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Recursive least squares identification for multirate multi-input single-output systems

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***Abstract*— This paper derives state-space models for multirate multi-input sampled-data systems. Based on the corresponding transfer function models, an auxiliary model based recursive least squares algorithm is presented to identify the model parameters of the multirate systems from the multirate input-output data. Further, convergence properties of the proposed algorithm are analyzed. An illustrative example is given.**

I. INTRODUCTION

THE conventional sampled-data control systems assume that the input updating period is equal to the output sampling period. However, in many industrial processes this is not such a case due to various limitations such as delays in sensors and laboratory analysis [1], [2]. For example, for a polymer reactor, the composition, density or molecular weight distribution measurements are typically obtained after several minutes, while the manipulated variables can be adjusted at relatively fast rates [3]. This leads to a dual-rate/multirate system. A natural question is how to establish the mathematical models and to estimate the model parameters for such a multirate system from multirate input-output data, which is the focus of this work.

In the area of multirate system identification, much work has been published. For example, Lu and Fisher used the projection and least-squares algorithms to estimate the parameters and intersample outputs for dual-rate deterministic systems [4], [5], Ding et al used the polynomial transformation technique to deal with the identification problem for dual-rate stochastic systems [6] and [7] discussed the parameter estimation of multirate multi-input multi-output systems also using the polynomial transformation techniques. By means of the lifting technique, Li et al [8] and Wang et al [9] used the available system states and the multirate input-output data to estimate parameters of the lifted state-space models for multirate systems. Also, Ding and Chen presented the combined parameter and state estimation algorithms of the lifted state-space models for general dual-rate systems based on the hierarchical identification principle [10]. However, most identification algorithms for multirate systems reported in the literatures are based on multirate single-input single-output systems. Recently, Shi et al gave a crosstalk identification algorithm for multirate xDSL systems [11] and Yu and Shi solved the $l_2 - l_\infty$ filter problem for lifted multirate systems [12].

The auxiliary model identification principle [13] has been used to solve the identification problem of nonlinear systems [14] and dual-rate systems [15], [16]. This paper considers identification problems for multirate multi-input sampled-data systems also using the auxiliary model identification principle.

The paper is organized as follows. Section II discusses the problem formulation related to a multirate multi-input system. Section III derives a discrete-time state-space model of the multirate multi-input system. Section IV discusses the auxiliary model based recursive least squares identification algorithm for multirate multi-input systems to identify parameters of the transfer function model derived from the state-space model. Section V analyzes the convergence of the parameter estimation given by the proposed algorithm. Section VI provides an illustrative example. Finally, concluding remarks are given in Section VII.

II. PROBLEM FORMULATION

The focus of this paper is a multirate multi-input system – as depicted in Figure 1, where P_c is a multi-input single-output continuous-time process with additive disturbance $v(t)$; the jth input updating period is $T_j := p_j h$, the input $u_j(t)$ to P_c is produced by a zero-order hold H_{T_j} with period T_j, processing a discrete-time signal $u_j(kT_j)$; $w(t)$ is the true output of P_c but unmeasurable; the continuous-time output signal $y(t)$ is sampled by a sampler S_T with period $T := qh$, yielding a discrete-time signal $y(kT)$. For convenience, let $p_1, p_2, \cdots, p_r$ be positive integers and q be the least common multiple of $(p_1, p_2, \cdots, p_r)$ [8]. Here, h is called the base period and T the frame period [10].

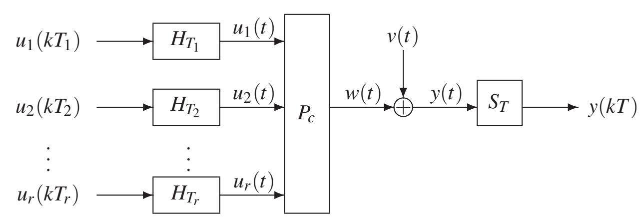

Fig. 1. The multirate multi-input sampled-data system

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Using the properties of the zero-order holds, for $j = 1, 2, \cdots, r$,

$$u_j(t) = u_j(kT_j), \quad kT_j \leqslant t < (k+1)T_j. \tag{1}$$

For such multirate systems, the input-output data available are

$$\{u_1(kT_1), u_2(kT_2), \cdots, u_r(kT_r), y(kT) : k = 1, 2, \cdots\}$$

which are referred to as the multirate measurement input-output data. Especially, for $r = 2$-input system, if $T_1 = 2$ s and $T_2 = 3$ s, the input-output data available are

$$\{u_1(0), u_1(2), u_1(4), u_1(6), \cdots, u_2(0), u_2(3), u_2(6), \cdots, y(0), y(6), y(12), \cdots\}.$$

The system depicted in Figure 1 is a linear periodically time-varying due to different updating and sampling periods [8], [10], [17]–[19]. For such a time-varying multirate multi-input system, our objectives are as follows:

- First, to establish the mathematical model from discrete-time input signals $[u_1(kT_1),\ u_2(kT_2),\ \cdots,\ u_r(kT_r)]$ with different periods to the output $y(kT)$, that is, to find the mapping relationship (state-space model) between the available input and output data.
- Second, by using the auxiliary model (or reference model) identification methods, to propose an algorithm to estimate the parameters of the transfer function model obtained from the state-space model.
- Third, to prove that the parameter estimation error consistently converges to zero under the persistent excitation condition.

III. The State-Space Models of Multirate Systems

Let us introduce some notations first. The symbol I stands for an identity matrix of appropriate sizes; $\mathbf{1}_{n\times m}$ is an $n \times m$ matrix whose elements are 1, $\mathbf{1}_n := \mathbf{1}_{n\times 1}$; the superscript T denotes the matrix transpose; the norm of the matrix X is defined by $\|X\|^2 = \mathrm{tr}[XX^{\mathrm{T}}] = \mathrm{tr}[X^{\mathrm{T}}X]$; let X be a square matrix, the symbols $\lambda_{\max}[X]$ and $\lambda_{\min}[X]$ represent the maximum and minimum eigenvalues of X, respectively; $|X| = \det[X]$ denotes the matrix determinant; E denotes the expectation operator; If $f(k) \to 0$ and $g(k) \to 0$ as $k \to \infty$, $f(k) = O(g(k))$ expresses that there exist constants $\delta_1 > 0$ and k_0 such that $\left|\frac{f(k)}{g(k)}\right| \leqslant \delta_1$ for $k \geqslant k_0$ and $f(k) = o(g(k))$ represents $\frac{f(k)}{g(k)} \to 0$ as $k \to \infty$.

Assume that the continuous-time process P_c has the following state-space model,

$$\begin{cases} \dot{x}(t) = A_c x(t) + \sum\limits_{j=1}^{r} B_{cj} u_j(t), \\ y(t) = Cx(t) + v(t), \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u_j(t)\ (j = 1, 2, \cdots, r)$ is the jth channel control input, $y(t) \in \mathbb{R}^1$ the output, $v(t) \in \mathbb{R}^1$ a stochastic noise with the zero mean and A_c, B_{cj} and C are the matrices of appropriate dimensions.

In order to derive the state-space model of the multirate multi-input system for the available multirate data, it is necessary to discretize P_c via the zero-order hold method with the sampling period T. This is the key point of this work.

Theorem 1: For the multirate multi-input system in Figure 1, let q be the least common multiple of $(p_1, p_2, \cdots, p_r)$, $\nu_j := q/p_j,\ j = 1, 2, \cdots, r$. Then, the state-space model of the multirate multi-input system can be expressed as

$$\begin{cases} x(kT+T) = Ax(kT) + \sum\limits_{j=1}^{r}\sum\limits_{i=1}^{\nu_j} B_{ji} u_j(kT + (i-1)T_j), \\ \quad y(kT) \quad = Cx(kT) + v(kT), \end{cases} \tag{3}$$

where

$$x(kT) := x(t)|_{t=kT}, \quad u_j(kT + iT_j) := u_j(t)|_{t=kT+iT_j},$$
$$y(kT) := y(t)|_{t=kT}, \quad v(kT) := v(t)|_{t=kT},$$

and

$$A_h := \mathrm{e}^{A_c h} \in \mathbb{R}^{n\times n}, \quad A := \mathrm{e}^{A_c qh},$$
$$B_{p_j} := \int_0^{T_j} \mathrm{e}^{A_c t}\mathrm{d}t B_{cj} \in \mathbb{R}^n,$$
$$B_{ji} := A_h^{q-ip_j} B_{p_j} \in \mathbb{R}^n,\ j = 1, 2, \cdots, r,\ i = 1, 2, \cdots, \nu_j.$$

Proof The state solution of (2) is given by

$$x(t) = \mathrm{e}^{A_c(t-t_0)} + \sum_{j=1}^{r}\int_{t_0}^{t} \mathrm{e}^{A_c(t-\tau)} B_{cj} u_j(\tau)\mathrm{d}\tau,$$

where t_0 is the initial time and t the current time.

In order to discretize P_c with the sampling period T, let $t_0 = kT$ and $t = kT + T$ in the above formula, we have

$$x(kT+T) = \mathrm{e}^{A_c T} x(kT) + \sum_{j=1}^{r}\int_{kT}^{kT+T} \mathrm{e}^{A_c(kT+T-\tau)} B_{cj} u_j(\tau)\mathrm{d}\tau. \tag{4}$$

For $j = 1, 2, \cdots, r$ and $i = 1, 2, \cdots, \nu_j$, using the properties of the zero holds in (1) gives

$$u_j(t) = u_j(kT + (i-1)T_j),\ kT + (i-1)T_j \leqslant t < kT + iT_j.$$

Thus

$$\begin{aligned} x(kT+T) &= \mathrm{e}^{A_c qh} x(kT) + \sum_{j=1}^{r}\left[\sum_{i=1}^{\nu_j}\int_{kT+(i-1)T_j}^{kT+iT_j} \mathrm{e}^{A_c(kT+T-\tau)}\mathrm{d}\tau B_{cj} u_j(kT + (i-1)T_j)\right] \\ &= Ax(kT) + \sum_{j=1}^{r}\left[\sum_{i=1}^{\nu_j} \mathrm{e}^{A_c(T-iT_j)}\int_{-T_j}^{0} \mathrm{e}^{-A_c t}\mathrm{d}t B_{cj} u_j(kT + (i-1)T_j)\right] \\ &= Ax(kT) + \sum_{j=1}^{r}\left[\sum_{i=1}^{\nu_j} \mathrm{e}^{A_c(T-iT_j)}\int_{0}^{T_j} \mathrm{e}^{A_c t}\mathrm{d}t B_{cj} u_j(kT + (i-1)T_j)\right] \\ &= Ax(kT) + \sum_{j=1}^{r}\left[\sum_{i=1}^{\nu_j} A_h^{q-ip_j} B_{p_j} u_j(kT + (i-1)T_j)\right] \\ &= Ax(kT) + \sum_{j=1}^{r}\sum_{i=1}^{\nu_j} B_{ji} u_j(kT - (i-1)T_j). \end{aligned} \tag{5}$$

Letting $t = kT$ in (2) gives the discretized output equation

$$y(kT) = Cx(kT) + v(kT). \tag{6}$$

Combining (5) with (6) leads to the results of Theorem 1. □

IV. Algorithm description

This section discusses the auxiliary model based recursive least squares algorithm of the transfer function model corresponding state-space model in (3) of the multirate system.

Let z^{-1} be a unit delay operator: $z^{-1}y(kT) = y(kT - T)$. From (3), we get

$$\begin{aligned} y(kT) &= \sum_{j=1}^{r}\sum_{i=1}^{\nu_j} C(zI-A)^{-1}B_{ji}u_j(kT+(i-1)T_j) \\ &\quad +v(kT) \\ &= \sum_{j=1}^{r}\sum_{i=1}^{\nu_j} \frac{C\,\mathrm{adj}[zI-A]B_{ji}}{\det[zI-A]} u_j(kT+(i-1)T_j) \\ &\quad +v(kT) \\ &= \sum_{j=1}^{r}\sum_{i=1}^{\nu_j} \frac{z^{-n}C\,\mathrm{adj}[zI-A]B_{ji}}{z^{-n}\det[zI-A]} u_j(kT+(i-1)T_j) \\ &\quad +v(kT) \\ &=: \frac{1}{\alpha(z)}\sum_{j=1}^{r}\sum_{i=1}^{\nu_j} \beta_{ji}(z)u_j(kT+(i-1)T_j) \\ &\quad +v(kT) \end{aligned} \tag{7}$$

with $\alpha(z)$ and $\beta_{ji}(z)$ being polynomials in z^{-1} of degree n and both being represented as

$$\begin{aligned} &\alpha(z) = z^{-n}\det[zI-A] \\ &\qquad =: 1 + \alpha_1 z^{-1} + \cdots + \alpha_n z^{-n}, \\ &\alpha_i \in \mathbb{R}^1,\ i = 1,2,\cdots,n; \\ &\beta_{ji}(z) = z^{-n}C\,\mathrm{adj}[zI-A]B_{ji} \\ &\qquad =: \beta_{ji}(1)z^{-1} + \cdots + \beta_{ji}(n)z^{-n}, \\ &\beta_{ji}(l) \in \mathbb{R}^1,\ j = 1,2,\cdots,r,\ i = 1,2,\cdots,\nu_j, \\ &l = 1,2,\cdots,n. \end{aligned}$$

The goal here is to estimate the parameters α_i and $\beta_{ji}(l)$ of the multirate model in (7) from multirate data $\{u_1(kT_1), u_2(kT_2), \cdots, u_r(kT_r), y(kT): k = 0,1,2,\cdots\}$.

Referring to Figure 1, define the unmeasurable inner variable (also called the unknown noise-free output or true output),

$$w(kT) := \frac{1}{\alpha(z)}\sum_{j=1}^{r}\sum_{i=1}^{\nu_j} \beta_{ji}(z)u_j(kT+(i-1)T_j),$$

or

$$\alpha(z)w(kT) = \sum_{j=1}^{r}\sum_{i=1}^{\nu_j} \beta_{ji}(z)u_j(kT+(i-1)T_j). \tag{8}$$

Then from (7), we have

$$y(kT) = w(kT) + v(kT). \tag{9}$$

Define the parameter vector θ and the information vector $\varphi_0(kT)$ as

$$\begin{aligned} \theta &:= [\alpha_1,\cdots,\alpha_n,\beta_{11}(1),\cdots\beta_{11}(n),\cdots, \\ &\quad \beta_{1\nu_1}(1),\cdots,\beta_{1\nu_1}(n),\cdots,\beta_{r1}(1),\cdots,\beta_{r1}(n), \\ &\quad \cdots,\beta_{r\nu_r}(1),\cdots,\beta_{r\nu_r}(n)]^{\mathrm{T}} \in \mathbb{R}^{n_0}, \\ \varphi_0(kT) &:= [\psi_0^{\mathrm{T}}(kT),\psi_u^{\mathrm{T}}(kT)]^{\mathrm{T}} \in \mathbb{R}^{n_0}, \\ n_0 &:= n + n(\nu_1+\nu_2+\cdots+\nu_r), \\ \psi_0(kT) &:= [-w(kT-T),\cdots,\ w(kT-nT)]^{\mathrm{T}} \in \mathbb{R}^n, \\ \psi_u(kT) &:= [u_1(kT-T),\cdots,u_1(kT-nT),\cdots, \\ &\quad u_1(kT+(\nu_1-1)T_j-T),\cdots, \\ &\quad u_1(kT+(\nu_1-1)T_j-nT),\cdots, \\ &\quad u_r(kT-T),\cdots,u_r(kT-nT),\cdots, \\ &\quad u_r(kT+(\nu_r-1)T_r-T),\cdots, \\ &\quad u_r(kT+(\nu_r-1)T_r-nT)]^{\mathrm{T}} \in \mathbb{R}^{n(\nu_1+\nu_2+\cdots+\nu_r)}. \end{aligned}$$

Equations (8) and (9) can be equivalently written as

$$w(kT) = \varphi_0^{\mathrm{T}}(kT)\theta, \quad y(kT) = \varphi_0^{\mathrm{T}}(kT)\theta + v(kT). \tag{10}$$

Notice that $\varphi_0(kT)$ contains unknown inner variables $w(kT - iT)$ $(i = 1,2,\cdots,n)$, so the standard recursive least squares (RLS) algorithm cannot be applied *directly* to obtain the estimate of the parameter vector θ in (10).

Based on the auxiliary model identification principle in [15], the unknown variables $w(kT-iT)$ in $\varphi_0(kT)$ (i.e., in $\psi_0(kT)$) are replaced with the outputs $w_a(kT-iT)$ of an auxiliary model, then the estimation problem of θ can be solved using $w_a(kT)$ instead of $w(kT)$. Here, $w_a(kT)$ is referred to as the estimate of $w(kT)$. Define

$$\begin{aligned} \varphi(kT) &:= [\psi^{\mathrm{T}}(kT),\psi_u^{\mathrm{T}}(kT)]^{\mathrm{T}}, \\ \psi(kT) &:= [-w_a(kT-T),\cdots,-w_a(kT-nT)]^{\mathrm{T}}. \end{aligned}$$

Let $\hat{\theta}(kT)$ represent the estimate of θ at time kT. We replace $\varphi_0(kT)$ with $\varphi(kT)$ and can obtain the following auxiliary model based recursive least squares (AM-RLS) algorithm for estimating the parameter vector θ for multirate system in (10) from the multirate data, which is abbreviated as the MR-AM-RLS algorithm,

$$\begin{aligned} \hat{\theta}(kT) &= \hat{\theta}(kT-T) + P(kT)\varphi(kT)[y(kT) \\ &\quad -\varphi^{\mathrm{T}}(kT)\hat{\theta}(kT-T)], \qquad (11) \\ P^{-1}(kT) &= P^{-1}(kT-T) + \varphi(kT)\varphi^{\mathrm{T}}(kT), \qquad (12) \\ \varphi(kT) &= [\psi^{\mathrm{T}}(kT),\psi_u^{\mathrm{T}}(kT)]^{\mathrm{T}}, \qquad (13) \\ \psi(kT) &= [-w_a(kT-T),\cdots,-w_a(kT-nT)]^{\mathrm{T}}, \qquad (14) \\ \psi_u(kT) &= [u_1^{\mathrm{T}}(kT),\cdots,u_1^{\mathrm{T}}(kT-nT), \\ &\quad \cdots,u_r^{\mathrm{T}}(kT),\cdots,u_r^{\mathrm{T}}(kT-nT)]^{\mathrm{T}}, \qquad (15) \\ w_a(kT) &= \varphi^{\mathrm{T}}(kT)\hat{\theta}(kT)\ \text{(auxiliary model)}. \qquad (16) \end{aligned}$$

$P(kT)$ is the covariance matrix of the parameter estimation error.

To initialize the MR-AM-RLS algorithm, the initial value $P(0)$ is generally taken to be $P(0) = p_0\mathbf{1}_{n_0\times n_0}$ with p_0 normally a large positive number (select $p_0 = 10^6$), and the initial value $\hat{\theta}(0)$ a zero vector or a small real vector, e.g., $\hat{\theta}(0) = \mathbf{1}_{n_0}/p_0$.

Note that the MR-AM-RLS algorithm can be implemented on-line.

V. Convergence of the parameter estimation

In this section, the main results of this paper are proved by formulating a martingale process and by using the martingale convergence theorem (Lemma D.5.3. in [20]).

Assume that $\{v(kT), \mathscr{F}_k\}$ is a martingale sequence defined on a probability space $\{\Omega, \mathscr{F}_k, P\}$, where $\{\mathscr{F}_k\}$ is the σ algebra sequence generated by $v(kT)$, i.e., $\mathscr{F}_k = \sigma(v(kT), v(kT-T), v(kT-2T), \cdots)$ [20]. The noise sequence $\{v(kT)\}$ satisfies

$$
\begin{aligned}
&(A1) \quad \mathrm{E}[v(kT)|\mathscr{F}_{k-1}] = 0, \text{ a.s.,}\\
&(A2) \quad \mathrm{E}\left[\|v(kT)\|^2|\mathscr{F}_{k-1}\right] \leqslant \sigma_v^2, \text{ a.s.}
\end{aligned}
$$

Define

$$
\begin{aligned}
P_0^{-1}(kT) &:= \sum_{i=1}^{k} \varphi_0(iT)\varphi_0^{\mathrm{T}}(iT) + \frac{1}{p_0}I,\\
r(kT) &:= \mathrm{tr}[P^{-1}(kT)], \quad r_0(kT) := \mathrm{tr}[P_0^{-1}(kT)].
\end{aligned}
$$

Define the parameter estimation error vector $\tilde{\theta}(kT)$ and a nonnegative definite function $V(kT)$ as

$$\tilde{\theta}(kT) := \hat{\theta}(kT) - \theta, \tag{17}$$

$$V(kT) := \tilde{\theta}^{\mathrm{T}}(kT)P^{-1}(kT)\tilde{\theta}(kT) \tag{18}$$

Lemma 1: Assume that there exist functions $f(k) \geqslant 0$ and $g(k) \geqslant 0$ such that $\lim_{k\to\infty} f(k) = f_0 < \infty$, $\sum_{k=1}^{\infty} g(k)$ is divergent and $\sum_{k=1}^{\infty} f(k)g(k)$ is convergent. Then $f_0 = 0$.

The proof of Lemma 1 is straightforward and hence omitted.

Lemma 2: For the MR-AM-RLS algorithm in (11)–(16), the following inequalities hold:

$$\sum_{i=1}^{\infty} \frac{\varphi^{\mathrm{T}}(iT)P(iT)\varphi(iT)}{\{\ln|P^{-1}(kT)|\}^c} < \infty, \text{ a.s., for any } c > 1.$$

The proof can be done in a similar way in [6] and is omitted.

Lemma 3: For the system in (10) and the MR-AM-RLS algorithm in (11)-(16), assume that $(A1)$ and $(A2)$ hold and

$$(A3) \quad H(z) = \frac{1}{\alpha(z)} - \frac{1}{2} \text{ is strictly positive real.}$$

Then the following inequality holds,

$$
\begin{aligned}
\mathrm{E}[V(kT) + S(kT)|\mathscr{F}_{k-1}] \leqslant\, & V(kT-T) + S(kT-T)\\
&+ 2\varphi^{\mathrm{T}}(kT)P(kT)\varphi(kT)\sigma_v^2,
\end{aligned}
$$

where

$$
\begin{aligned}
S(kT) &:= 2\sum_{i=1}^{k} \tilde{u}(kT)\tilde{y}(kT), \quad \tilde{u}(kT) := -\varphi^{\mathrm{T}}(kT)\tilde{\theta}(kT),\\
\tilde{y}(kT) &:= \frac{1}{2}\varphi^{\mathrm{T}}(kT)\tilde{\theta}(kT) + [y(kT) - w_a(kT) - v(kT)].
\end{aligned}
$$

Here, $(A3)$ guarantees that $S(kT) \geqslant 0$.

The assumptions in $(A1)$ and $(A2)$ imply that $v(kT)$ is a uncorrelated noise sequence with zero mean, time-varying but a bounded variance. The following Theorem 2 show that the convergent rate of the MR-AM-RLS parameter estimation is the ratio of the logarithm of the maximum eigenvalue to the minimum eigenvalue of the covariance matrix $P_0^{-1}(kT)$.

Theorem 2: For the system in (10) and the MR-AM-RLS algorithm in (11)-(16), suppose that $(A1)$-$(A3)$ and

$$(A4) \quad [\ln r_0(kT)]^c = o(\lambda_{\min}[P_0^{-1}(kT)]) \text{ for any } c > 1,$$

$\alpha(z)$ is stable (i.e. all zeros of $\alpha(z)$ are inside the unit circle). Then the parameter estimation error satisfies

$$\|\hat{\theta}(kT) - \theta\|^2 = O\left(\frac{\{\ln \lambda_{\max}[P_0^{-1}(kT)]\}^c}{\lambda_{\min}[P_0^{-1}(kT)]}\right) \text{ for any } c > 1.$$

Assume that there exist positive constants c_0, c_1, c_2 and k_0 such that for $k \geqslant k_0$, the following generalized persistent excitation condition (unbounded condition number) holds

$$(C1) \quad c_1 I \leqslant \frac{1}{k}\sum_{i=1}^{k} \varphi_0(iT)\varphi_0^{\mathrm{T}}(iT) \leqslant c_2 k^{c_0} I, \text{ a.s.} \tag{19}$$

Then for any $c > 1$, we have

$$\|\tilde{\theta}(kT)\|^2 = O\left(\frac{[\ln k]^c}{k}\right) \to 0, \text{ a.s.}$$

This shows that the estimation error $\|\tilde{\theta}(kT)\|$ converges to zero as k goes to infinity.

VI. Example

Example Consider a 3-input single-output system with three-input updating periods $T_1 = 2h$, $T_2 = 3h$ and $T_3 = h$ and the output sampling period $T = 6h$, take $h = 2$, the corresponding transfer model is taken to be

$$
\begin{aligned}
y(kT) = \frac{1}{\alpha(z)}\,&[\beta_{11}(z)u_1(kT) + \beta_{12}(z)u_1(kT+T_1)\\
&+ \beta_{13}(z)u_1(kT+2T_1) + \beta_{21}(z)u_2(kT)\\
&+ \beta_{22}(z)u_2(kT+T_2) + \beta_{31}(z)u_3(kT)\\
&+ \beta_{32}(z)u_3(kT+T_3) + \beta_{33}(z)u_3(kT+2T_3)\\
&+ \beta_{34}(z)u_3(kT+3T_3) + \beta_{35}(z)u_3(kT+4T_3)\\
&+ \beta_{36}(z)u_3(kT+5T_3)] + v(kT),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha(z) &= 1+\alpha_1 z^{-1}+\alpha_2 z^{-2}+\alpha_3 z^{-3} \\
&= 1+0.600z^{-1}+0.400z^{-2}+0.300z^{-3}, \\
\beta_{11}(z) &= \beta_{11}(1)z^{-1}+\beta_{11}(2)z^{-2}+\beta_{11}(3)z^{-3} \\
&= -0.200z^{-1}+0.500z^{-2}-0.400z^{-3}, \\
\beta_{12}(z) &= \beta_{12}(1)z^{-1}+\beta_{12}(2)z^{-2}+\beta_{12}(3)z^{-3} \\
&= -0.650z^{-1}+0.600z^{-2}+0.100z^{-3}, \\
\beta_{13}(z) &= \beta_{13}(1)z^{-1}+\beta_{13}(2)z^{-2}+\beta_{13}(3)z^{-3} \\
&= 1.000z^{-1}-0.500z^{-2}+0.400z^{-3}, \\
\beta_{21}(z) &= \beta_{21}(1)z^{-1}+\beta_{21}(2)z^{-2}\beta_{21}(3)z^{-3} \\
&= -0.400z^{-1}+0.800z^{-2}-0.200z^{-3}, \\
\beta_{22}(z) &= \beta_{22}(1)z^{-1}+\beta_{22}(2)z^{-2}+\beta_{22}(3)z^{-3} \\
&= 0.700z^{-1}+0.200z^{-2}+0.300z^{-3}, \\
\beta_{31}(z) &= \beta_{31}(1)z^{-1}+\beta_{31}(2)z^{-2}+\beta_{31}(3)z^{-3} \\
&= 1.000z^{-1}-0.900z^{-2}+0.200z^{-3}, \\
\beta_{32}(z) &= \beta_{32}(1)z^{-1}+\beta_{32}(2)z^{-2}+\beta_{32}(3)z^{-3} \\
&= 1.000z^{-1}-0.750z^{-2}+0.125z^{-3}, \\
\beta_{33}(z) &= \beta_{33}(1)z^{-1}+\beta_{33}(2)z^{-2}+\beta_{33}(3)z^{-3} \\
&= 1.200z^{-1}-1.080z^{-2}+0.240z^{-3}, \\
\beta_{34}(z) &= \beta_{34}(1)z^{-1}+\beta_{34}(2)z^{-2}+\beta_{34}(3)z^{-3} \\
&= 1.200z^{-1}-0.780z^{-2}+0.120z^{-3}, \\
\beta_{35}(z) &= \beta_{35}(1)z^{-1}+\beta_{35}(2)z^{-2}+\beta_{35}(3)z^{-3} \\
&= 0.800z^{-1}-0.880z^{-2}+0.200z^{-3}, \\
\beta_{36}(z) &= \beta_{36}(1)z^{-1}+\beta_{36}(2)z^{-2}+\beta_{36}(3)z^{-3} \\
&= 0.600z^{-1}-0.700z^{-2}+0.400z^{-3},.
\end{aligned}
$$

Then the identification model can be expressed as

$$
w(kT)=\varphi_0^{\mathrm{T}}(kT)\theta, \quad y(kT)=w(kT)+v(kT),
$$

where

$$
\begin{aligned}
\theta = [&\alpha_1,\alpha_2,\alpha_3,\beta_{11}(1),\beta_{11}(2),\beta_{11}(3), \\
&\beta_{12}(1),\beta_{12}(2),\beta_{12}(3),\beta_{13}(1),\beta_{13}(2),\beta_{13}(3), \\
&\beta_{21}(1),\beta_{21}(2),\beta_{21}(3),\beta_{22}(1),\beta_{22}(2),\beta_{22}(3), \\
&\beta_{31}(1),\beta_{31}(2),\beta_{31}(3),\beta_{32}(1),\beta_{32}(2),\beta_{32}(3) \\
&\beta_{33}(1),\beta_{33}(2),\beta_{33}(3),\beta_{34}(1),\beta_{34}(2),\beta_{34}(3), \\
&\beta_{35}(1),\beta_{35}(2),\beta_{35}(3),\beta_{36}(1),\beta_{36}(2),\beta_{36}(3)]^{\mathrm{T}} \\
= [&0.600,0.400,0.300,-0.200,0.500,-0.400, \\
&-0.650,0.600,0.100,1.000,-0.500,0.400, \\
&-0.400,0.800,-0.200,0.700,0.200,0.300, \\
&1.000,-0.900,0.200,1.000,-0.750,0.125, \\
&1.200,-1.080,0.240,1.200,0.780,0.120, \\
&0.800,-0.880,0.200,0.600,-0.700,0.400]^{\mathrm{T}}, \\
\varphi_0(kT) = [&-w(kT-T),-w(kT-2T),-w(kT-3T), \\
&u_1(kT-T),u_1(kT-2T),u_1(kT-3T) \\
&u_1(kT+T_1-T),u_1(kT+T_1-2T), \\
&u_1(kT+T_1-3T),u_1(kT+2T_1-T), \\
&u_1(kT+2T_1-2T),u_1(kT+2T_1-3T), \\
&u_2(kT-T),u_2(kT-2T),u_2(kT-3T), \\
&u_2(kT+T_2-T),u_2(kT+T_2-2T), \\
&u_2(kT+T_2-3T),u_3(kT-T),u_3(kT-2T), \\
&u_3(kT-3T),u_3(kT+T_3-T), \\
&u_3(kT+T_3-2T),u_3(kT+T_3-3T), \\
&u_3(kT+2T_3-T),u_3(kT+2T_3-2T), \\
&u_3(kT+2T_3-3T),u_3(kT+3T_3-T), \\
&u_3(kT+3T_3-2T),u_3(kT+3T_3-3T), \\
&u_3(kT+4T_3-T),u_3(kT+4T_3-2T), \\
&u_3(kT+4T_3-3T),u_3(kT+5T_3-T), \\
&u_3(kT+5T_3-2T),u_3(kT+5T_3-3T)]^{\mathrm{T}}.
\end{aligned}
$$

The inputs $\{u_1(kT_1)\}$, $\{u_2(kT_2)\}$ and $\{u_3(kT_3)\}$ are taken as uncorrelated persistent excitation signal sequences with zero mean and unit variances, $\{v(kT)\}$ as a white noise sequence with zero mean and variances $\sigma^2=0.10^2$. Applying the MR-AM-RLS algorithm to estimate the parameters of this system, the parameter estimates and their errors with different noise variances are shown in Table I and the parameter estimation errors $\delta := \|\hat{\theta}(k)-\theta\|/\|\theta\|$ versus k are shown in Figure 2.

TABLE I

THE PARAMETERS AND THEIR ESTIMATES ($\sigma^2=0.10^2$)

k	2000	3000	4000	5000	True values
α_1	0.59064	0.58818	0.59262	0.59493	0.60000
α_2	0.39931	0.39644	0.39789	0.39848	0.40000
α_3	0.30487	0.30330	0.30198	0.30179	0.30000
$\beta_{11}(1)$	-0.19898	-0.19841	-0.19791	-0.19877	-0.20000
$\beta_{11}(2)$	0.50297	0.50312	0.50009	0.49983	0.50000
$\beta_{11}(3)$	-0.40172	-0.40262	-0.39989	-0.39985	-0.40000
$\beta_{12}(1)$	-0.64184	-0.64186	-0.64477	-0.64779	-0.65000
$\beta_{12}(2)$	0.61093	0.61267	0.60790	0.60724	0.60000
$\beta_{12}(3)$	0.08422	0.08560	0.09156	0.09446	0.10000
$\beta_{13}(1)$	0.99335	0.99108	0.99323	0.99509	1.00000
$\beta_{13}(2)$	-0.50696	-0.51159	-0.50810	-0.50530	-0.50000
$\beta_{13}(3)$	0.40967	0.41117	0.40503	0.40309	0.40000
$\beta_{21}(1)$	-0.39902	-0.40161	-0.40015	-0.40174	-0.40000
$\beta_{21}(2)$	0.81099	0.81161	0.80724	0.80464	0.80000
$\beta_{21}(3)$	-0.20072	-0.20330	-0.20158	-0.19968	-0.20000
$\beta_{22}(1)$	0.70476	0.70407	0.70305	0.70230	0.70000
$\beta_{22}(2)$	0.19982	0.19719	0.19812	0.19968	0.20000
$\beta_{22}(3)$	0.29450	0.29641	0.29823	0.29758	0.30000
$\beta_{31}(1)$	0.99737	0.99508	0.99650	0.99780	1.00000
$\beta_{31}(2)$	-0.90722	-0.90927	-0.90505	-0.90375	-0.90000
$\beta_{31}(3)$	0.21178	0.21229	0.20770	0.20616	0.20000
$\beta_{32}(1)$	0.99885	0.99920	0.99996	1.00055	1.00000
$\beta_{32}(2)$	-0.76305	-0.76497	-0.75834	-0.75482	-0.75000
$\beta_{32}(3)$	0.14226	0.13890	0.13414	0.13198	0.12500
$\beta_{33}(1)$	1.20561	1.20748	1.20611	1.20480	1.20000
$\beta_{33}(2)$	-1.09781	-1.10057	-1.09229	-1.08900	-1.08000
$\beta_{33}(3)$	0.25458	0.25389	0.24895	0.24637	0.24000
$\beta_{34}(1)$	1.19517	1.19757	1.19846	1.19775	1.20000
$\beta_{34}(2)$	-0.80143	-0.80418	-0.79638	-0.79370	-0.78000
$\beta_{34}(3)$	0.12734	0.12791	0.12484	0.12267	0.12000
$\beta_{35}(1)$	0.79868	0.79929	0.79714	0.79743	0.80000
$\beta_{35}(2)$	-0.89246	-0.89369	-0.89024	-0.88831	-0.88000
$\beta_{35}(3)$	0.21041	0.21226	0.20882	0.20580	0.20000
$\beta_{36}(1)$	0.59680	0.60080	0.60179	0.60149	0.60000
$\beta_{36}(2)$	-0.71227	-0.71322	-0.70787	-0.70643	-0.70000
$\beta_{36}(3)$	0.41039	0.41212	0.40573	0.40255	0.40000
δ (%)	1.44891	1.57504	1.00093	0.74490	

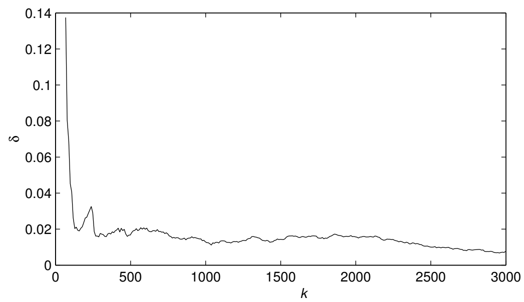

Fig. 2. The parameter estimation errors δ versus k

From Table I and Figure 2, it is clear that the parameter estimation errors by the MR-AM-RLS algorithm become (generally) smaller and smaller and go to zero with k increasing. This shows that the proposed algorithm can effectively estimate the parameters of the multirate systems.

VII. Conclusion

The state-space model of a multirate multi-input system is derived by discretizing a continuous-time process with the frame sampling period and the MR-AM-RLS algorithm is presented to estimate the model parameters for such multirate systems and convergence performance of the proposed algorithm is analyzed. Since a multi-input, multi-output system can be decomposed many multi-input, single-output systems, the identification method in this paper can be extended to identify multirate multi-input, multi-output systems.

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