# Stability of Impulsive Systems driven by Renewal Processes

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*Abstract*— Necessary and sufficient conditions are provided for stochastic stability and mean exponential stability of impulsive systems with jumps triggered by a renewal process, that is, the intervals between jumps are independent and identically distributed. The conditions for stochastic stability can be efficiently tested in terms of the feasibility of a set of LMIs or in terms of an algebraic test. The relation between the different stability notions for this class of systems is also discussed. The results are illustrated through their application to the stability analysis of networked control systems. We present two benchmark examples for which one can guarantee stability for inter-sampling times roughly twice as large as in a previous paper.

## I. INTRODUCTION

We consider a linear impulsive system taking the form

$$\begin{aligned}
\dot{x}(t) &= Ax(t), \quad t \neq t_k \\
x(t_k) &= Jx(t_k^-), \quad x(0) = x_0, \ t_0 \leq 0 < t_1, \\
&\qquad\qquad t \in \mathbb{R}_{\geq 0}, \ k \in \mathbb{Z}_{\geq 0},
\end{aligned} \tag{1}$$

where the times between consecutive jumps $\{h_k := t_{k+1} - t_k, k \geq 0\}$ are independent and identically distributed (i.i.d.) with a common cumulative distribution $F$. Following [1], we call (1) an impulsive system driven by a renewal process, motivated by the fact that the process

$$N(t) := \max\{k \in \mathbb{Z}_{\geq 0} : t_k \leq t\} \tag{2}$$

that counts the number of jumps up to time $t$ is a renewal process [2].

Pioneering work on systems with i.i.d. intervals between sampling times can be found, e.g., in [3] and [4]. Necessary and sufficient conditions for the stability of (1) are known in the literature for the particular cases where the interval between jumps is constant and exponentially distributed (e.g. [5]). A nonlinear version of (1), is considered in [1], and *sufficient* conditions are provided for a stability definition that implies mean exponential stability. In [6], a class of networked control systems with i.i.d. transmission times is considered, and *sufficient* conditions are given for almost sure stability and mean-square stability.

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In the present paper we provide *necessary and sufficient* conditions for stochastic stability and mean exponential stability of the system (1). These stability definitions are similar to other definitions in the literature (e.g. [7]). The conditions for stochastic stability are given both in terms of a feasibility test of a set of LMIs and in terms of an algebraic test. These conditions are shown to be related to the stability of a properly defined discrete-time system. Regarding mean exponential stability, we use a stochastic Lyapunov function approach [8]. The relation between stochastic stability, mean exponential stability, and mean square stability is discussed for the class of systems considered. Although the approaches used to establish stochastic stability and mean exponential stability are different, we show that these two stability notions are equivalent in many cases of interest.

To illustrate the use of the stability conditions, we consider a networked control system in which a remote controller receives and processes sensor information and sends actuation signals through a communication network with stochastic inter-sampling times [1]. The equations of the feedback connection can be modeled by an extension of the system (1), in which the matrix $J$ depends on the index $k$. Besides illustrating the results, this example allow us to verify that the results in [1] were indeed conservative. In fact, with the results in the present paper one can guarantee stability for inter-sampling roughly twice as large as in [1].

The paper is organized as follows. Section II presents some preliminaries. In Section III we state and prove our main results concerning stochastic and mean exponential stability of (1), and discuss the relation between the stability notions. Section IV presents further conditions for stochastic stability of (1) and extensions of the results to the case of index dependent jump matrices. An example is presented in Section V followed by the conclusions in Section VI.

*Notation:* For a given matrix $A$, its transpose is denoted by $A^T$, its adjoint by $A^*$, its spectral radius by $\sigma(A)$, and its trace by $\text{tr}(A)$. The kronecker product is denoted by $\otimes$.

## II. PRELIMINARIES

In addition to the process $x(t)$ defined in (1), it is useful to define a timer process $\tau(t)$, which keeps track of the time elapsed since the last jump. The augmented process $\mathbf{x}(t) = [x(t)^T \ \tau(t)]^T \in \mathbb{R}^n \times \mathbb{R}$ is then defined as

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{\tau}(t) \end{bmatrix} &= \begin{bmatrix} Ax(t) \\ 1 \end{bmatrix}, \quad t \neq t_k \\
\begin{bmatrix} x(t_k) \\ \tau(t_k) \end{bmatrix} &= \begin{bmatrix} Jx(t_k^-) \\ 0 \end{bmatrix}, \ \begin{bmatrix} x(0) \\ \tau(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \tau_0 \end{bmatrix} \ t_0 \leq 0 < t_1 \\
&\qquad\qquad t \in \mathbb{R}_{\geq 0}, \ k \in \mathbb{Z}_{\geq 0},
\end{aligned} \tag{3}$$

where we denote by $t_0$ the last jump time before the initial time $t = 0$, that is, $-\tau_0 = t_0 \leq 0 < t_1$. The process $\mathbf{x}(t)$ can be shown to be a Markov Process [9]. For simplicity, we assume constant initial conditions $(x_0, \tau_0)$ but there is no difficulty in considering an initial distribution for $(x_0, \tau_0)$.

We denote by $(\Omega, \mathcal{B}, \mathcal{P})$ the underlying probability space of the stochastic process $\mathbf{x}(t; \omega)$, $\omega \in \Omega$. The dependence of $\mathbf{x}$ on $\omega$ will often be dropped, as for other random variables. The expected value will be denoted by $\mathbb{E}(.)$ and the probability of a given event $A$ by $\text{Prob}[A] = \mathcal{P}[A]$.

All stochasticity in $\mathbf{x}(t; \omega)$ derives from the random variables $h_k(\omega)$ that determine the time intervals between consecutive jumps. These variables are assumed i.i.d. with a common distribution $F$ that has support on a given interval $[0, T], T \in \mathbb{R}_{>0} \cup \{+\infty\}$, which means that

$$0 < F(x) < 1, \quad x \in [0, T), \quad F(T) = 1. \quad (4)$$

We assume some regularity on $F$, namely that this function can be written as $F = F_1 + F_2$, where $F_1$ is an absolutely continuous function $F_1(x) = \int_0^x f(s)ds$, for some density function $f(x) \geq 0$, and $F_2$ is a piecewise constant increasing function that captures possible atom points $\{a_i > 0\}$, where the cumulative distribution places mass $\{w_i\}$. The integral with respect to the monotone function $F$ is then defined as

$$\int_0^T G(x)F(dx) = \int_0^T G(x)f(x)dx + \sum_i w_i G(a_i),$$

where $G(x)$ is generally a matrix-valued function.

For absolutely continuous distributions $F(x) = F_1(x)$, the *hazard rate* $\lambda(x)$ is defined as,

$$\lambda(x) = \frac{-S'(x)}{S(x)}, x \in [0, T), \quad (5)$$

where $S(x)$ is the *survivor function* $S(x) = 1 - F(x)$.

We define the random variable $h_{\tau_0}(\omega) = t_1$ to be the time interval between the initial time $t = 0$ and the time of the next jump $t_1$. The distribution of this random variable is determined by the survivor function $S_{h_{\tau_0}}(x)$ given by

$$S_{h_{\tau_0}}(x) := \text{Prob}[h_{\tau_0} > x] = \text{Prob}[h_0 > x + \tau_0 | h_0 > \tau_0]$$
$$= \frac{S(x + \tau_0)}{S(\tau_0)}.$$

The value at time $t$ of a sample path of (1) starting at an initial condition $(x_0, \tau_0)$ is denoted by $x(t; x_0, \tau_0)$ and is given by

$$x(t; x_0, \tau_0) = \Phi_{\tau_0}(t)x_0, \quad (6)$$

where the transition function $\Phi_{\tau_0}(t)$ is given by

$$\Phi_{\tau_0}(t) := \Phi(t, 0) := \begin{cases} \Phi(t, t_r)\Phi(t_r, t_1)\Phi(t_1, 0), & r > 1 \\ \Phi(t, t_1)\Phi(t_1, 0), & r = 1 \\ \exp(At), & r = 0 \end{cases}, \quad (7)$$

where $r = \max\{k \in \mathbb{Z}_{\geq 0} : t_k \leq t\}$ and

$$\Phi(t, t_r) := \exp(A(t - t_r)), \quad \Phi(t_1, 0) := J\exp(Ah_{\tau_0}),$$
$$\Phi(t_r, t_1) := \Pi_{j=1}^{r-1}\Phi(t_{j+1}, t_j), \quad \Phi(t_{j+1}, t_j) := J\exp(Ah_j).$$

We consider three stability notions for (1), which are consistent with those appearing in the literature (e.g.,[7]).

*Definition 1:* The system (1) is said to be

(i) *Mean Square Stable (MSS)* if for any $(x_0, \tau_0)$,

$$\lim_{t \to +\infty} \mathbb{E}[x(t; x_0, \tau_0)^T x(t; x_0, \tau_0)] = 0.$$

(ii) *Stochastic Stable (SS)* if there exists a positive constant $\kappa(x_0, \tau_0)$ such that for any $(x_0, \tau_0)$

$$\int_0^{+\infty} \mathbb{E}[x(t; x_0, \tau_0)^T x(t; x_0, \tau_0)]dt < \kappa(x_0, \tau_0).$$

(iii) *Mean Exponentially Stable (MES)* if for any $(x_0, \tau_0)$

$$\exists_{c,\alpha} \mathbb{E}[x(t; x_0, \tau_0)^T x(t; x_0, \tau_0)] \leq c \exp(-\alpha t)x_0^T x_0, \forall_{t \geq 0}.$$

## III. MAIN RESULTS

In this section we provide conditions for stochastic stability and mean exponential stability of the system (1) and discuss the relation between them. The proofs are deferred to Subsection III-D.

### A. Stochastic Stability

The following theorem provides testable necessary and sufficient conditions for stochastic stability of (1) and is the main result of the paper.

*Theorem 2:* Consider the following conditions

$(A)$ $M(s) := \int_0^s \exp(Aw)^T \exp(Aw)dw$ is $F-$integrable

   i.e., $\int_0^T M(s)F(ds)$ is bounded ;

$(B)$ $\exists_{P>0} : \int_0^T (J\exp(As))^T PJ\exp(As)F(ds) - P < 0;$

$(C)$ $\sigma(H) < 1$, where

$$H := \int_0^T (J\exp(As))^T \otimes (J\exp(As))^T F(ds).$$

Then the following three statements are equivalent

i) The system (1) is stochastic stable.

ii) (A) and (B) hold.

iii) (A) and (C) hold.

Notice that condition (B) is an LMI condition since given a basis $\{B_i\}, i \in \{1, \ldots, m := \frac{n(n+1)}{2}\}$ for the linear space of $n \times n$ symmetric matrices, we can express $P = \sum_{i=1}^m \pi_i B_i$ and rewrite (B) as $\exists_{\pi_i \in \mathbb{R}: i \in \{1, \ldots, m\}}$ :

$$\sum_{i=1}^m \pi_i(\int_0^T (J\exp(As))^T B_i J\exp(As)F(ds) - B_i) < 0, \quad (8)$$
$$\sum_{i=1}^m \pi_i B_i > 0$$

The integrals involved in (C) and (8) may be efficiently computed numerically, and in some cases even symbolically.

The need for the condition (A) arises from the fact that, when (i) the impulsive system matrix $A$ is unstable, (ii) the interval distribution does not have finite support, i.e., $T = +\infty$, and (iii) the jumps are somewhat infrequent, then it

might not be possible for the impulsive system to be stable even with a full reset at jump times: $J = 0$. To gain intuition on why (A) is needed and on why (B) and (C) could not suffice, suppose that $J = 0$ (in which case (B) and (C) hold trivially), $T = \infty$ and, for simplicity, take $\tau_0 = 0$. Then, $\mathbb{E}(x(t)^T x(t))$ can be computed as

$$\mathbb{E}(x(t)^T x(t)) = \sum_{k=0}^{\infty} \mathbb{E}(x(t)^T x(t)|N(t) = k)\text{Prob}[N(t) = k] \quad (9)$$
$$= \mathbb{E}(x(t)^T x(t)|N(t) = 0)\text{Prob}[N(t) = 0]$$
$$= x_0^T \exp(At)^T \exp(At)x_0 S(t)$$

Suppose also that $F(s)$ is continuously differentiable $F'(s) = -S'(s) = f(s)$ and that we integrate condition (A) by parts, yielding

$$\int_0^T M(s)(-S'(s))ds = -M(s)(S(s))|_{s=0}^{s=T} + \int_0^T M'(s)(S(s))ds$$
$$= \int_0^T \exp(As)^T \exp(As)S(s)ds, \quad (10)$$

where we used the fact that $M(0) = 0$ and $S(T) = 0$. It is then clear that, for this particular case, condition (A) must hold for $\int_0^T \mathbb{E}(x(s)^T x(s))ds$ to be bounded, and therefore for (1) to be stochastic stable.

### B. Mean Exponential Stability

The next theorem provides necessary and sufficient conditions for mean exponential stability of (1). We assume in this section that $F$ is continuously differentiable and therefore that the hazard rate $\lambda(x)$ is continuous.

*Theorem 3:* The system (1) is mean exponentially stable if and only if there exists a symmetric matrix-valued function $P(\tau)$, $\tau \in [0, T)$ and constants $c_1 > 0, c_2 > 0$ such that for every $S_1 > 0, S_2 > 0$

$$c_1 I < P(\tau) < c_2 I, \quad (11)$$
$$\dot{P}(\tau) = -S_1 - A^T P(\tau) - P(\tau)A - \lambda(\tau)(J^T P(0)J - P(\tau) + S_2). \quad (12)$$

The proof, which considers a Lyapunov function taking the form $x(t)^T P(\tau(t))x(t)$, is omitted due to space limitations. This theorem provides further insight on the stability of (1), but the conditions are not as straightforward to check as the conditions of Theorem 2. However, in the special case where the intervals between jumps are exponentially distributed, these conditions become simpler. This is stated in the next corollary.

*Corollary 4:* The system (1) with $F(x) = 1 - \exp(\lambda x)$ is mean exponentially stable if and only if

$$\forall_{S>0} \exists_{P>0} : A^T P + PA + \lambda(J^T PJ - P) = -S.$$

In the next section we show that in some cases of interest, mean exponential stability is equivalent to stochastic stability and therefore we can use the conditions of Theorem 3 to test mean exponential stability.

### C. Relation between stability notions

The following example shows that, in general, the stability notions of Definition 1 are not equivalent for the system (1).

*Example 5:* Consider the system (1) with $A = a > 0$, $J = j \in \mathbb{R}$, and suppose that the survivor function of the time intervals between jumps is given by $S(t) = \exp(-2at)\frac{1}{t+1}$, $t \in [0, T)$, $T = +\infty$. The condition (A) of Theorem 2, which can be tested using (10), does not hold since $\int_0^\infty \exp(2as)S(s)ds = +\infty$ and therefore the system in not stochastic stable. However, making $j = 0$, considering an arbitrary initial condition $(x_0, \tau_0)$, and using a similar reasoning to (9), we have that $\mathbb{E}(x^2(t)) = \exp(2at)\frac{S(t+\tau_0)}{S(\tau_0)}x_0^2 = \frac{\tau_0+1}{t+\tau_0+1}x_0^2$, which tends to zero for any $\tau_0 \in [0, T)$, and therefore the system is mean square stable.

If we have instead $S(t) = \exp(-2at)\frac{1}{t^2+1}$, condition (A) of Theorem 2 holds and condition (B) of the same theorem also holds for sufficiently small $j$, and therefore the system is stochastic stable when $j = 0$. However, making $j = 0$ and $\tau_0 = 0$, $\mathbb{E}(x^2(t)) = \frac{1}{t^2+1}x_0^2$ does not decrease exponentially, and therefore the system is not mean exponentially stable.

From the Definition 1 it is evident that mean exponential stability implies stochastic stability, but the previous example shows that the converse implication does not hold in general. The next lemma states equivalence between the two stability notions for an important subclass of systems.

*Lemma 6:* Suppose the cumulative distribution between jumps $F$ is continuously differentiable and has finite support, that is, $T < +\infty$. Then the system (1) is stochastic stable if and only if it is mean exponentially stable.

Another important case for which the different notions of stability collapse is that of exponentially distributed time intervals between jumps, $F(x) = 1 - \exp(-\lambda x)$, $x \in [0, T)$, $T = +\infty$. In this case, the distribution does not have finite support but, as stated in the following Lemma, the three stability notions in Definition 1 are equivalent.

*Lemma 7:* Suppose the cumulative distribution between jumps is exponential, that is, $F(x) = 1 - \exp(\lambda x)$. Then the system (1) is stochastic stable if and only if it is mean exponentially stable and if and only if it is mean square stable.

### D. Proofs

Due to space limitations, we prove only Theorem 2. As a preliminary to the proof, we introduce the following function from the space of $n \times n$ matrices into itself

$$L(U) = \int_0^T (J\exp(As))^T UJ\exp(As)F(ds), \quad (13)$$

and we denote by $L^k(U)$ the composition obtained by applying $k$ times this function, e.g., $L^2(U) = L(L(U))$. By convention $L^0(U) = U$. We denote by $\nu$ the operator that transforms a matrix into a vector $\nu(A) = \nu([a_1 \ldots a_n]) = [a_1^T \ldots a_n^T]^T$. Using the property [10, Lemma 4.3.1]

$$\nu(AXB) = (B^T \otimes A)\nu(X), \quad (14)$$

we obtain $\nu(L(U)) = H\nu(U)$, or equivalently,

$$L(U) = \nu^{-1} \circ H \circ \nu(U), \qquad (15)$$

where $H$ is described in (C) of Theorem 2.

We denote by $S_+^n(\mathbb{C})$ the space of $n \times n$ self-adjoint positive semi-definite complex matrices. We need the following proposition which can be derived from [11, Prop. 1].

*Proposition 8:* For $B \in \mathbb{R}^{n^2 \times n^2}$ the following are equivalent

(a) $\sigma(B) < 1$
(b) $\lim_{k \to +\infty} B^k \nu(X) = 0$, for every $X \in S_+^n(\mathbb{C})$.

*Proof:* (of Theorem 2) We start by characterizing further (i). By definition, the system (1) is stochastic stable if and only if for any $(x_0, \tau_0)$, $x_0^T P(\tau_0) x_0$ is bounded, where

$$P(\tau_0) := \int_0^{+\infty} \mathbb{E}[\Phi_{\tau_0}(s)^T \Phi_{\tau_0}(s)] ds, \ \ \tau_0 \in [0, T), \quad (16)$$

and $\Phi_{\tau_0}(s)$ is given by (7). The expectation is taken with respect to the independent random variables $h_{\tau_0}, \{h_k, k \geq 1\}$. Computing separately the expectation with respect to $h_{\tau_0}$, defining $F_{\tau_0}(x) := \frac{F(x+\tau_0)}{S(\tau_0)}$, $x \in [0, T - \tau_0)$, and noticing that for a scalar function $g(x)$

$$\mathbb{E}_{h_{\tau_0}}(g(h_{\tau_0})) = \int_0^{T-\tau_0} g(s) F_{\tau_0}(ds) \leq \int_0^T g(s) \frac{F(ds)}{S(\tau_0)}$$
$$= \frac{\mathbb{E}_{h_0}(g(h_0))}{S(\tau_0)},$$

we get

$$\mathbb{E}_{h_{\tau_0}}[x_0^T \mathbb{E}_{\{h_k, k \geq 1\}}[\Phi_{\tau_0}(s)^T \Phi_{\tau_0}(s)] x_0] \leq$$
$$\frac{(\mathbb{E}_{\{h_k, k \geq 0\}}(x_0^T \Phi_{\tau_0=0}(s)^T \Phi_{\tau_0=0}(s)) x_0)}{S(\tau_0)},$$

and therefore $x_0^T P(\tau_0) x_0 \leq \frac{x_0^T P(0) x_0}{S(\tau_0)}$. Notice that (4) implies $S(\tau_0) > 0, \tau_0 \in [0, T)$. It is then clear that (i) holds if and only if $P := P(0)$ is bounded. Suppose this is the case, i.e., that (i) holds and therefore that $P$ is bounded. We notice that we can interchange the integral and expectation operations in the expression for $P$, that is, (16) with $\tau_0 = 0$. In fact, denoting by $M(s) := \Phi_{\tau_0=0}(s)^T \Phi_{\tau_0=0}(s)$, we have that the diagonal entries of $M(s)$ verify $M_{ii}(s) \geq 0$, and are integrable since $P$ is bounded, and therefore we can apply the Fubini Theorem. Since $M(s)$ is positive semidefinite the off-diagonal elements are dominated by the diagonal elements and therefore are also integrable. Besides interchanging the integral and expectation operations, suppose we also partition the region of integration into the intervals $[t_k, t_{k+1})$, yielding

$$P = \mathbb{E}[\sum_{k=0}^{+\infty} \int_{t_k}^{t_{k+1}} \Phi(s,0)^T \Phi(s,0) ds] \qquad (17)$$

$$= \mathbb{E}[\sum_{k=0}^{+\infty} \Phi(t_k,0)^T (\int_{t_k}^{t_{k+1}} \Phi(s,t_k)^T \Phi(s,t_k) ds) \Phi(t_k,0)] \quad (18)$$

$$= \sum_{k=0}^{+\infty} \mathbb{E}[\Phi(t_k,0)^T U \Phi(t_k,0)] \qquad (19)$$

where

$$U := \mathbb{E}[\int_{t_k}^{t_{k+1}} \Phi(s,t_k)^T \Phi(s,t_k) ds]$$
$$= \int_0^T [\int_0^s \exp(Aw)^T \exp(Aw) dw] F(ds)$$

is positive definite, and the remaining matrices are defined in (7). The interchange of the expected value and summation in (18) can be proved to be valid, using the Lebesgue monotone convergence theorem and similar arguments to the ones just used for the interchange of the integral and expectation in (16). It is clear from (19) that $U$ must be bounded, or equivalently (A) must hold. Due to the fact that $h_k$ are i.i.d, (19) can be written as

$$P = \sum_{k=0}^{+\infty} L^k(U), \qquad (20)$$

where $L$ is defined by (13). This expression reveals that (i) holds if and only if the summation (20) is bounded.

To prove that (i) implies (ii), notice that

$$P = U + \sum_{k=1}^{+\infty} L^k(U) = U + L(\sum_{k=0}^{+\infty} L^k(U)) = U + L(P),$$

from which we conclude that $L(P) - P = -U < 0$, i.e., (ii).

To prove that (ii) implies (iii) we consider the system

$$x_{k+1} = H^T x_k, \ \ X_{k+1} = L^*(X_k), \ \ \nu(X_k) = x_k, \ \ X_0 \in S_+^n(\mathbb{C}), \qquad (21)$$

where $L^*(X_k) := \nu^{-1} \circ H^T \circ \nu(X_k)$ is given by

$$L^*(X_k) = \int_0^T J \exp(As) X_k (J(\exp(As))^T F(ds),$$

and for any $Y, Z \in S_+^n(\mathbb{C})$ verifies

$$\text{tr}(L(Z)^* Y) = \text{tr}(Z^* L^*(Y)). \qquad (22)$$

Notice that $X_k \geq 0$ for a given $k$ implies $X_{k+1} = L^*(X_k) \geq 0$, and therefore, by induction, we conclude that $X_k \geq 0$ for all $k$. We show that (ii) implies that this system is stable by considering a Lyapunov function $V(x_k) = \text{tr}(P\nu^{-1}(x_k))$ for (21), where $P$ verifies (B), that is $P > 0$ and $L(P) - P < 0$. In fact, this function $V$ is radially unbounded and positive definite for $X_k \geq 0$, and verifies $V(0) = 0$. Using (B) and (22), we have that for any $X_k \in S_+^n(\mathbb{C}) - \{0\}$

$$V(x_{k+1}) - V(x_k) = \text{tr}(P\nu^{-1}(H^T \nu(X_k))) - \text{tr}(P X_k)$$
$$= \text{tr}(PL^*(X_k)) - \text{tr}(P X_k) = \text{tr}(L(P) X_k) - \text{tr}(P X_k)$$
$$= \text{tr}((L(P) - P) X_k) = -\text{tr}(Z X_k) < 0,$$

where $Z := -(L(P) - P) > 0$. Therefore (21) is stable for any symmetric positive semidefinite initial condition $X_0 \in S_+^n(\mathbb{C})$, which by Proposition 8 is equivalent to $\sigma(H) = \sigma(H^T) < 1$ which is (iii).

Finally, to prove that (iii) implies (i), notice that $\sigma(H) < 1$ implies that

$$(I - H)^{-1} = \sum_{k=0}^{+\infty} H^k(I) \qquad (23)$$

is bounded [10, Theorem 6.2.8]. By applying $\nu^{-1}(.)\nu$ on both sides of (23) we conclude that $(I-L)^{-1} = \sum_{k=0}^{+\infty} L^k(I)$ is also bounded. Since $\exists_c : \sum_{k=0}^{+\infty} L^k(U) < c \sum_{k=0}^{+\infty} L^k(I)$, this implies that (20) is bounded. We can invert the steps (20)-(16) showing that $x_0^T P(\tau_0)x_0$ is bounded and therefore (i) holds, which concludes the proof. ∎

## IV. EQUIVALENT CONDITIONS AND EXTENSIONS

In this section we further characterize the stochastic stability of system (1) by providing conditions in terms of the stability of two suitably constructed discrete-time stochastic processes. These discrete-time stochastic processes are obtained by sampling the system (1) at jump times $z_{1k} = x(t_k)$ and $z_{2k} = x(t_k^-)$ and can be described by

$$
\begin{aligned}
z_{1k+1} &= J \exp(Ah_k)z_{1k} \\
z_{2k+1} &= \exp(Ah_k)J z_{2k}
\end{aligned}
\tag{24}
$$

The notions of stability for a discrete-time stochastic process are analogous to those in Definition 1, provided that one replaces the continuous time $t \in \mathbb{R}$ by the discrete time $k \in \mathbb{N}$.

*Theorem 9:* Consider the following set of four matrix-valued functions

$$
\mathcal{K} = \{J \exp(Ah), \exp(Ah)J, (J \exp(Ah))^T, (\exp(Ah)J)^T\}.
$$

and suppose that the condition (A) of Theorem 2 holds. Then the following statements are equivalent

i) The system (1) is stochastic stable.
ii) The discrete-time system

$$
z_{k+1} = K(h_k)z_k,
\tag{25}
$$

is V-stable where $h_k$ are the time intervals described in Section I, $K(h)$ can be any of the matrices of the set $\mathcal{K}$, and V can be replaced by any of the stability notions under consideration: mean exponential, stochastic, or mean square.
iii) $\forall_{Q>0}, \exists_{P>0} : L(P) - P = -Q$, where $L$ can be any of the operators $\{\int_0^T K(s)^T P K(s) F(ds) : K \in \mathcal{K}\}$
iv) $\sigma(H) < 1$ where $H$ can be any of the matrices $\{\int_0^T K(s)^T \otimes K(s)^T F(ds) : K \in \mathcal{K}\}$.

Regarding statement ii), assuming that the condition (A) of Theorem 2 holds, stability of (25) for any one of the stability notions and for any one of the matrices $K$ in the set $\mathcal{K}$ suffices to guarantee stochastic stability of (1) and, in fact, also guarantees stability of (25) for all other stability notions and for all other matrices $K$ in $\mathcal{K}$.

### A. Index Dependent Jump Matrices

In this section we extend Theorem 2 to the case of jump matrices that depend on the jump index $k$, i.e.,

$$
\begin{aligned}
\dot{x}(t) &= Ax(t), \quad t \neq t_k \\
x(t_k) &= J_k x(t_k^-), \quad x(0) = x_0, \ t_0 \leq 0 < t_1,
\end{aligned}
\tag{26}
$$
$$
t \in \mathbb{R}_{\geq 0}, \ k \in \mathbb{Z}_{\geq 0},
$$

where the times between consecutive jumps $h_k := t_{k+1} - t_k$ are still as defined in Section I. We assume that the dependence of $J_k$ on $k$ is $K$ periodic, that is, $J_k = J_{k+K}, \forall_k$. The next lemma extends Theorem 2 to the periodic system (26) and is used in the example presented in the next Section.

*Lemma 10:* The following are equivalent

i) The system (26) is stochastic stable.
ii) (A) of Theorem 2 holds and

$$
\exists_{P_r>0} : \int_0^T (J_r \exp(As))^T P_{[r+1]} J_r \exp(As)F(ds) - P_{[r]} < 0;
$$

where $r \in \{1,\ldots,K\}$, $[r] = 1$ if $r = K + 1$, $[r] = r$ otherwise .
iii) (A) of Theorem 2 holds and $\sigma(H) < 1$, where for $D_r = \int_0^T (J_r \exp(As))^T \otimes (J_r \exp(As))^T F(ds)$,

$$
H = \begin{bmatrix}
0 & D_1 & 0 & \ldots & 0 \\
0 & 0 & D_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & D_{K-1} \\
D_K & 0 & \ldots & 0 & 0
\end{bmatrix}.
$$

## V. EXAMPLE

We illustrate the results of the present paper by eliminating the conservativeness of the results in [1] for two benchmark problems. In [1], a networked control system is considered, for which a plant and a remote controller are connected through a communication network. The linear plant and remote linear controller take the form:

$$
\dot{x}_P = A_P x_P + B_P \hat{u} \qquad y = C_P x_P
\tag{27}
$$

$$
\dot{x}_C = A_C x_C + B_C \hat{y} \qquad u = C_C x_C + D_C \hat{y}
\tag{28}
$$

where $x_P$ and $x_C$ are the states of the plant and the controller; $\hat{u}$ and $y$ the plant's input and output; $\hat{y}$ and $u$ the controller's input and output. Ignoring network delay, between the sampling times $\{t_k, k \in \mathbb{Z}_{\geq 0}\}$ both $\hat{u}$ and $\hat{y}$ are held constant

$$
\hat{u}(t) = \hat{u}(t_k), \ \ \hat{y}(t) = \hat{y}(t_k), t \in [t_k, t_{k+1}), \ \ k \in \mathbb{Z}_{\geq 0}
$$

The components of the signals $u(t)$ and $y(t)$ are not necessarily both sampled and sent to the network at every sampling time. Defining

$$
e = \begin{bmatrix} e_u \\ e_y \end{bmatrix} = \begin{bmatrix} \hat{u} - u \\ \hat{y} - y \end{bmatrix},
$$

this is captured by

$$
\begin{bmatrix} e_u(t_k) \\ e_y(t_k) \end{bmatrix} = \begin{bmatrix} R_k^u & 0 \\ 0 & R_k^y \end{bmatrix} \begin{bmatrix} e_u(t_k^-) \\ e_y(t_k^-) \end{bmatrix},
$$

where the matrices $R_k^u = \mathrm{diag}([\pi_{1k} \ldots \pi_{nk}])$, $\pi_{jk} \in \{0,1\}$, and $R_k^y = \mathrm{diag}([\kappa_{1k} \ldots \kappa_{nk}])$, $\kappa_{jk} \in \{0,1\}$ choose what controller and plant outputs, respectively, are sampled at each sampling time and therefore specify the network protocol.

## A. Batch Reactor

This example considers the control of a linearized model of an open loop unstable batch reactor, described by (27), where

$$A_P = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix},$$

$$B_P = \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}, \quad C_P = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

As in [1], we assume that only the outputs are sent through the network, using a round-robin protocol, meaning that the matrix $R_k^y$ that specifies the network protocol is given by $R_k^y = R_0$, when $k$ even and $R_k^y = R_1$, when $k$ odd, where

$$R_0 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R_1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The intervals $h_k$ between consecutive output sampling times $t_k$ are assumed i.i.d. and following a distribution $F$. The system is controller by a PI controller, described by (28), where

$$A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_C = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}, D_C = \begin{bmatrix} 0 & -2 \\ 5 & 0 \end{bmatrix}.$$

The dynamic equations for $x := [x_P^T \ x_C^T \ e_y^T]^T$ take the form (1) with

$$A = \begin{bmatrix} A_{xx} & A_{xe} \\ A_{ex} & A_{ee} \end{bmatrix}, \qquad J_k = \begin{bmatrix} I & 0 \\ 0 & R_{[k]} \end{bmatrix}$$

where $[k] = 0$, when $k$ even and $[k] = 1$, when $k$ odd, and

$$A_{xx} = \begin{bmatrix} A_P + B_P D_C C_P & B_P C_C \\ B_C C_P & A_C \end{bmatrix}, A_{xe} = \begin{bmatrix} B_P D_C \\ B_C \end{bmatrix},$$

$$A_{ex} = \begin{bmatrix} -C_P & 0 \end{bmatrix} A_{xx}, \qquad A_{ee} = \begin{bmatrix} -C_P & 0 \end{bmatrix} A_{xe}.$$

To compare our results with the ones in [1] we consider uniformly and exponentially distributed time intervals $h_k$. Notice that in these two cases stochastic stability and mean exponential stability are equivalent and therefore we can use the computationally efficient conditions in Lemma 10. We note that the nomenclature used in [1] is mean square stability, but the results provided there are, in fact, sufficient conditions for mean exponential stability as defined in the present paper, and therefore the result are comparable. The results are summarized in Table I.

### TABLE I
STABILITY CONDITIONS FOR THE BATCH REACTOR EXAMPLE

| | Necessary and sufficient conditions | Results taken from [1] |
|---|---|---|
| Maximum support $T$ of Uniform Distribution | 0.112 | 0.0517 |
| Maximum average $\lambda$ of Exponential Distribution | 0.0417 | 0.0217 |

## B. CH-47 Tandem-Rotor Helicopter

The second example regards the control of a CH-47 tandem-rotor helicopter. The example is completely analogous to the previous one except that the controller is static. The system is described by (27) with

$$A_P = \begin{bmatrix} -0.02 & 0.005 & 2.4 & -32 \\ -0.14 & 0.44 & -1.3 & -30 \\ 0 & 0.018 & -1.6 & 1.2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B_P = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix},$$

$$C_P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 57.3 \end{bmatrix}$$

and the controller is described by

$$u = D_C \hat{y}, D_C = \begin{bmatrix} -12.7177 & -45.0824 \\ 63.5163 & 25.9144 \end{bmatrix}$$

The results are shown in Table (II).

### TABLE II
STABILITY CONDITIONS FOR THE CH-47 HELICOPTER EXAMPLE

| | Necessary and sufficient conditions | Results taken from [1] |
|---|---|---|
| Maximum support $T$ of Uniform Distribution | $3.11 \times 10^{-3}$ | $1.48 \times 10^{-3}$ |
| Maximum average $\lambda$ of Exponential Distribution | $1.21 \times 10^{-3}$ | $6.21 \times 10^{-4}$ |

## VI. CONCLUSIONS AND FUTURE WORK

The analysis of linear impulsive system with i.i.d. time intervals between consecutive jumps was considered, motivated by the application of this class of systems in networked control systems. Necessary and sufficient conditions were provided for stochastic stability and mean exponential stability, and the relation between the two definitions was addressed. Natural directions for future work include considering mean square stability and analyzing the system (1) with the introduction of a control input.

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