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Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Networked Control Systems with Random Delays Modeled by Markov Chains

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***Abstract*— This paper considers the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of networked control systems where random time delays existing in sensor-to-controller (S-C) and controller-to-actuator (C-A) links are modeled by Markov chains. The designed output feedback controller is two-mode-dependent, which depends on the available S-C and C-A delay information. The closed-loop system is formulated as a special jump linear system. The definitions of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are further proposed to reflect the special characteristics of the system. The $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems are solved via the linear matrix inequality (LMI) optimization approaches. Simulation examples illustrate the effectiveness of the proposed methods.**

I. INTRODUCTION

With the development of network techniques, the integration of networks into control systems has attracted much attention recently. Networked control system (NCS) is a type of distributed system, in which the information of system components is exchanged via the communication networks. Compared with the traditional control systems, the main advantages of NCSs include low cost, easy diagnosis, and high reliability. Hence, NCSs have many industrial applications [1]–[3], and the stability and control problems in NCSs have received increasing attentions in the past decade.

The existence of communication networks also brings some problems. One of the main issues is the network-induced time delay, which is a source of poor performance and instability in control systems. Ample papers have been found in the literature to model and control NCSs in the presence of time delays. Markov and Bernoulli processes are widely used to model the network-induced time delay and packet loss [2], [4]–[7]. The using of Markov process takes the dependencies between time delays into account, and also can include the packet dropout naturally [4]. The Bernoulli process can be taken as a special case of Markov process [8].

Various control methods have been proposed to cope with the network-induced delays modeled by Markov chains. Generally, these controllers can be classified into three categories based on their dependency on the delay information.

- **Mode-independent controller.** The controller does not depend on either sensor-to-controller (S-C) or controller-to-actuator (C-A) delays. In [4], the authors model the time delays as Markov chains and propose the mode-independent output feedback controller design method. In [8], the authors consider the Markovian packet loss process and a mode-independent state feedback controller is designed.
- **One-mode-dependent controller.** The controller only depends on the S-C delays. In [4], a one-mode-dependent state feedback controller is designed to stabilize the NCSs with S-C delays modeled by Markov chains. In [2], an $\mathcal{H}_\infty$ controller is designed for the vehicle control problems under the framework of Markovian jump linear systems (MJLSs) [9]–[11]; only the S-C delay is considered. In [6], both the S-C and C-A packet dropouts modeled as Markov chains are considered and a one-mode-dependent state feedback controller design method is provided; the authors introduce a new classification of NCSs to simplify the modeling and avoid incorporating the C-A dropouts in the controller.
- **Two-mode-dependent controller.** The controller depends on both S-C and C-A delays. In [5], the authors propose a two-mode-dependent state feedback controller design method to guarantee the stochastic stability. The controller depends on the current S-C delay (τ_k) and the preceding C-A delay (d_{k-1}). In [7], the Markov processes are used to model the continuous S-C and C-A delays, and further a two-mode-dependent state feedback controller depending on both current S-C and C-A delays is designed to maintain the stability.

It is worth noting that the two-mode-dependent controller can include the mode-independent and one-mode-dependent controllers as special cases. Hence, it can reduce the conservativeness and achieve better performance. However, the two-mode-dependent controller design has not been fully investigated, especially for the practical controller and the control synthesis. In our previous work [12], we propose that at the current time k, the C-A delay $d_{k-\tau_k-1}$ can be obtained by the controller instead of d_{k-1} in [5] as the preceding C-A delay obtained by the embedded processor at the plant node has to be transmitted through the S-C link as shown in Fig. 1. This moves a step towards the practical application of two-mode-dependent controller and further the output feedback controller is designed for the stochastic stability [12]. The contributions of this paper lie in that the control performance ($\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control) is included in the controller design while only the stabilization problem is considered in the previous work [5], [7], [12], [13].

The remainder of this paper is organized as follows. Section II formulates the problem and presents the objectives of this work. In Section III, the definitions of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are proposed, and further the $\mathcal{H}_2$ and mixed

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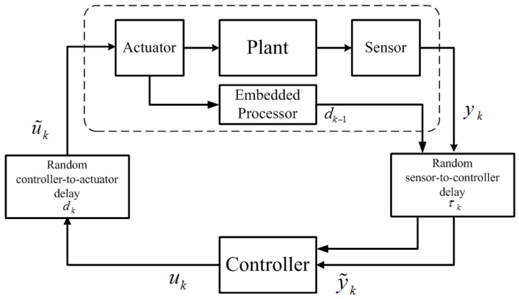

Fig. 1. Diagram of a networked control system.

$\mathcal{H}_2/\mathcal{H}_\infty$ control problems are solved in terms of linear matrix inequalities (LMIs) with nonconvex constraints. The comparison and design examples are given in Section IV. Finally, the concluding remarks are addressed in Section V.

II. Problem Formulation

Consider the NCS setup in Fig. 1. The discrete-time linear time-invariant plant model is

$$x(k+1) = Ax(k) + B\tilde{u}(k) + J\omega(k), \tag{1a}$$
$$y(k) = Cx(k), \tag{1b}$$

where $x(k) \in \mathbb{R}^n$, $\tilde{u}(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$, $\omega(k) \in \mathbb{R}^l$ and A, B, C, and J are known real matrices with appropriate dimensions. Bounded random delays exist in the links from sensor to controller and controller to actuator as shown in Fig. 1. Here, $\tau \geq \tau_k \geq 0$ represents the S-C delay and $d \geq d_k \geq 0$ stands for the C-A delay. In this paper, τ_k and d_k are modeled as two homogeneous Markov chains [4], [5], [12] that take values in $\mathcal{M} = \{0, 1, ..., \tau\}$ and $\mathcal{N} = \{0, 1, ..., d\}$, and their transition probability matrices are $\Lambda = [\lambda_{ij}]$ and $\Pi = [\pi_{rs}]$, respectively. That means τ_k and d_k jump from mode i to j and from mode r to s, respectively, with probabilities λ_{ij} and π_{rs}, which are defined by

$$\lambda_{ij} = \Pr(\tau_{k+1} = j | \tau_k = i), \;\; \pi_{rs} = \Pr(d_{k+1} = s | d_k = r)$$

with the constraints $\lambda_{ij}, \pi_{rs} \geq 0$ and

$$\sum_{j=0}^{\tau} \lambda_{ij} = 1, \qquad \sum_{s=0}^{d} \pi_{rs} = 1 \tag{2}$$

for all $i, j \in \mathcal{M}$ and $r, s \in \mathcal{N}$.

It is noticed that when the controller is designed at current time k, the S-C delay τ_k can be obtained using the time-stamping technique and the embedded processor can calculate the preceding C-A time delay d_{k-1}. Furthermore, by considering the random delays in the S-C link, $d_{k-\tau_k-1}$ can be obtained by the controller at current time k for sure [12]. The dynamic output controller is designed based on the available information $(\tau_k, d_{k-\tau_k-1})$, and thus has the following form:

$$z(k+1) = F(\tau_k, d_{k-\tau_k-1})z(k) + G(\tau_k, d_{k-\tau_k-1})\tilde{y}(k), \tag{3a}$$
$$u(k) = H(\tau_k, d_{k-\tau_k-1})z(k) + T(\tau_k, d_{k-\tau_k-1})\tilde{y}(k), \tag{3b}$$

where $z(k) \in \mathbb{R}^n$ is the state vector of the output feedback controller; and F, G, H, and T are appropriately dimensioned matrices to be designed.

Consider the time delays in S-C link, we have

$$x(k+1) = Ax(k) + B\tilde{u}(k) + J\omega(k), \tag{4a}$$
$$\tilde{y}(k) = y(k-\tau_k) = Cx(k-\tau_k). \tag{4b}$$

At sampling time k, if augment the state variable as

$$\tilde{X}(k) = \begin{bmatrix} x(k)^{\mathrm{T}} & y(k-1)^{\mathrm{T}} & y(k-2)^{\mathrm{T}} & \cdots & y(k-\tau)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

then

$$\tilde{X}(k+1) = \tilde{A}\tilde{X}(k) + \tilde{B}\tilde{u}(k) + \tilde{J}_1\omega(k), \tag{5a}$$
$$\tilde{y}(k) = \tilde{C}_1(\tau_k)\tilde{X}(k), \tag{5b}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ C & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \;\; \tilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \;\; \tilde{J}_1 = \begin{bmatrix} J \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{C}_1(\tau_k) = \begin{cases} \begin{bmatrix} C & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, & \text{for } \tau(k) = 0 \\ \underbrace{\begin{bmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{bmatrix}}_{(1+\tau_k)th\ block\ being\ identity}, & \text{for } \tau(k) > 0. \end{cases}$$

Similarly, at sampling time k, augment the state variable as

$$\tilde{Z}(k) = \begin{bmatrix} z(k)^{\mathrm{T}} & u(k-1)^{\mathrm{T}} & u(k-2)^{\mathrm{T}} & \cdots & u(k-d)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

then we have

$$\tilde{Z}(k+1) = \tilde{F}(\tau_k, d_{k-\tau_k-1})\tilde{Z}(k) + \tilde{G}(\tau_k, d_{k-\tau_k-1})\tilde{y}(k), \tag{6a}$$
$$\tilde{u}(k) = \tilde{H}(\tau_k, d_k, d_{k-\tau_k-1})\tilde{Z}(k) + \tilde{T}(\tau_k, d_k, d_{k-\tau_k-1})\tilde{y}(k), \tag{6b}$$

where

$$\tilde{F}(\tau_k, d_{k-\tau_k-1}) = \begin{bmatrix} F(\tau_k, d_{k-\tau_k-1}) & 0 & \cdots & 0 & 0 \\ H(\tau_k, d_{k-\tau_k-1}) & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

$$\tilde{G}(\tau_k, d_{k-\tau_k-1}) = \begin{bmatrix} G(\tau_k, d_{k-\tau_k-1}) \\ T(\tau_k, d_{k-\tau_k-1}) \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{H}(\tau_k, d_{k-\tau_k-1}, d_k) = \begin{cases} \begin{bmatrix} H(\tau_k, d_{k-\tau_k-1}) & 0 & \cdots & 0 \end{bmatrix}, & \text{for } d(k) = 0, \\ \underbrace{\begin{bmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{bmatrix}}_{(1+d_k)th\ block\ being\ identity}, & \text{for } d(k) > 0, \end{cases}$$

$$\tilde{T}(\tau_k, d_{k-\tau_k-1}, d_k) = \begin{cases} T(\tau_k, d_{k-\tau_k-1}), & \text{for } d(k) = 0, \\ 0, & \text{for } d(k) > 0. \end{cases}$$

Combining (5) and (6), and letting the state variable as

$$X(k) = \begin{bmatrix} \tilde{X}(k)^{\mathrm{T}} & \tilde{Z}(k)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

we have the following closed-loop system dynamics:

$$X(k+1) = \left[\bar{A} + \bar{B}K(\tau_k, d_{k-\tau_k-1}, d_k)\bar{C}(\tau_k)\right]X(k) + \tilde{J}\omega(k), \tag{7a}$$
$$y(k) = \tilde{C}X(k), \tag{7b}$$

where

$$\bar{A}=\begin{bmatrix}\tilde{A} & 0\\ 0 & 0\end{bmatrix},\bar{B}=\begin{bmatrix}0 & \tilde{B}\\ I & 0\end{bmatrix},\bar{C}(\tau_k)=\begin{bmatrix}0 & I\\ \tilde{C}_1(\tau_k) & 0\end{bmatrix},$$

$$\begin{aligned}&K(\tau_k,d_{k-\tau_k-1},d_k)\\ &\quad=\begin{bmatrix}\tilde{F}(\tau_k,d_{k-\tau_k-1}) & \tilde{G}(\tau_k,d_{k-\tau_k-1})\\ \tilde{H}(\tau_k,d_{k-\tau_k-1},d_k) & \tilde{J}(\tau_k,d_{k-\tau_k-1},d_k)\end{bmatrix},\\ &\tilde{J}=\begin{bmatrix}J^{\mathrm{T}} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}^{\mathrm{T}},\\ &\tilde{C}=\begin{bmatrix}C & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}.\end{aligned}$$

Remark 1 The closed-loop system in (7) cannot be transformed to a standard MJLS as in [2], [4] because the system depends on τ_k, d_k, and $d_{k-\tau_k-1}$, and $d_{k-\tau_k-1}$ is related to τ_k and d_k. Hence, the results on MJLSs cannot be directly applied. The special feature results from that the controller depends on $d_{k-\tau_k-1}$. This makes the system more complex and the control of the special system more challenging.

Remark 2 The controller (3) is two-mode-dependent. It can include the mode-independent and one-mode-dependent controllers as special cases, as shown in Remark 2 in [12]. Hence, the two-mode-dependent controller can reduce the conservativeness and achieve better performance, which will be shown in numerical examples.

To the best of the authors' knowledge, to incorporate the control performance ($\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms) into the two-mode-dependent controller design has not been fully investigated. The objective of this paper is to solve the $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ two-mode-dependent control problems for networked systems. For $\mathcal{H}_2$ control synthesis, we aim to design the output feedback controller (3) to guarantee that:

- The closed-loop system in (7) is stochastically stable;
- The $\mathcal{H}_2$ norm of the system is minimized.

For mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control synthesis, we aim to design the output feedback controller (3) to guarantee that:

- The closed-loop system in (7) is stochastically stable;
- The $\mathcal{H}_2$ norm of the system is minimized while the $\mathcal{H}_\infty$ norm of the system is lower than the prescribed level.

III. $\mathcal{H}_2$ AND MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL

A. Stability analysis

Definition 1: [12] The system in (7) with $\omega(k)=0$ for all $k\geq 0$ is said to be stochastically stable if for every finite $X_0=X(0)$, initial mode $\tau_0=\tau(0)\in\mathcal{M}$, and $d_{-\tau_0-1}=d(-\tau_0-1)\in\mathcal{N}$, there exists a finite $W>0$ such that the following holds:

$$\mathcal{E}\left\{\sum_{k=0}^{\infty}\|X(k)\|^2|X_0,\tau_0,d_{-\tau_0-1}\right\}<{X_0}^{\mathrm{T}}WX_0. \tag{8}$$

The following theorem provides the sufficient and necessary condition for the stochastic stability of the system in (7).

Theorem 1: [12] Under the proposed output feedback control law (3), the resulting closed-loop system in (7) is stochastically stable if and only if there exists symmetric $P(i,r)>0$ such that the following matrix inequality:

$$\begin{aligned}&\sum_{j=0}^{\tau}\sum_{s_1=0}^{d}\sum_{s_2=0}^{d}\lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2s_1}^{j}\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}}\\ &\quad\times P(j,s_2)\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]-P(i,r)<0\end{aligned} \tag{9}$$

holds for all $i\in\mathcal{M}$ and $r\in\mathcal{N}$.

B. Definitions of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms

For a stable discrete-time LTI system, the classical $\mathcal{H}_2$ norm has the following time-domain interpretation: The l_2 norm of the output equals the $\mathcal{H}_2$ norm of the system if the input is the unit impulse [14]. A definition of $\mathcal{H}_2$ norm for MJLSs is given in [9]. However, as the closed-loop system in (7) under consideration is a special discrete-time jump linear system, the definitions of the classical $\mathcal{H}_2$ norm and MJLS $\mathcal{H}_2$ norm are not suitable. Following the general definitions of the $\mathcal{H}_2$ norm, define the following $\mathcal{H}_2$ norm for the system in (7) to take the special feature into account, which can be used as a performance index.

Definition 2: We define the $\mathcal{H}_2$ norm of system in (7) with $X(0)=0$ as

$$\|H_{y\omega}\|_2^2=\sum_{s_o=1}^{l}\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\alpha_{(i_o,r_o)}\|\mathcal{E}(y_{s_o,i_o,r_o})\|_2^2, \tag{10}$$

where y_{s_o,i_o,r_o} is the output sequence of the system in (7) when

(1) the input sequence is given by $\omega=(\omega(0),\omega(1),\cdots)$, $\omega(0)=e_s$, $\omega(k)=0$, $k>0$, $e_s\in\mathbb{R}^l$ the unitary vector formed by one at the s_oth position and zero elsewhere;
(2) $\tau(0)=i_o$;
(3) $d(-\tau_0-1)=r_o$.

The initial distribution for (τ_0, $d_{-\tau_0-1}$) is given by $\alpha=(\alpha_{(i_o,r_o)})$, where $i_o\in\mathcal{M}$, $r_o\in\mathcal{N}$ and $\sum_{i_o\in\mathcal{M},r_o\in\mathcal{N}}\alpha_{(i_o,r_o)}=1$.

Remark 3 When $\tau=0$, $d=0$, the Definition 2 of $\mathcal{H}_2$ norm is reduced to the classical $\mathcal{H}_2$ norm. Hence, the definition can be viewed as a generalization of the $\mathcal{H}_2$ norm from LTI systems to the special system. Moreover, when $d=0$, the Definition 2 is reduced to the $\mathcal{H}_2$ norm for MJLSs [9].

The definition of classical $\mathcal{H}_\infty$ norm for LTI systems can be interpreted as a measure of robust stability that represents the worst-case energy attenuation for any energy-bounded disturbance. Following the time-domain interpretation, the $\mathcal{H}_\infty$ norm for the special system in (7) is defined as follows.

Definition 3: Let $X(0)=0$ and define the $\mathcal{H}_\infty$ norm as

$$\|H_{y\omega}\|_\infty=\sup_{\tau(0)\in\mathcal{M}}\ \sup_{d(-\tau_0-1)\in\mathcal{N}}\ \sup_{\omega\in l_2(0,\infty)}\frac{\|y\|_2}{\|\omega\|_2}. \tag{11}$$

C. $\mathcal{H}_2$ control

The following theorem establishes the relationship between the $\mathcal{H}_2$ norm and the state-space model of the jump linear system in (7).

Theorem 2: The $\mathcal{H}_2$ norm of system in (7) can be computed as follows.

$$\begin{aligned} &\|H_{y\omega}\|_2^2 \\ &= \sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d}\alpha_{(i_o,r_o)}\lambda_{i_oj_o}\Pi_{r_os_{o2}}^{1+i_o-j_o}\mathrm{tr}\left\{\tilde{J}^{\mathrm{T}}S(j_o,s_{o2})\tilde{J}\right\}, \end{aligned} \tag{12}$$

where $S(j_o, s_{o2}) > 0$ is the solution of the following discrete-time Lyapunov equation

$$\begin{aligned} &S(i,r) \\ &= \sum_{j=0}^{\tau}\sum_{s_1=0}^{d}\sum_{s_2=0}^{d}\lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2s_1}^{j}\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}} \\ &\quad\times S(j,s_2)\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]+\tilde{C}^{\mathrm{T}}\tilde{C}, \end{aligned} \tag{13}$$

for $i \in \mathcal{M}, r \in \mathcal{N}$.

Proof: The system in (7) depends on three parameters τ_k, d_k, and $d_{k-\tau_k-1}$. As these three parameters are interdependent, the multi-step jump of Markov chain happens in the system evolvement [12]. The Proposition 1 in [12] can be used to handle the multi-step jump. Further, considering the system dynamics and the Definition 2, the theorem can be completed. The detailed proof is omitted here for the length limitation of the paper. ■

Now, we are in a good position to solve the $\mathcal{H}_2$ control problem for this special system. The objective is to design a controller in (3) such that the $\mathcal{H}_2$ norm of system in (7) is minimized. The $\mathcal{H}_2$ control for LTI systems has been studied [15], [16] and the $\mathcal{H}_2$ control for MJLSs has been investigated in [11]. In the following, the $\mathcal{H}_2$ control problem will be transformed to an optimization problem.

Theorem 3: Under the proposed output feedback control law (3), the closed-loop system in (7) is stochastically stable and $\|H_{y\omega}\|_2 < \beta$, if and only if there exist matrices $F(i,r)$, $G(i,r)$, $H(i,r)$, $T(i,r)$, and symmetric matrices $\bar{X}(j,s_2) > 0$, $P(i,r) > 0$ satisfying the following inequalities with nonconvex constraints

$$\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d}\alpha_{(i_o,r_o)}\lambda_{i_oj_o}\Pi_{r_os_{o2}}^{1+i_o-j_o}\mathrm{tr}\{\tilde{J}^{\mathrm{T}}P(j_o,s_{o2})\tilde{J}\} < \beta^2, \tag{14a}$$

$$\begin{bmatrix} -P(i,r)+\tilde{C}^{\mathrm{T}}\tilde{C} & V(i,r)^{\mathrm{T}} \\ V(i,r) & -X(i,r) \end{bmatrix} < 0, \tag{14b}$$

$$\bar{X}(j,s_2)P(j,s_2) = I, \tag{14c}$$

for all $i,j \in \mathcal{M}$ and $r, s_2 \in \mathcal{N}$, where

$$V(i,r) = \begin{bmatrix} V_0(i,r)^{\mathrm{T}} & V_1(i,r)^{\mathrm{T}} & \cdots & V_\tau(i,r)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$V_j(i,r) = \begin{bmatrix} V_{j,0}(i,r)^{\mathrm{T}} & V_{j,1}(i,r)^{\mathrm{T}} & \cdots & V_{j,d}(i,r)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$V_{j,s_2}(i,r) = \begin{bmatrix} (\lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_20}^{j})^{\frac{1}{2}}\left[\bar{A}+\bar{B}K(i,r,0)\bar{C}(i)\right] \\ (\lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_21}^{j})^{\frac{1}{2}}\left[\bar{A}+\bar{B}K(i,r,1)\bar{C}(i)\right] \\ \vdots \\ (\lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2d}^{j})^{\frac{1}{2}}\left[\bar{A}+\bar{B}K(i,r,d)\bar{C}(i)\right] \end{bmatrix},$$

$$X(i,r) = \mathrm{diag}\{X_0(i,r) \quad X_1(i,r) \quad \cdots \quad X_\tau(i,r)\},$$

$$X_j(i,r) = \mathrm{diag}\{X_{j,0}(i,r) \quad X_{j,1}(i,r) \quad \cdots \quad X_{j,d}(i,r)\},$$

$$X_{j,s_2}(i,r) = \mathrm{diag}\left\{\underbrace{\bar{X}(j,s_2)\ \bar{X}(j,s_2)\ \cdots\ \bar{X}(j,s_2)}_{d+1}\right\}. \tag{15}$$

Proof: The proof is omitted due to the limited space. ■

D. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control

The following theorem provides the sufficient condition for mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control.

Theorem 4: If

$$\begin{aligned} &\sum_{j=0}^{\tau}\sum_{s_1=0}^{d}\sum_{s_2=0}^{d}\lambda_{ij}\Pi_{rs_2}^{1+i-j}\Pi_{s_2s_1}^{j}\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]^{\mathrm{T}} \\ &\times P(j,s_2)\left[\bar{A}+\bar{B}K(i,r,s_1)\bar{C}(i)\right]-P(i,r)+\tilde{C}^{\mathrm{T}}\tilde{C} \\ &+\frac{1}{\gamma^2}P(i,r)\tilde{J}\tilde{J}^{\mathrm{T}}P(i,r) < 0, \end{aligned} \tag{16}$$

then

1) The system in (7) is stochastically stable;
2) $\|H_{y\omega}\|_\infty < \gamma$;
3) $\|H_{y\omega}\|_2^2 \le \sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d}\alpha_{(i_o,r_o)}\lambda_{i_oj_o}\Pi_{r_os_{o2}}^{1+i_o-j_o} \times \mathrm{tr}\left\{\tilde{J}^{\mathrm{T}}P(j_o,s_{o2})\tilde{J}\right\}.$

Proof: Omitted here for the length limitation of the paper. ■

Then it can be transformed to an equivalent condition expressed as a set of LMIs with nonconvex constraints.

Theorem 5: Under the proposed output feedback control law (3), the closed-loop system in (7) is stochastically stable and $\|H_{y\omega}\|_\infty < \gamma$, $\|H_{y\omega}\|_2 < \beta$ if there exist matrices $F(i,r)$, $G(i,r)$, $H(i,r)$, $T(i,r)$, and symmetric matrices $\bar{X}(j,s_2) > 0$, $P(i,r) > 0$, satisfying:

$$\sum_{i_o=0}^{\tau}\sum_{r_o=0}^{d}\sum_{j_o=0}^{\tau}\sum_{s_{o2}=0}^{d}\alpha_{(i_o,r_o)}\lambda_{i_oj_o}\Pi_{r_os_{o2}}^{1+i_o-j_o}\mathrm{tr}\{\tilde{J}^{\mathrm{T}}P(j_o,s_{o2})\tilde{J}\} < \beta^2, \tag{17a}$$

$$\begin{bmatrix} -P(i,r)+\tilde{C}^{\mathrm{T}}\tilde{C} & V(i,r)^{\mathrm{T}} & \frac{1}{\gamma}P(i,r)\tilde{J} \\ V(i,r) & -X(i,r) & 0 \\ \frac{1}{\gamma}\tilde{J}^{\mathrm{T}}P(i,r) & 0 & -I \end{bmatrix} < 0, \tag{17b}$$

$$\bar{X}(j,s_2)P(j,s_2) = I, \tag{17c}$$

for all $i,j \in \mathcal{M}$ and $r,s_2 \in \mathcal{N}$, with matrices defined in (15).

Proof: The proof can be completed by using the Schur complement and letting $\bar{X}(j,s_2) = P(j,s_2)^{-1}$. ■

The conditions (14) and (17) contain a set of LMIs and nonconvex constraints. This can be solved by the product reduction algorithm (PRA) [17]. Detailed procedure about how to apply PRA to this problem can be referred to [18].

Now, the $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ two-mode-dependent control design for networked systems can be summarized as: For $\mathcal{H}_2$ control, minimize β subject to (14); for mixed

$\mathcal{H}_2/\mathcal{H}_\infty$ control, let γ be a certain given value, minimize β subject to (17).

Remark 4 As the mode-independent controller and one-mode-dependent controller can be viewed as the special cases of the two-mode-dependent controller with certain constraints. The $\mathcal{H}_2$ control method in Theorem 3 and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control method in Theorem 5 can also handle the mode-independent controller and one-mode-dependent controller design.

IV. Numerical examples

Consider an inverted pendulum system shown in Fig. 2, where θ is the angular position of the pendulum, and u is the input torque. The state variables are chosen as $[\ \theta^{\mathrm{T}} \quad \dot{\theta}^{\mathrm{T}}\ ]^{\mathrm{T}}$. The output is $y = [\ \theta\ ]$. The parameters here are: $m = 0.1\ kg$, $L = 1\ m$. The output feedback controller is designed using the discrete-time model (sampling time $T_s = 0.05s$):

$$x(k+1) = A_d x(k) + B_d u(k) + J_d \omega(k), \tag{18a}$$

$$y(k) = C_d x(k), \tag{18b}$$

where

$$A_d = \begin{bmatrix} 1.0123 & 0.0502 \\ 0.4920 & 1.0123 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.0125 \\ 0.5020 \end{bmatrix},$$

$$J_d = \begin{bmatrix} 0.100 \\ 0.100 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{19}$$

The eigenvalues of A_d are 0.7312 and 1.3676. Hence, the discrete-time system is unstable.

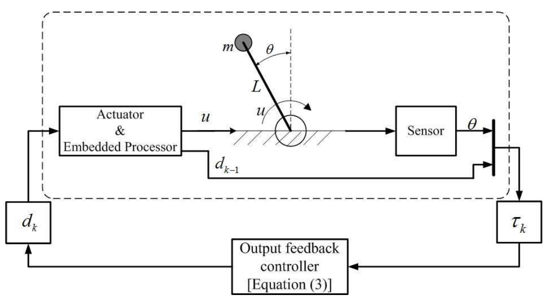

Fig. 2. An inverted pendulum system.

The random delays involved in this NCS are assumed to be $\tau_k \in \{0, 1, 2\}$ and $d_k \in \{0, 1\}$, and their transition probability matrices are given by

$$\Lambda = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}.$$

The initial distribution for $(\tau_0, d_{-\tau_0-1})$ is equal for every $(\alpha_{(i_o,r_o)})$, where $i_o \in \mathcal{M}$, $r_o \in \mathcal{N}$, which means $\alpha_{(i_o,r_o)} = \frac{1}{6}$ in the following examples.

A. $\mathcal{H}_2$ control

Firstly, apply the $\mathcal{H}_2$ control method in Theorem 3 to design a two-mode-dependent output feedback controller, and the minimum $\mathcal{H}_2$ norm $\beta_{\min}$ is 0.277. Secondly, design the one-mode-dependent controller that only depends on τ_k, and by using the Theorem 3, the minimum $\mathcal{H}_2$ norm $\beta_{\min}$ is 0.285. Thirdly, the mode-independent controller is designed and the minimum $\mathcal{H}_2$ norm $\beta_{\min}$ is 0.292. The comparison for three types of controllers is provided in Table I. Obviously, the two-mode-dependent controller provides more freedom for the controller design and outperforms both the one-mode-dependent and mode-independent controllers.

TABLE I

Comparison of $\mathcal{H}_2$ control performance

Controller Type	Two-mode -dependent	One-mode -dependent	Mode -independent
Minimum $\mathcal{H}_2$ norm	0.277	0.285	0.292

B. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control

In this example, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design will be considered. The system matrices are shown in (19). γ is set to be 2. By using Theorem 5, we can obtain that the minimum value of $\mathcal{H}_2$ norm $\beta_{\min}$ is 0.286 and the corresponding controller is

$$F(0,0) = \begin{bmatrix} 0.9221 & 0.6237 \\ -1.3405 & -0.6901 \end{bmatrix}, \ G(0,0) = \begin{bmatrix} 2.5033 \\ -0.6847 \end{bmatrix},$$

$$H(0,0) = \begin{bmatrix} 0.7807 & -0.7254 \end{bmatrix}, \ T(0,0) = \begin{bmatrix} -8.1888 \end{bmatrix},$$

$$F(0,1) = \begin{bmatrix} 0.6818 & 0.3012 \\ -0.7984 & -0.0182 \end{bmatrix}, \ G(0,1) = \begin{bmatrix} 2.6225 \\ -0.7534 \end{bmatrix},$$

$$H(0,1) = \begin{bmatrix} -1.0050 & -2.6649 \end{bmatrix}, \ T(0,1) = \begin{bmatrix} -7.8262 \end{bmatrix},$$

$$F(1,0) = \begin{bmatrix} 1.0712 & 0.8433 \\ -0.9852 & -0.2523 \end{bmatrix}, \ G(1,0) = \begin{bmatrix} 2.6060 \\ -0.7075 \end{bmatrix},$$

$$H(1,0) = \begin{bmatrix} -1.5050 & -3.6200 \end{bmatrix}, \ T(1,0) = \begin{bmatrix} -8.2682 \end{bmatrix},$$

$$F(1,1) = \begin{bmatrix} 1.1030 & 0.9091 \\ -1.1002 & -0.4253 \end{bmatrix}, \ G(1,1) = \begin{bmatrix} 2.6079 \\ -0.7610 \end{bmatrix},$$

$$H(1,1) = \begin{bmatrix} -1.0833 & -3.1329 \end{bmatrix}, \ T(1,1) = \begin{bmatrix} -8.2084 \end{bmatrix},$$

$$F(2,0) = \begin{bmatrix} 1.5169 & 1.2761 \\ -1.2053 & -0.4733 \end{bmatrix}, \ G(2,0) = \begin{bmatrix} 2.0694 \\ -0.4630 \end{bmatrix},$$

$$H(2,0) = \begin{bmatrix} -2.6483 & -4.5070 \end{bmatrix}, \ T(2,0) = \begin{bmatrix} -6.2058 \end{bmatrix},$$

$$F(2,1) = \begin{bmatrix} 1.7234 & 1.3308 \\ -1.2488 & -0.4495 \end{bmatrix}, \ G(2,1) = \begin{bmatrix} 1.5251 \\ -0.2581 \end{bmatrix},$$

$$H(2,1) = \begin{bmatrix} -3.3034 & -4.8754 \end{bmatrix}, \ T(2,1) = \begin{bmatrix} -5.0632 \end{bmatrix}.$$

To illustrate the performance of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control, a set of input signals are chosen as follows:

$$\omega(k) = \begin{cases} 1, & \text{for } 1 \le k \le 10, \\ -1, & \text{for } 21 \le k \le 30, \\ 0, & \text{otherwise.} \end{cases}$$

Figs. 3 and 4 show the network-induced delays τ_k and d_k, respectively. The responses of θ and $\dot{\theta}$ are shown in Figs. 5 and 6. It is observed that the system is stabilized. By calculation, we have $\|\omega\|_2 = 4.4721$, $\|y\|_2 = 2.2307$, which yields

$$\frac{\|y\|_2}{\|\omega\|_2} = 0.4988 < \gamma = 2.$$

The l_2 norm of the impulse response according to Definition 2 is evaluated as

$$\sqrt{\sum_{s_o=1}^{l} \sum_{i_o=0}^{\tau} \sum_{r_o=0}^{d} \alpha_{(i_o,r_o)} \|\mathcal{E}(y_{s_o,i_o,r_o})\|_2^2} = 0.2372 < 0.286.$$

These results show the effectiveness of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control method.

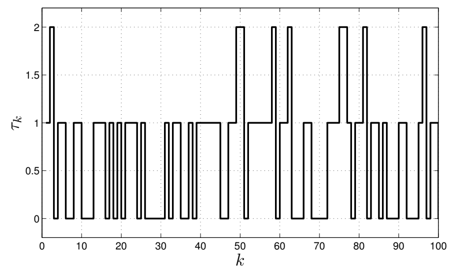

Fig. 3. S-C delay τ_k.

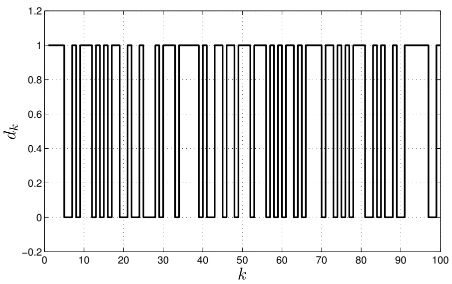

Fig. 4. C-A delay d_k.

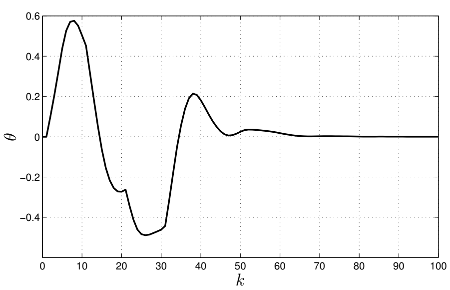

Fig. 5. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: The response of θ.

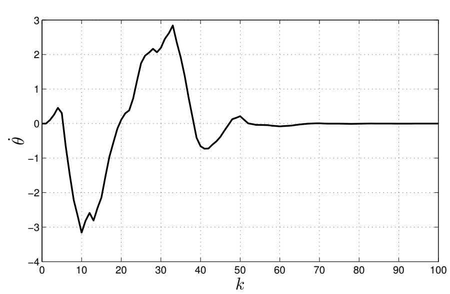

Fig. 6. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: The response of $\dot{\theta}$.

V. Conclusion

This paper proposes the $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control synthesis methods for NCSs with time delays modeled as Markov chains. The designed two-mode-dependent controller can reduce the conservativeness. The formulated closed-loop system is a special jump linear system, which cannot be converted to the classical MJLS. The $\mathcal{H}_2$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problems are transformed to the optimization problems in terms of LMIs with nonconvex constraints. Numerical examples show the advantage of two-mode-dependent controller and also verify the proposed methods. It will be interesting to develop the robust $\mathcal{H}_\infty$ two-mode-dependent control design for networked systems with model uncertainties.

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