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System Reliability Estimation and Confidence Regions from Subsystem and Full System Tests

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Abstract—This paper develops a rigorous and practical method for estimating the reliability—with confidence regions—of a complex system based on a combination of full system and subsystem (and/or component or other) tests. It is assumed that the system is composed of multiple processes (e.g., the subsystems and/or components within subsystems), where the subsystems may be arranged in series, parallel (i.e., redundant), combination series/parallel, or other mode. Maximum likelihood estimation (MLE) is used to estimate the overall system reliability. Interestingly, for a given number of subsystems and/or components, the likelihood function does *not* change with the system configuration; rather, only the optimization constraints change, leading to an appropriate MLE. The MLE approach is well suited to providing asymptotic or finite-sample confidence bounds through the use of Fisher information or bootstrap Monte Carlo-based sampling.

Keywords: System identification, system reliability, parameter estimation, optimization, bootstrap, maximum likelihood, Fisher information matrix, data fusion.

1. Introduction

This paper considers the problem of estimating the reliability for a complex system based on a combination of information from tests on the subsystems, components, or other processes within the system, and, if available, tests on the full system. A key motivation for this setting comes from the fact that it is often difficult or infeasible to directly evaluate the reliability of complex systems through a large number of full system tests alone. Such a difficulty may arise, for example, when the full system is very costly or dangerous to operate and/or when each full system test requires the destruction of the system itself. Nevertheless, it is also often the case that there are at least a few tests of the full system available; it is obviously desirable to include such information in the overall reliability assessment. Such full systems tests are often critical to help guard against possible mismodeling of the relationship between the subsystems and full system in calculating overall reliability. This paper develops a method based on principles of maximum likelihood for estimating the overall system reliability from a *combination* of full system and subsystem or other tests.[1]

Certainly, other approaches exist for estimating system reliability when the subsystems are independent (see, e.g., Hwang et al., 1981; and Ramírez-Márquez and Jiang, 2006). However, these approaches do not allow for easy inclusion of limited full system tests when available, and do not generalize to include systems where the subsystems may be statistically dependent. (Note that the inequality-based reliability method of Hill and Spall, 2007, *does* allow for such dependence in producing a bound to the full system reliability, but this method requires certain pairwise subsystem tests that may not be feasible in practice.)

A key part of the approach here is the calculation of uncertainty (confidence) bounds on the estimates. We discuss the Fisher information matrix as a basis for asymptotic bounds and also discuss a bootstrap-based method for computing confidence regions when the asymptotic bounds are inappropriate. This bootstrap approach deals with the inadequacies of traditional methods based on the asymptotic normality of the MLE.

Several other approaches have been proposed to deal with the inadequacy of asymptotic normality in the context of using subsystem tests to estimate full system reliability. For example, Myhre and Saunders (1968) use the asymptotic chi-squared distribution of the log-likelihood ratio to deal with the problem of having confidence intervals outside the unit interval [0, 1]. Easterling (1972) treats the system reliability derived from subsystem estimates as an estimate from data having a binomial distribution. Then it is possible to use standard results on the exact distribution of the binomial estimate to get the confidence interval, yielding an exact solution in the special case where the system is composed of one subsystem (i.e., system = subsystem) and an intuitively appealing approximation when there are two or more

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[1]To avoid the cumbersome need to repeatedly refer to tests on subsystems, components, and other processes within the system as the key source of information other than full system tests, we will usually only refer to subsystem tests; "subsystem tests" in this context should be considered a proxy for all possible test information short of full system tests.

subsystems. Coit (1997) considers the case where there are a "large" number of subsystems in either a series or parallel configuration; in the series configuration, the logarithm of the system reliability is approximately normally distributed by central limit theorem effects. Hence, a log-normal distribution is assumed for the system reliability, providing the basis for the confidence interval. Ramírez-Márquez and Jiang (2006) focus on methods for estimating the variance of the reliability estimates, and then use these variance estimates together with normal or binomial approximations to the distribution of the estimates to form confidence intervals.

2. The Likelihood Function and MLE Formulation

2.1 General Formulation

Consider a system composed of p processes, typically subsystems and/or components of subsystems. The subsystems may be arranged in series, parallel (i.e., redundant), combination series/parallel, or other form (e.g., standby systems), subject to being able to write down a probabilistic characterization of the system that leads to a likelihood function (Leemis, 1995, Chap. 2, includes a thorough discussion of the many types of systems frequently encountered in practice). While we assume that the *test data* for estimating the reliability are statistically independent, we do not, in general, require that the processes be statistically independent *when operating as part of the full system*. (In principle, it is also possible to formulate a likelihood function based on test data that are statistically dependent. That is, the testing outcome for process j may be statistically dependent on the outcome for process i, for some $i \neq j$. We do not pursue that extension here.) In the discussion below, the term "operationally [in]dependent" is used to refer to the case where the processes are either statistically independent or dependent, as relevant, when operating as part of the full system.

The general MLE formulation involves a parameter vector $\boldsymbol{\theta}$, representing the parameters to be estimated, together with an associated log-likelihood criterion $\mathcal{L}(\boldsymbol{\theta})$. Let ρ and ρ_j represent the reliabilities (success probabilities) for the full system and for process j, respectively, $j = 1, 2, ..., p$. The vector $\boldsymbol{\theta} = [\rho_1, \rho_2, \ldots, \rho_p]^T$; ρ is not included in $\boldsymbol{\theta}$ because it will be uniquely determined, or at least bounded, from the ρ_j and possibly other information via relevant constraints. When ρ is uniquely determined by a function of $\boldsymbol{\theta}$, then the estimate $\hat{\rho}$, as determined from applying this function to the MLE of $\boldsymbol{\theta}$ (say $\hat{\boldsymbol{\theta}}$), is the MLE of ρ. This invariance of MLE applies even though the mapping from $\boldsymbol{\theta}$ to ρ is not generally one-to-one and may not be continuous (see, e.g., Zehna, 1966).

Ultimately, we are interested in an estimate and confidence region for ρ, as derived from the MLE for $\boldsymbol{\theta}$. The specific definition of $\boldsymbol{\theta}$ will depend on the details of the system. Interestingly, for a given definition of $\boldsymbol{\theta}$, the definition of $\mathcal{L}(\boldsymbol{\theta})$ will *not* depend on how the subsystems are arranged in the full system. That is, $\mathcal{L}(\boldsymbol{\theta})$ is the same regardless of whether, say, the subsystems are in series or parallel. However, the MLE *will* change as a function of the system arrangement. This is a consequence of the constraints in the optimization problem that is solved to produce the MLE, as illustrated below.

It is generally assumed, at a minimum, that success/failure data are available on the p processes within the full system. As mentioned above, it is also generally assumed that success/failure data are available directly on the full system. In cases involving dependent subsystems, it may be desirable that the information from the p processes include some data other than direct subsystem success/failure data in order to obtain the information needed for characterizing the nature of the dependence (we say "desirable" because it may be possible to estimate *bounds* to the system reliability in the absence of such information). For example, in the dependent-subsystem case discussed in Section 3, obtaining data on one critical component appearing within multiple subsystems allows for an MLE of ρ; the absence of such data allows the analyst to estimate a *lower bound* to ρ.

We now present the general MLE optimization problem. Let Θ represent the feasible region for the elements of $\boldsymbol{\theta}$. To ensure that relevant logarithms are defined and that the appropriate derivatives exist, it is assumed, at a minimum, that the feasible region Θ includes the restriction that $0 < \rho_j < 1$ for all j (other restrictions may be included as appropriate). The general MLE formulation is:

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta} \in \Theta} \mathcal{L}(\boldsymbol{\theta}) \quad \text{subject to } f(\boldsymbol{\theta}, \rho) \geq 0, \tag{2.1}$$

where $f(\cdot)$ is some function reflecting the constraints associated with the operation of the full system. In some common cases (e.g., fully series and fully parallel cases, with the processes corresponding to the subsystems as in Subsection 2.2), the inequality in the constraint can be replaced with an equality.

Let X represent the number of successes in n independent, identically distributed (i.i.d.) experiments with the full system and X_j represent the number of successes in n_j i.i.d. experiments with process j, $j = 1, 2, ..., p$. Note that in the discussion below there is no notational distinction between a random variable (vector) and its realization, with the expectation that the distinction should be clear from the context. Let $\boldsymbol{Y}$ represent the full set of data $\{X, X_1, X_2, \ldots, X_p\}$. From the assumption of independence of all test data, the probability mass function, say $p(\boldsymbol{Y} \mid \boldsymbol{\theta}, \rho)$, is:

$$p(\boldsymbol{Y} \mid \boldsymbol{\theta}, \rho) = \underbrace{\binom{n}{X} \rho^{X} (1-\rho)^{(n-X)}}_{\text{system}} \times \underbrace{\binom{n_1}{X_1} \rho_1^{X_1} (1-\rho_1)^{(n_1-X_1)} \cdots \binom{n_p}{X_p} \rho_p^{X_p} (1-\rho_p)^{(n_p-X_p)}}_{p \text{ processes}}, \tag{2.2}$$

leading to the log-likelihood function $\mathcal{L}(\boldsymbol{\theta}) \equiv \log p(\boldsymbol{Y} \mid \boldsymbol{\theta}, \rho)$:

$$\mathcal{L}(\boldsymbol{\theta}) = X \log \rho + (n - X) \log(1-\rho) + \sum_{j=1}^{p} \left[X_j \log \rho_j + (n_j - X_j) \log(1-\rho_j) \right] + \text{constant}, \tag{2.3}$$

where the constant is not dependent on $\boldsymbol{\theta}$. Note that the generic forms for the above likelihood and log-likelihood apply regardless of the specific layout of the subsystems (series, parallel, combination series/parallel, or operationally dependent). However, the relationship between ρ and the ρ_j differs according to the layout of the processes. In finding the MLE of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}$, this relationship manifests itself as constraints in an optimization problem.

Typically, the MLE is determined via finding a root of the score equation $\partial \mathcal{L}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} = \mathbf{0}$, or a normalized form of this equation, in the asymptotic sample size case, where the score vector is

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \partial \mathcal{L} / \partial \rho_1 \\ \partial \mathcal{L} / \partial \rho_2 \\ \vdots \\ \partial \mathcal{L} / \partial \rho_p \end{bmatrix}. \tag{2.4}$$

The elements of the score vector depend on the constraints. A common special case is where the constraints lead to a unique differentiable function $h(\cdot)$ that relates $\boldsymbol{\theta}$ to ρ: $\rho = h(\boldsymbol{\theta})$. In that case, (2.3) leads to the following form of the score vector in (2.4):

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = \frac{X}{\rho} \frac{\partial h}{\partial \boldsymbol{\theta}} - \frac{n - X}{1 - \rho} \frac{\partial h}{\partial \boldsymbol{\theta}} + \begin{bmatrix} \dfrac{X_1}{\rho_1} - \dfrac{n_1 - X_1}{1 - \rho_1} \\ \vdots \\ \dfrac{X_p}{\rho_p} - \dfrac{n_p - X_p}{1 - \rho_p} \end{bmatrix}. \tag{2.5}$$

We will see several illustrations of the form in (2.5) in the examples and theoretical results below.

2.2 Fully Series and Fully Parallel Cases

To illustrate the general formulation of Subsection 2.1, we consider here the two most common "extreme" cases of fully series systems and fully parallel systems with the p processes corresponding to p operationally independent subsystems. The results below can extend readily to common cases of mixed (operationally independent) series/parallel systems (see, e.g., Leemis, 1995, Sect. 2.1, for a description of such general systems). The results here illustrate the setting mentioned in the context of (2.5) above, where there exists a differentiable function $\rho = h(\boldsymbol{\theta})$.

From (2.1), the MLE in the series-subsystem case is found according to

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta} \in \Theta} \mathcal{L}(\boldsymbol{\theta}) \quad \text{subject to } \rho = \prod_{j=1}^{p} \rho_j,$$

while the MLE in the parallel-subsystem case is found according to

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta} \in \Theta} \mathcal{L}(\boldsymbol{\theta}) \quad \text{subject to } 1 - \rho = \prod_{j=1}^{p} (1 - \rho_j).$$

In the above series and parallel cases, it is straightforward to determine the score vector using (2.5).

Making the substitution $\rho = \prod_{j=1}^{p} \rho_j$ in eqn. (2.5), the $j = 1, 2, \ldots, p$ elements of the score vector in (2.4) (or (2.5)) for the series case are:

$$\frac{\partial \mathcal{L}}{\partial \rho_j} = \frac{X + X_j}{\rho_j} - \frac{(n - X)\rho}{(1-\rho)\rho_j} - \frac{(n_j - X_j)}{(1-\rho_j)}. \tag{2.6}$$

Likewise, making the substitution $1 - \rho = \prod_{j=1}^{p} (1 - \rho_j)$ in eqn. (2.5), the elements of the score vector in (2.4) (or (2.5)) for the parallel case are:

$$\frac{\partial \mathcal{L}}{\partial \rho_j} = \frac{X_j}{\rho_j} + \frac{X}{\rho} \prod_{\substack{i=1 \\ i \neq j}}^{p} (1 - \rho_i) - \frac{(n + n_j - X - X_j)}{(1 - \rho_j)}. \tag{2.7}$$

Except for the degenerate settings of $X = n$ in the series case and $X = 0$ in the parallel case, the solution to $\partial \mathcal{L}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta} = \mathbf{0}$ must generally be found by numerical search methods. (The two degenerate cases yield $\hat{\rho}_j = (n + X_j)/(n + n_j)$ and $\hat{\rho}_j = X_j/(n + n_j)$, respectively, both of which are the natural—intuitively obvious—solutions.)

3. DEPENDENT SUBSYSTEMS

There are obviously innumerable ways in which subsystems can interact while operating as part of a full system. The reliability analysis for each such system must be handled separately based on the information available. While the reliability analysis with operationally dependent

subsystems is usually more difficult than with operationally independent subsystems, the MLE may still be available if the problem can be cast in the form of (2.1) with an appropriate constraint. For example, in systems that may be represented as a series of $m \le p$ generally dependent subsystems, the following expression relates the full system reliability to conditional subsystem reliabilities:

$$\rho = P(S_1 = 1)P(S_2 = 1 \mid S_1 = 1)P\left(S_3 = 1 \middle| \{S_1 = 1\} \cap \{S_2 = 1\}\right) \cdots P\left(S_m = 1 \middle| \bigcap_{j=1}^{m-1} \{S_j = 1\}\right), \quad (3.1)$$

where $S_j = 0$ or 1 is the indicator of whether subsystem j is, respectively, a failure or success. Hence, the analyst may have the information needed to implement (2.1) with an equality constraint that uniquely defines ρ if data and/or prior information are available to characterize the conditional probabilities on the right-hand side of (3.1). Further information on the relatively common setting where dependence gets introduced through shared components is available from the author (not included here due to ACC 2009 space constraints).

4. Theoretical Properties

This section summarizes the convergence properties associated with the MLE formulation above. Note that standard i.i.d. MLE theory (e.g., Serfling, 1980, Sect. 4.2) does not apply because of the different success/failure probabilities associated with the different subsystems. Nevertheless, the structure associated with (2.1) and (2.3) allows us to show that the MLE for ρ will converge to the true full system reliability under reasonable conditions.

First, however, we present a result giving conditions under which there is a unique function $h(\boldsymbol{\theta})$ relating θ to ρ.

Lemma 1. Suppose that the constraint in problem statement (2.1) can be represented as an equality $f(\boldsymbol{\theta}, \rho) = 0$ with f being a continuously differentiable function in both $\boldsymbol{\theta} \in \Theta$ and $0 < \rho < 1$. For a fixed $\boldsymbol{\theta}' \in \Theta$, suppose $\partial f(\boldsymbol{\theta}', \rho)/\partial \rho \neq 0$ almost surely (a.s.) at ρ such that $f(\boldsymbol{\theta}', \rho) = 0$. Then there exists an open neighborhood of $\boldsymbol{\theta}'$ and a unique continuously differentiable function h such that for all $\boldsymbol{\theta}$ in this neighborhood, $\rho = h(\boldsymbol{\theta})$ and

$$\frac{\partial h(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -\frac{\partial f(\boldsymbol{\theta}, \rho)}{\partial \boldsymbol{\theta}} \left(\frac{\partial f(\boldsymbol{\theta}, \rho)}{\partial \rho} \right)^{-1} \quad (4.1)$$

Proof. The result is a consequence of the implicit function theorem (e.g., Apostol, 1974, Sect. 13.4). *Q.E.D.*

We now present the following strong (a.s.) convergence result for the MLE of ρ. Let ρ^* be the true value of the full system reliability and $\boldsymbol{\theta}^* = [\rho_1^*, \rho_2^*, \ldots, \rho_p^*]^T$ be the corresponding true value of $\boldsymbol{\theta}$. Further, let $\boldsymbol{A}_N = \text{diag}\left[\text{var}(\partial \mathcal{L}/\partial \rho_1), \text{var}(\partial \mathcal{L}/\partial \rho_2), \ldots, \text{var}(\partial \mathcal{L}/\partial \rho_p) \right]$ be a diagonal matrix used to normalize for the variability of the elements in $\partial \mathcal{L}/\partial \boldsymbol{\theta}$, where $N = \{n, n_1, n_2, \ldots, n_p\}$ is the collective sample size. Proposition 1 below establishes conditions for $\hat{\rho} = \rho^* + o(1)$ being a root of the normalized score equation, $\boldsymbol{A}_N^{-1} \partial \mathcal{L}/\partial \boldsymbol{\theta} = \boldsymbol{0}$, where $o(1)$ is a term going to zero as N gets large in the sense described below. As usual when working with score vectors, however, the Proposition does not guarantee that this solution is unique and/or a global maximum of $\mathcal{L}(\boldsymbol{\theta})$.

Proposition 1. For constants $0 < C^- \le C^+ < 1$, suppose the feasible region Θ is such that $\Theta = \{\boldsymbol{\theta}: C^- \le \rho_j \le C^+ \text{ for all } j\}$ and that $C^- \le \rho \le C^+$. Further, suppose that the constraint in problem statement (2.1) can be represented as an equality $f(\boldsymbol{\theta}, \rho) = 0$ with f being a continuously differentiable function in both $\boldsymbol{\theta} \in \Theta$ and $C^- \le \rho \le C^+$. For some strictly positive constants C and C' and all $\boldsymbol{\theta} \in \Theta$ and $C^- \le \rho \le C^+$, suppose $C < \left|\partial f(\boldsymbol{\theta}, \rho)/\partial \rho\right| \le C'$ and $C \le \left|\partial f(\boldsymbol{\theta}, \rho)/\partial \rho_j\right| \le C'$ for all j. Then, for the problem described in (2.1) and (2.3), $\hat{\rho} = \rho^* + o(1)$ is an a.s. solution to $\boldsymbol{A}_N^{-1} \partial \mathcal{L}/\partial \boldsymbol{\theta} = \boldsymbol{0}$ as $n + n_1 \to \infty$, $n + n_2 \to \infty$, ..., $n + n_p \to \infty$, where, for each j, one of the following three possibilities holds: (i) $n_j/n = o(1)$, (ii) $n/n_j = o(1)$, or (iii) $n_j/n = O(1)$ and $n/n_j = O(1)$.

Comment. The multiple limits $n + n_1 \to \infty$, $n + n_2 \to \infty$, ..., $n + n_p \to \infty$ are true if and only if one of the following three (mutually exclusive) possibilities occur: (a) $n \to \infty$ and $n_j < \infty$ for all j, (b) $n < \infty$ and $n_j \to \infty$ for all j, or (c) $n \to \infty$ and $n_j \to \infty$ for at least one j. The proof considers these three cases in turn subject to the additional constraints (i) – (iii) in the Proposition statement.

Proof. The conditions of the Proposition are stronger than those of Lemma 1. Hence, there exists a differentiable function h such that for all $\boldsymbol{\theta}$ in an open neighborhood of $\hat{\boldsymbol{\theta}}$, $\rho = h(\boldsymbol{\theta})$ with derivative given by (4.1). Thus, the score vector is given by (2.5). The remainder of the proof is available upon request (not included here due to space constraints).

5. The Fisher Information Matrix

The Fisher information matrix is helpful in at least two respects in the reliability estimation problem: (i) It can be used to determine when the estimation problem in Section 2 is well posed (i.e., when $\boldsymbol{\theta}$ is identifiable) through an evaluation of the conditions ensuring that the information matrix is positive definite (e.g., Goodwin and Payne, 1977, pp. 104 and 139) and (ii) the inverse mean information matrix is the covariance matrix appearing in the asymptotic distribution of the appropriately normalized MLE. Hence, when combined with the asymptotic normal distribution, the information matrix may be used in constructing confidence regions for the MLE when the sample size is sufficiently large. More generally, the information matrix provides a summary of the amount of

information in the data relative to $\boldsymbol{\theta}$ (e.g., Spall, 2003, Sect. 13.3). We restrict our attention below to the fully series and fully parallel subsystems cases (Subsection 2.2), but the analysis can be modified in a straightforward manner for certain other cases, including hybrid series-parallel subsystems cases.

The $p \times p$ Fisher information matrix $\boldsymbol{F}(\boldsymbol{\theta})$ for a twice-differentiable log-likelihood function is given by

$$\boldsymbol{F}(\boldsymbol{\theta}) = E\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}^T} \right) = -E\left(\frac{\partial^2 \mathcal{L}}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} \right), \tag{5.1}$$

where $\partial^2 \mathcal{L} / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T$ appearing after the last equality above corresponds to the Hessian matrix of the log-likelihood function.

Let us first compute $\boldsymbol{F}(\boldsymbol{\theta})$ for the series case using the Hessian-based form appearing after the second equality in (5.1). Because $\Theta = \{0 < \rho_j < 1 \text{ for all } j\}$, it is known that the Hessian matrix is continuous and, consequently, symmetric. From (2.6), the elements of the negative Hessian for the series case of interest are:

$$-\frac{\partial^2 \mathcal{L}}{\partial \rho_j \partial \rho_k} = \begin{cases} \dfrac{X + X_j}{\rho_j^2} + \dfrac{(n - X)\rho^2}{(1-\rho)^2 \rho_j^2} + \dfrac{(n_j - X_j)}{(1-\rho_j)^2} & \text{when } j = k, \\ \dfrac{(n - X)\rho}{(1-\rho)^2 \rho_j \rho_k} & \text{when } j \neq k. \end{cases}$$

Then, the corresponding elements of the information matrix $\boldsymbol{F}(\boldsymbol{\theta}) = \left[F_{jk}(\boldsymbol{\theta}) \right]$ are:

$$F_{jk}(\boldsymbol{\theta}) = \begin{cases} \dfrac{n\rho + n_j \rho_j}{\rho_j^2} + \dfrac{n\rho^2}{(1-\rho)\rho_j^2} + \dfrac{n_j}{(1-\rho_j)} & \text{when } j = k, \\ \dfrac{n\rho}{(1-\rho)\rho_j \rho_k} & \text{when } j \neq k. \end{cases} \tag{5.2}$$

Likewise, the Hessian can be used to compute $\boldsymbol{F}(\boldsymbol{\theta})$ in the parallel subsystem case. From (2.7), the elements of the negative Hessian in the parallel case are:

$$-\frac{\partial^2 \mathcal{L}}{\partial \rho_j \partial \rho_k} = \begin{cases} \dfrac{X_j}{\rho_j^2} + \dfrac{(n + n_j - X - X_j)}{(1-\rho_j)^2} + \dfrac{X(1-\rho)^2}{(1-\rho_j)^2 \rho^2} & \text{when } j = k, \\ \dfrac{(1-\rho)(\rho^2 - X\rho + X)}{(1-\rho_j)(1-\rho_k)\rho^2} & \text{when } j \neq k, \end{cases}$$

leading to the following elements of the information matrix:

$$F_{jk}(\boldsymbol{\theta}) = \begin{cases} \dfrac{n_j}{\rho_j} + \dfrac{n(1-\rho) + n_j \rho (1-\rho_j)}{(1-\rho_j)^2 \rho} & \text{when } j = k, \\ \dfrac{(1-\rho)(\rho - n\rho + n)}{(1-\rho_j)(1-\rho_k)\rho} & \text{when } j \neq k. \end{cases} \tag{5.3}$$

Then, the expression in (5.2) or (5.3) can be used to determine if the information matrix is positive definite, thereby characterizing the identifiability of $\boldsymbol{\theta}$. It is clear that both n and the n_j can contribute to the positive definiteness of $\boldsymbol{F}(\boldsymbol{\theta})$. For example, if the n_j dominate n, then increasing all n_j at the same rate (in the sense that $n_j / n_k = O(1)$ and $n_k / n_j = O(1)$ for all j, k) is sufficient to achieve the positive definiteness for sufficiently large sample sizes. It is also possible to have $n \to \infty$ subject to the n_j growing sufficiently rapidly as well. In a practical application, it will be necessary to assume a value for $\boldsymbol{\theta}$ prior to carrying out the estimation in order to evaluate $\boldsymbol{F}(\boldsymbol{\theta})$. This value for $\boldsymbol{\theta}$ may be chosen conservatively or at a "typical" level in determining identifiability.

The other main interest for application of the information matrix is determining approximate confidence regions. However, one of the complications in using the standard asymptotic normality results is the multiple samples sizes, n, n_1,..., n_p. Fortunately, the form for $\boldsymbol{F}(\boldsymbol{\theta})$ provides clarification with respect to the mix of sample sizes.

Recall that the standard *generic* form for the asymptotic distribution of MLEs is,

$$\sqrt{\text{sample size}}(\text{MLE} - \text{true value}) \xrightarrow{\text{dist}} N(\boldsymbol{0}, \bar{\boldsymbol{F}}^{-1}) \tag{5.4}$$

where $\xrightarrow{\text{dist}}$ denotes convergence in distribution and $\bar{\boldsymbol{F}}$ is the limit of the mean information matrix (i.e., the limit of the information matrix averaged over the sample size) (e.g., Hoadley, 1971; Rao, 1973, pp. 415–417). In well-posed problems, $\bar{\boldsymbol{F}}$ is a finite-magnitude positive definite matrix. Hence, to within slower growth terms, the magnitude of the *unaveraged* Fisher information matrix $\boldsymbol{F}(\boldsymbol{\theta})$ must grow linearly with the increase in the relevant sample size. In the context of the typical forms for $\boldsymbol{F}(\boldsymbol{\theta})$ above, it is clear that both n and the n_j can contribute to the growth in magnitude for $\boldsymbol{F}(\boldsymbol{\theta})$. For example, as above, if the n_j dominate n, then increasing all n_j at the same rate (in the sense that $n_j / n_k = O(1)$ and $n_k / n_j = O(1)$ for all j, k) is sufficient to achieve the necessary growth in the magnitude of $\boldsymbol{F}(\boldsymbol{\theta})$. It is also possible to have $n \to \infty$ subject to the n_j growing sufficiently rapidly as well. The author is currently pursuing the issue of asymptotic normality in greater detail.

6. Bootstrap Confidence Intervals

It is well known that the traditional asymptotic normality-based methods are often inadequate in constructing confidence intervals for reliability estimates. Two factors contribute to this inadequacy: (i) sample sizes that are too small to justify the asymptotic normality and (ii) confidence intervals from the asymptotic normality that fall outside of the interval [0, 1] as a consequence of the need to approximate the true asymmetric distribution with the symmetric normal distribution. The latter factor is

exacerbated by the fact that practical reliability estimates are often very near unity. We now present a bootstrap-based method for constructing confidence intervals for the full system reliability estimate $\hat{\rho}$ under the assumption that $\hat{\rho}$ is uniquely determined from $\hat{\boldsymbol{\theta}}$. Lemma 1 presented sufficient conditions for such a function via the implicit function theorem.

Bootstrap methods are well-known Monte Carlo procedures for creating important statistical quantities of interest when analytical methods are infeasible (e.g., Efron and Tibshirani, 1986; Ljung, 1999, pp. 304 and 334; and Aronsson et al., 2006). The steps below describe a "parametric bootstrap" approach to constructing confidence intervals for $\hat{\rho}$. Parametric bootstrap methods sample from a specified distribution based on using the estimated parameter values; a standard bootstrap method samples from the raw data.

Step 0: Treat the MLE $\hat{\boldsymbol{\theta}}$, and associated $\hat{\rho}$, as the true value of $\boldsymbol{\theta}$ and ρ.

Step 1: Generate (by Monte Carlo) a set of bootstrap data of the same collective sample size $N = \{n,\ n_1,\ n_2,\ ...,\ n_p\}$ as the real data $\boldsymbol{Y}$ using the assumed probability mass function in (2.2) and the value of $\boldsymbol{\theta}$ and ρ from Step 0.

Step 2: Calculate the MLE of $\boldsymbol{\theta}$, say $\hat{\boldsymbol{\theta}}_{\text{boot}}$, from the bootstrap data $\boldsymbol{Y}$ in Step 1, and then calculate the corresponding full system reliability MLE, $\hat{\rho}_{\text{boot}}$.

Step 3: Repeat Steps 1 and 2 a large number of times (perhaps 1000) and rank order the resulting $\hat{\rho}_{\text{boot}}$ values; one- or two-sided confidence intervals are available by determining the appropriate quantiles from the ranked sample of $\hat{\rho}_{\text{boot}}$ values.

7. Concluding Remarks

We have described above an MLE-based approach for estimating the reliability of a complex system by combining data from full system reliability tests and subsystem or other tests. The method applies in general systems, where the subsystems may be arranged in series, parallel, combination series/parallel, or other mode.

This MLE approach provides a means of estimating the reliability of systems with relatively few (or even no) full system tests through the knowledge obtained via subsystem tests. By appropriately formulating constraints in an optimization problem, the approach accommodates general relationships between the subsystems and full system, including statistical dependencies among subsystems operating within the full system. Interestingly, the likelihood function has the same general form across all settings; only the constraints in the optimization problem change. The method includes asymptotic (Fisher information-based) and finite-sample (bootstrap) methods for characterizing the uncertainty via confidence regions.

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