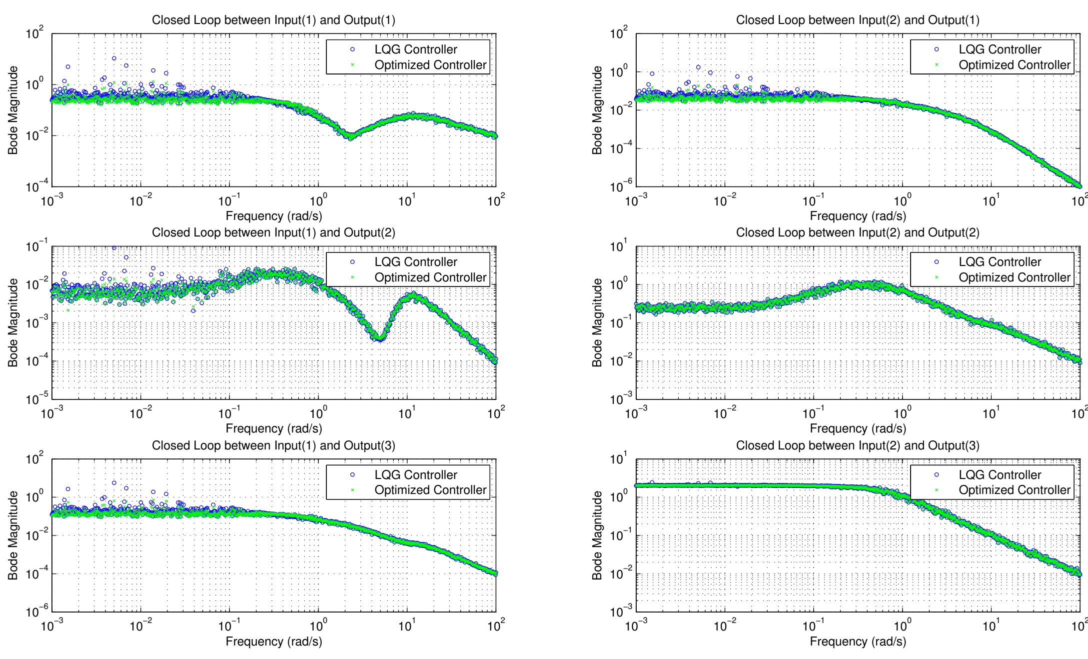

Fig. 3. Example 1. Closed-loop Bode magnitude plot associated with Figures 2.In it, we can see that the LQG controller has a few FRF points of high amplitude at low frequency, especially in the 1^{st} input channel. However, the optimized controller was able to suppress these points more effectively.

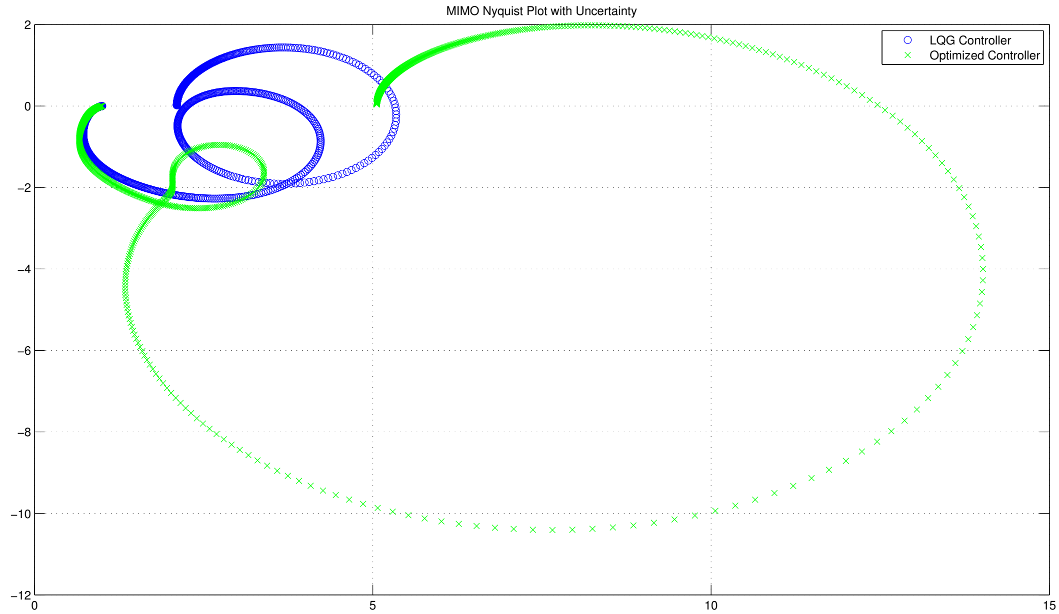

Fig. 4. Example 2. MIMO Nyquist Plot of the nominal system. It shows that incorporating the stability cost in the optimization pushes the curve away from the origin, thus giving us greater robustness against plant uncertainty.

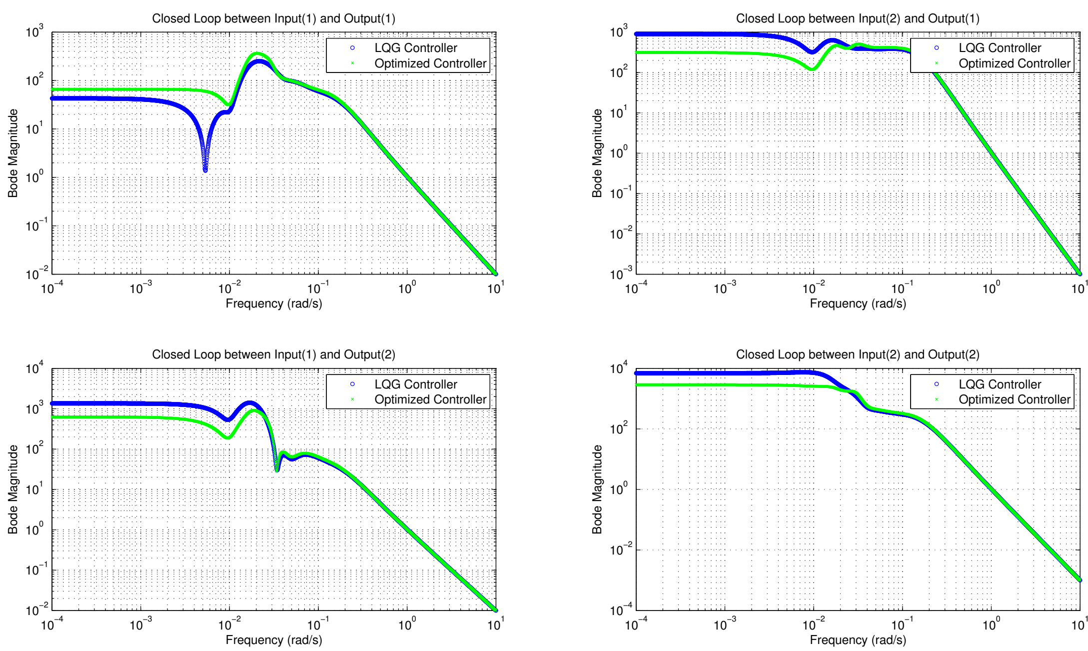

Fig. 5. Example 2. Closed-loop Bode magnitude plot associated with Figure 4. This figure shows that the optimized controller gives comparable performance while improving robustness to plant uncertainty