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Output Linear Controller for a Class of Nonlinear Systems Using the Invariant Ellipsoid Technique

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***Abstract*— This paper deals with the problem of robust stabilization of an uncertain nonlinear system with output measurements using the invariant ellipsoid method. The non-linear system is uncertain but bounded according to a 'quasi-lipschitz' condition and the output measurements are subjected to perturbations bounded by ellipsoids.The invariant ellipsoid method allows to obtain the robust linear feedback as a solution of the special linear optimization problem with bilinear constraints. The methods for solving this optimization problem involves the LMI technique. The stabilization of the double mass-spring system is considered as an illustrative example.**

Keywords:linear matrix inequalities, output regulation, uncertain systems.

I. INTRODUCTION

The model of a system is only an approximate representation of its full behavior. As a consequence of that, uncertain parameters and perturbations appears. So, it is desired that a control law is designed according to the characteristics of the unknown variables in order to provide robustness against them.

A class of state feedback controls in order to guarantee uniform ultimate boundedness (UUB) for uncertain dynamic systems were presented in [1] and summarized in [7] where additional results were presented for the analysis of perturbed systems subjected to nonvanishing perturbations.

For the case of L_2 perturbations, the explicit solutions for the optimal controller with the H_2 theory were obtained as it was shown in [2] and [3]. The restriction of bounded perturbations was changed in [4] where only the maximum amplitude of the perturbations (l_1 norm) was considered, although the optimal controller designed with the l_1 scheme can be of high order. In [9] the problem of 'peak to peak' gain minimization with *Linear Matrix Inequalities* (LMI) was considered.

The notion of invariant sets and its close connection with the Lyapunov theory was also exploited for the analysis and control of dynamical systems such as constrained control, robustness analysis, and disturbance rejection as was stated in [10].

The concept of invariance set states that any element of this set for $t = 0$, will remain in the invariance set for all $t \in$ R. Ellipsoids are a simple way to characterize the invariant set because only a center and a matrix are required, although they are conservative approximations of the invariant set.

Many robust control problems can be formulated in terms of linear matrix inequalities (LMI) and solved with semidefinite programming [8]. A significantly wider class of problems can be formulated in terms of *Bilinear Matrix Inequalities* (BMI) as in [5], but only in very few cases (such as static state feedback and dynamic output feedback) it is possible to convexify the problem and derive equivalent LMIs.

In [11] the static output feedback stabilization was solved for an uncertain linear time invariant (LTI) system using LMIs without minimizing any criterion. In [13] and [14] the stabilization problem was stated for an LTI system under bounded perturbations and noisy output measurements using a linear control law based on the observed state. This was done using the so-called invariant ellipsoid method (Lyapunov function approach and the invariant ellipsoid concept) that allows to design a control such that minimices the invariant ellipsoid for the controlled system. The solution to this problem was reduced to an optimization problem with linear objective and LMI constraints.

This paper is concerned with the problem of stabilization of a nonlinear system in the presence of bounded perturbations and output measurements. The nonlinearities in the system come from uncertainties in a nominal LTI system. The stabilization scheme is the invariant ellipsoid method to construct a Luenberger observer and the observed state feedback controller that minimices the invariant ellipsoid as in [13]. Due to the nonlinearities of the system, the constraints of the optimization problem are bilinear. However, with the use of some appropriate transformations, the obtained BMI constraints are shown to be dependent only on two scalar parameters, i.e., for fixed scalar parameters, the BMI constraints becomes LMI and the optimization problem can be solved through standard semidefinite programming technique. The outline of this paper is as follows. In section 2, the problem statement and basic assumptions are introduced. Then, in section 3 the main results and its proofs are given; some numerical aspects of the robust feedback synthesis are given in section 4. Next, section 5 presents the application of the obtained result to the stabilization of a double mass-spring system. Finally, the conclusions are given.

II. PROBLEM STATEMENT

Consider the nonlinear dynamic system with linear state-output mapping given by

$$\begin{aligned} \dot{x} &= f(x,t) + Bu \\ y &= Cx + w_y \end{aligned} \tag{1}$$

where $x \in \mathrm{R}^n$ is the state vector, $u \in \mathrm{R}^m$ is the control input, $y \in \mathrm{R}^k$ is the system output, $w_y \in \mathrm{R}^k$ is the output perturbation, $f : \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n$ is an unknown (from a class given below) nonlinear function, $B \in \mathrm{R}^{n\times m}$, $C \in \mathrm{R}^{k\times n}$ are the system matrices.

The following assumptions will be in force throughout:

- the output disturbances w_y are inside of a bounded ellipsoid, that is,
$$\|w_y\|^2_{K_\eta} = w_y^T K_\eta w_y \le 1 \tag{2}$$
where $K_\eta > 0$ is given;
- the nonlinear function $f(x,t)$ is quasi-Lipschitz, namely, it belongs to the class of functions satisfying
$$\|f(x,t) - Ax\|^2_{K_f} \le \delta + \|x\|^2_{K_x}$$
where $0 < K_f \in \mathrm{R}^{n\times n}$, $0 < K_x \in \mathrm{R}^{n\times n}$, $A \in \mathrm{R}^{n\times n}$ are known matrices, $\delta \ge 0$; without loss of generality, after the normalization of this inequality it is sufficient to consider only two-valued case: $\delta = (0;1)$.
- The pair (A,B) is controllable and the pair (A,C) is observable.

Using denotation $w_x := f(x,t) - Ax$ the system (1) can be rewritten in the form

$$\dot{x} = Ax + Bu + w_x \tag{3}$$

$$y = Cx + w_y \tag{4}$$

with

$$\|w_x\|^2_{K_f} \le \delta + \|x\|^2_{K_x} \tag{5}$$

Here we only consider the linear feedback controls

$$u = K\hat{x}, \quad K \in \mathrm{R}^{m\times n} \tag{6}$$

with respect to observed state $\hat{x} \in \mathrm{R}^n$ which are obtained by the classical Luenberger observer having the structure

$$\dot{\hat{x}} = A\hat{x} + Bu + F(y - C\hat{x}), \quad F \in \mathrm{R}^{n\times p} \tag{7}$$

The robust stabilization of the system (3),(4) will be realized using the method of *Invariant Ellipsoids* (see, for example, [13]). Here we just present some basic ideas of this method.

The ellipsoid

$$\varepsilon(0,P) = \left\{x \in \mathrm{R}^n : x^T P^{-1} x \le 1\right\}, \; P > 0$$

with center in the origin and shape matrix P is said to be *state-invariant* for the system (3),(4) under the disturbances (2) and nonlinearities (5) if the condition $x(0) \in \varepsilon(0,P)$ implies $x(t) \in \varepsilon(0,P)$ for all $t \ge 0$; in other words, any trajectory of the system starting in the invariant ellipsoid stays in it for all $t \ge 0$. The trajectory starting outside of the ellipsoid converges to this ellipsoid (asymptotically or in finite time) and this is done with the aid of the second method of Lyapunov for designing a observed state controller.

This invariant ellipsoid can be considered as a characteristic of the influence of the uncertainties in a system. So, the minimum (in some sense) invariant ellipsoid corresponds to a robust feedback control. Hence, the main problem is to design the observer-based linear feedback control providing the convergence of any trajectory of the system (3),(4),(7) to the 'minimum' invariant ellipsoid. Here we will use the trace as the criterion of the ellipsoids minimality.

III. MAIN RESULTS

Define the state estimation error as $e := x - \hat{x}$. Then its time derivative satisfies

$$\dot{e} = (A - FC)e + w_x - Fw_y$$

Introduce the extended vector $z := \begin{pmatrix} \hat{x} & e \end{pmatrix}^T$ where $z \in \mathrm{R}^{2n}$. Then it follows

$$\dot{z} = \hat{A}z + \hat{F}w \tag{8}$$

where $\hat{A} := \begin{pmatrix} A + BK & FC \\ 0 & A - FC \end{pmatrix}$, $\hat{F} := \begin{pmatrix} 0 & F \\ I & -F \end{pmatrix}$ and $w := \begin{pmatrix} w_x \\ w_y \end{pmatrix}$.

Our aim here is to find the control gain matrix K and the observer gain matrix F providing a 'good enough' stabilization as well as state estimation of the system (8), or more exactly, to design K and F such that the corresponding invariant ellipsoid, called below '*quasi-minimal*', would contain the minimal one.

Theorem 1. If the following optimization problem

$$\mathrm{tr}(X_1) + \mathrm{tr}(H) \to \min \tag{9}$$

subject to

$$\begin{pmatrix} R_1 & Y_2^T C & 0 & Y_2^T & X_2 X_1 \\ C^T Y_2 & \Psi & X_2 & -Y_2^T & I \\ 0 & X_2 & -\tau_2 K_f & 0 & 0 \\ Y_2 & -Y_2 & 0 & -\tau_3 K_\eta & 0 \\ X_1 X_2 & I & 0 & 0 & -\frac{1}{\tau_2} K_x^{-1} \end{pmatrix} \le 0 \tag{10}$$

$$\begin{pmatrix} -R_1 - 2X_2 & I \\ I & X_1 A^T + AX_1 + Y_1 B^T + BY_1^T + \tau_1 X_1 \end{pmatrix} \le 0$$

$$\begin{pmatrix} H & I \\ I & X_2 \end{pmatrix} \ge 0, \quad \tau_1 \ge \delta\tau_2 + \tau_3, \; \tau_2 \ge 0, \tau_3 \ge 0$$

$$X_1 > 0, X_2 > 0, H > 0$$

with

$$\Psi := A^T X_2 + X_2 A - Y_2^T C - C^T Y_2 + \tau_1 X_2$$

has a solution with respect to the matrix variables $H, X_1, X_2, R_1, R_2 \in \mathrm{R}^{n\times n}$, $Y_1 \in \mathrm{R}^{n\times m}$, $Y_2 \in \mathrm{R}^{k\times n}$ and the scalar variables τ_1, τ_2 and τ_3, then the ellipsoid with the matrix

$$P = \begin{pmatrix} X_1 & 0 \\ 0 & X_2^{-1} \end{pmatrix} \tag{11}$$

is quasi-minimal invariant ellipsoid of the system (8) with (2), (5) with the feedback control gain matrix

$$K = \left(X_1^{-1} Y_1\right)^T \tag{12}$$

and the observer gain matrix

$$F = \left(Y_2 X_2^{-1}\right)^T \tag{13}$$

Proof: Define the Lyapunov function as

$$V(z) := (z, P^{-1} z), \; P^{-1} = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

where $P > 0$ is the matrix of an invariant ellipsoid should be minimized. Then

$$\begin{gathered} \dot{V} = (z, P^{-1}\dot{z}) + (\dot{z}, P^{-1} z) \\ = (z, P^{-1}(\hat{A}z + \hat{F}w)) + ((\hat{A}z + \hat{F}w), P^{-1} z) \\ = z^T[\hat{A}^T P^{-1} + P^{-1}\hat{A}]z + w^T \hat{F}^T P^{-1} z + z^T P^{-1} \hat{F} w \end{gathered}$$

or expressing in its quadratic form

$$\dot{V} = \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix}^T \begin{pmatrix} \hat{A}^T P^{-1} + P^{-1}\hat{A} & P^{-1}\hat{F} \\ \hat{F}^T P^{-1} & 0 \end{pmatrix} \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix} \leq 0 \tag{14}$$

This ellipsoid $(z, P^{-1}z) \leq 1$ will be invariant if and only if outside of it we have $\dot{V} \leq 0$, that is, for z satisfying

$$z^T P^{-1} z \geq 1 \tag{15}$$

Together with (14) and (15) we also have

$$w_x K_f w_x \leq \delta + x^T K_x x \text{ and } \|w_y\|_{K_\eta}^2 \leq 1 \tag{16}$$

To fullfill all this constraints let us apply the, so-called, S-procedure (see, for example, [15]). Define

$$A_0 := \begin{pmatrix} \hat{A}^T P^{-1} + P^{-1}\hat{A} & P^{-1}\hat{F} \\ \hat{F}^T P^{-1} & 0 \end{pmatrix}, \; \alpha_0 := 0$$

Since

$$z^T P^{-1} z \geq 1 \Leftrightarrow \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix}^T A_1 \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix} \leq -1 := \alpha_1$$

$$A_1 := \begin{pmatrix} -P^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Representing (16) in the form

$$w_x K_f w_x \leq \delta + z^T K_z z, \qquad K_z = \begin{pmatrix} K_x & K_x \\ K_x & K_x \end{pmatrix}$$

one has

$$\begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix}^T A_2 \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix} \leq \delta := \alpha_2,$$

$$A_2 := \begin{pmatrix} -K_z & 0 & 0 \\ 0 & K_f & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And analogously,

$$\|w_y\|_{K_\eta}^2 \leq 1 \Leftrightarrow \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix}^T A_3 \begin{pmatrix} z \\ w_x \\ w_y \end{pmatrix} \leq 1 := \alpha_3$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_\eta \end{pmatrix}$$

Finally, by the S-procedure if there exist $\tau_1 \geq 0$, $\tau_2 \geq 0$ and $\tau_3 \geq 0$ such that

$$\begin{gathered} \alpha_0 \geq \tau_1\alpha_1 + \tau_2\alpha_2 + \tau_3\alpha_3 \\ A_0 \leq \tau_1 A_1 + \tau_2 A_2 + \tau_3 A_3 \end{gathered} \tag{17}$$

then the conditions (14), (15) and (16) hold and the ellipsoid with the matrix P is an invariant ellipsoid of our system. The first condition in (17) can be represented as

$$\tau_1 \geq \delta\tau_2 + \tau_3, \; \tau_i \geq 0, \; i = 1, 2, 3 \tag{18}$$

For the second one, define

$$Q := A_0 - \tau_1 A_1 - \tau_2 A_2 - \tau_3 A_3 \leq 0$$

or

$$Q = \begin{pmatrix} \Theta_1 & P_1 F C + \tau_2 K_x & 0 & P_1 F \\ C^T F^T P_1 + \tau_2 K_x & \Theta_2 & P_2 & -P_2 F \\ 0 & P_2 & -\tau_2 K_f & 0 \\ F^T P_1 & -F^T P_2 & 0 & -\tau_3 K_\eta \end{pmatrix}$$

with

$$\Theta_1 \; : \; = A_K^T P_1 + P_1 A_K + \tau_2 K_x, A_K = A + BK + \frac{\tau_1}{2} I$$

$$\Theta_2 \; : \; = A_F^T P_2 + P_2 A_F + \tau_2 K_x, A_F = A - FC + \frac{\tau_1}{2} I$$

Applying the quadratic non-singular transformation

$$T_1 = \begin{pmatrix} P_1^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

to the matrix Q we get

$$Q_1 = T_1 Q T_1^T = \begin{pmatrix} \Psi_1 & \begin{matrix} FC+ \\ \tau_2 P_1^{-1} K_x \end{matrix} & 0 & F \\ \begin{matrix} C^T F^T + \\ \tau_2 K_x P_1^{-1} \end{matrix} & \Psi_2 & P_2 & -P_2 F \\ 0 & P_2 & -\tau_2 K_f & 0 \\ F^T & -F^T P_2 & 0 & -\tau_3 K_\eta \end{pmatrix} \leq 0$$

$$\Psi_1 := P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1} + \tau_2 P_1^{-1} K_x P_1^{-1}$$
$$\Psi_2 := A_F^T P_2 + P_2 A_F + \tau_1 P_2 + \tau_2 K_x$$

Obviously that

$$Q_1 = \tilde{Q} + \begin{pmatrix} P_1^{-1} \\ I \\ 0 \\ 0 \end{pmatrix} (\tau_2 K_x) \begin{pmatrix} P_1^{-1} & I & 0 & 0 \end{pmatrix} \leq 0 \quad (19)$$

where

$$\tilde{Q} = \begin{pmatrix} \Xi_1 & FC & 0 & F \\ C^T F^T & \Xi_2 & P_2 & -P_2 F \\ 0 & P_2 & -\tau_2 K_f & 0 \\ F^T & -F^T P_2 & 0 & -\tau_3 K_\eta \end{pmatrix}$$
$$\Xi_1 := P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1}$$
$$\Xi_2 := A_F^T P_2 + P_2 A_F + \tau_1 P_2$$

Using the Schur complement to (19) we obtain

$$Q_2 = \begin{pmatrix} \Pi_1 & FC & 0 & F & P_1^{-1} \\ C^T F^T & \Pi_2 & P_2 & \text{-}P_2 F & I \\ 0 & P_2 & \text{-}\tau_2 K_f & 0 & 0 \\ F^T & -F^T P_2 & 0 & \text{-}\tau_3 K_\eta & 0 \\ P_1^{-1} & I & 0 & 0 & \text{-}\frac{1}{\tau_2} K_x^{-1} \end{pmatrix} \leq 0$$
$$\Pi_1 := P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1}$$
$$\Pi_2 := A_F^T P_2 + P_2 A_F + \tau_1 P_2$$

Analogously, apply the transformation

$$T_2 = \begin{pmatrix} P_2 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$$

to Q_2 we obtain

$$Q_3 = T_2 Q_2 T_2^T = \begin{pmatrix} \Delta_1 & P_2 FC & 0 & P_2 F & P_2 P_1^{-1} \\ C^T F^T P_2 & \Delta_2 & P_2 & \text{-}P_2 F & I \\ 0 & P_2 & \text{-}\tau_2 K_f & 0 & 0 \\ F^T P_2 & -F^T P_2 & 0 & \text{-}\tau_3 K_\eta & 0 \\ P_1^{-1} P_2 & I & 0 & 0 & \text{-}\frac{1}{\tau_2} K_x^{-1} \end{pmatrix} \leq 0 \quad (20)$$

$$\Delta_1 \ : \ = P_2(P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1}) P_2$$
$$\Delta_2 \ : \ = A_F^T P_2 + P_2 A_F + \tau_1 P_2$$

By Λ-inequality (see [15])

$$XY^T + YX^T \leq X\Lambda X^T + Y\Lambda^{-1} Y^T \quad (21)$$

valid for any $X \in R^{n\times k}$, $Y \in R^{n\times k}$ and any $0 < \Lambda = \Lambda^T \in R^{k\times k}$ being applied for $X = P_2$ and $Y = I$ it follows

$$X + X^T \leq X\Lambda X^T + \Lambda^{-1}$$

that for

$$\Lambda := -(P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1})$$

implies

$$\begin{aligned} &P_2(P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1}) P_2 \\ &\leq \text{-}P_2\text{-}P_2\text{-}(P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1})^{-1} \end{aligned} \quad (22)$$

Applying then the Schur complement to the matrix inequality

$$-2P_2 - (P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1})^{-1} \leq R_1 \quad (23)$$

we get

$$\begin{pmatrix} -R_1 - 2P_2 & I \\ I & P_1^{-1}A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1} \end{pmatrix} \leq 0 \quad (24)$$

Defining

$$X_1 := P_1^{-1}, Y_1 := P_1^{-1} K^T, X_2 := P_2, \ Y_2 := F^T P_2$$

using (22), (23), (24) and the matrix inequality

$$\begin{bmatrix} X & Y^T \\ Y & Z \end{bmatrix} \leq \begin{bmatrix} X' & Y^T \\ Y & Z \end{bmatrix}$$

valid for any $X = X^T$, $Z = Z^T$ and $X' = X'^T \geq X$, then (20) can be restricted by

$$\begin{pmatrix} R_1 & Y_2^T C & 0 & Y_2^T & X_2 X_1 \\ C^T Y_2 & \Sigma_1 & X_2 & \text{-}Y_2^T & I \\ 0 & X_2 & \text{-}\tau_2 K_f & 0 & 0 \\ Y_2 & -Y_2 & 0 & \text{-}\tau_3 K_\eta & 0 \\ X_1 X_2 & I & 0 & 0 & \text{-}\frac{1}{\tau_2} K_x^{-1} \end{pmatrix} \leq 0$$
$$\Sigma_1 := A^T X_2 + X_2 A \text{-} Y_2^T C \text{-} C^T Y_2 + \tau_1 X_2$$
$$\begin{pmatrix} \text{-}R_1\text{-}2X_2 & I \\ I & X_1 A^T + AX_1 + Y_1 B^T + BY_1^T + \tau_1 X_1 \end{pmatrix} \leq 0$$

The minimization problem to be solved is

$$\mathrm{tr}(X_1) + \mathrm{tr}(X_2^{-1}) \to \min \quad (25)$$

subject to the constraints (10). Introduce the following additional constraint

$$H \geq X_2^{-1} \Leftrightarrow \begin{pmatrix} H & I \\ I & X_2 \end{pmatrix} \geq 0$$

This allows to reduce the optimization problem (25) to a linear one

$$\mathrm{tr}(X_1) + \mathrm{tr}(H) \to \min$$

■

Remark 1. The problem (9) is, in fact, a bilinear optimization problem because of the term X_1X_2 participating in the matrix constraint (10). The numerical solution of bilinear optimization problems is usually a non trivial task and the feasible set for a BMI is nonconvex.

Remark 2. The proof of the theorem 1 was mainly based on the S-procedure, Schur complement and Λ-inequality [15].

Remark 3. Λ-inequality gives only the upper estimates of the matrix inequalities so we can only guarantee that the obtained solution gives the quasi-minimal invariant ellipsoid.

IV. ROBUST FEEDBACK SYNTHESIS: NUMERICAL ASPECTS

Semi-definite relaxations (as in [6]) and the solution through nonlinear programming methods (e.g. 'branch-bound' algorithm) can be considered as two alternatives to solve bilinear optimization problems. The Matlab-Tool 'PENBMI' is highly sensitive to the initial point selection which is desired to be close to a solution [12].

The following lemma constraints the feasibility domain of the previous theorem simplifying the problem of a feasible starting point finding.

Lemma 1. The set of variables satisfying (10) contains the set of ones satisfying

$$\begin{pmatrix} R_1 & Y_2^T C & 0 & Y_2^T & 0 \\ C^T Y_2 & \Phi & X_2 & -Y_2^T & I \\ 0 & X_2 & -\tau_2 K_f & 0 & 0 \\ Y_2 & -Y_2 & 0 & -\tau_3 K_\eta & 0 \\ 0 & I & 0 & 0 & R_2 \end{pmatrix} \le 0 \qquad (26)$$

$$\begin{pmatrix} -R_1-2X_2 & I \\ I & X_1 A^T + A X_1 + Y_1 B^T + B Y_1^T + \tau_1 X_1 + \Lambda \end{pmatrix} \le 0$$

$$\begin{pmatrix} -\frac{1}{\tau_2} K_x^{-1} - R_2 & X_1 \\ X_1 & -\Lambda \end{pmatrix} \le 0, \ \begin{pmatrix} H & I \\ I & X_2 \end{pmatrix} \ge 0$$

$$\tau_1 \ge \delta\tau_2 + \tau_3, \ \tau_i \ge 0, \ i = 1, 2, 3$$

with

$$\Phi := A^T X_2 + X_2 A - Y_2^T C - C^T Y_2 + \tau_1 X_2$$

Proof: By the Λ-inequality (21) with

$X^T := \begin{pmatrix} 0 & 0 & 0 & 0 & X_1 \end{pmatrix}$ and $Y := \begin{pmatrix} X_2 & 0 & 0 & 0 & 0 \end{pmatrix}$ the matrix inequality (20) containing Q_3 can be estimated as

$$Q_3 \le Q_3' \le 0$$

where

$$Q_3' := \begin{pmatrix} \Gamma_1 & P_2 F C & 0 & P_2 F & 0 \\ C^T F^T P_2 & \Gamma_2 & P_2 & -P_2 F & I \\ 0 & P_2 & -\tau_2 K_f & 0 & 0 \\ F^T P_2 & -F^T P_2 & 0 & -\tau_3 K_\eta & 0 \\ 0 & I & 0 & 0 & \Gamma_3 \end{pmatrix} \le 0 \qquad (27)$$

with:

$$\begin{aligned} \Gamma_1 &:= P_2 (P_1^{-1} A_K^T + A_K P_1^{-1} + \tau_1 P_1^{-1} + \Lambda) P_2 \\ \Gamma_2 &:= A_F^T P_2 + P_2 A_F + \tau_1 P_2 \\ \Gamma_3 &:= -\frac{1}{\tau_2} K_x^{-1} + P_1^{-1} \Lambda^{-1} P_1^{-1} \end{aligned}$$

that, in view of (24),

$$\begin{pmatrix} -R_1-2X_2 & I \\ I & X_1 A^T + A X_1 + Y_1 B^T + B Y_1^T + \tau_1 X_1 + \Lambda \end{pmatrix} \le 0$$

The term Γ_3 in (27) can be bounded by R_2 as

$$P_1^{-1} \Lambda^{-1} P_1^{-1} - \frac{1}{\tau_2} K_x^{-1} \le R_2 \qquad (28)$$

Applying the Schur complement we can express (28) as

$$\begin{pmatrix} -R_2 - \frac{1}{\tau_2} K_x^{-1} & P_1^{-1} \\ P_1^{-1} & -\Lambda \end{pmatrix} \le 0$$

■

Remark 4. Notice that for fixed scalar parameters τ_1 and τ_2 the matrix inequalities (26) become LMIs. They can be solved using packages such as SeDuMi Toolbox, YALMIP Toolbox and the standard MATLAB LMI-toolbox.

Remark 5. The solutions obtained for the optimization problem (9) under (26) for fixed τ_1 and τ_2 can also be seen as suboptimal solutions.

V. NUMERICAL EXAMPLE

Consider the model of a double mass-spring system consisting in two unit masses, m_1 and m_2, connected by a elastic spring with spring constant k, sliding without friction along a fixed horizontal rod with control input u. and the following state space vector $x = \begin{pmatrix} x_1 & x_2 & v_1 & v_2 \end{pmatrix}^T$ where x_1, v_1 and x_2, v_2 are values of coordinate and velocity for the bodies. The corresponding state space representation:

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{pmatrix} u + w_x$$

The spring constant is an uncertain parameter that introduces nonlinearities in the system and is given by

$$0.8 \le k \le 1.2$$

One can rearrange the nonlinearities w_x to the form (5) with

$$K_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \delta = 0$$

$$K_x = \begin{pmatrix} 0.08 & -0.08 & 0 & 0 \\ -0.08 & 0.08 & 0 & 0 \\ 0 & 0 & 1e\text{-}10 & 0 \\ 0 & 0 & 0 & 1e\text{-}10 \end{pmatrix}$$

Also assume the following state-output mapping

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + w_y$$

where $w_y \in \mathrm{R}^2$ is an output noise which can be estimated as

$$w_y^T K_\eta w_y \le 1, \ K_\eta = \begin{pmatrix} 530 & 25 \\ 25 & 1960 \end{pmatrix}$$

A linear feedback is designed according to the main results of the previous section. The obtained parameters

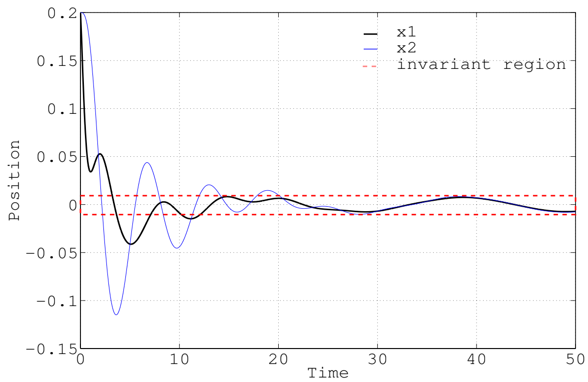

Fig. 1: Position of the double mass-spring model

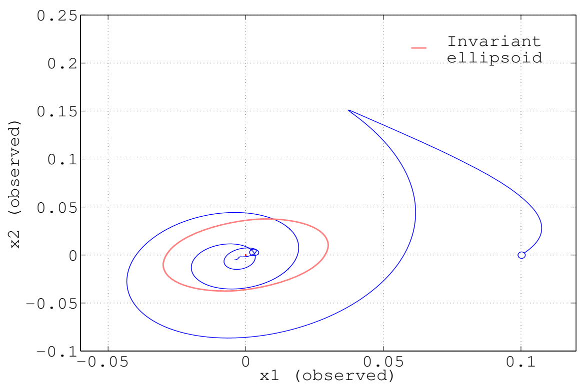

Fig. 2: Invariant ellipsoid for the observed state

K and F realizing the robust output linear controller are

$$K = \begin{pmatrix} \text{-15.353} & 3.243 & \text{-10.275} & \text{-10.908} \end{pmatrix}$$
$$F = \begin{pmatrix} 1.298 & 0.177 & 0.530 & 0.153 \\ 0.141 & 1.269 & 0.203 & 0.233 \end{pmatrix}^T$$

The obtained performance index (9) is 0.0117.

The figure 1 shows the position of the double mass-spring model and the figure 2 shows the projection of the invariant ellipsoid for $\hat{x}$ in the phase plane $(\hat{x}_1, \hat{x}_2)$.

Remark 6. As it can be seen from the above numerical simulation the closed loop system has a good stabilization performance, but the convergence processes in general are slow. This situation seems to be usual for robust linear controllers being applied to nonlinear models containing external noise or disturbances.

VI. CONCLUSIONS

Here the stabilization scheme for a class of nonlinear systems under the presence of uncertainties and disturbances is presented. As a principle part it contains the linear output controller using the current state estimates and designed based on the invariant ellipsoid technique. It minimizes the effect of the perturbations and nonlinearities in the system. Due to nonlinearities and perturbations the constraints involved into this optimization problem turns out to be bilinear matrix inequalities. Using appropriate transformations and fixing the scalar parameters (τ_1 and τ_2), the bilinear constraints become LMI. The robustness property of this scheme is confirmed by the application to the double mass-spring model.

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