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Decentralised Static Output feedback Stabilisation of Networks with $\mathscr{H}_2$ Performance

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Abstract— **In this paper global stabilisation of a complex network is attained by applying local decentralised static output feedback control, ensuring guaranteed bounds on quadratic performance. Necessary conditions for stabilisation of a complex network with quadratic performance are derived as a convex LMI representation. Strict positive realness conditions on the node level dynamics allow nonlinearities/uncertainties, which satisfy sector conditions, to be considered. A randomly generated academic example with 10 nodes is used to demonstrate the efficacy of the approach.**

I. Introduction

How multiple dynamical systems connected over an arbitrary network achieve global stabilisation with gauranteed performance is a significant research problem. Many researchers have made contributions to the control of network systems and cooperative control problems (see [5], [6] for an overview). In comparison to conventional control problems, the control of networks is much more demanding. One key issue is how the information topology of the network distribution, which plays a key role in determining the dynamical behaviour, can be suitably exploited in the problem. Making use of decentralised control strategies is attractive from the perspective of limited computing power and sensing capability.

A particular representation of the scale-free dynamical network [4] in a modified form will be utilised in this paper. According to [9], broadly speaking, state agreement, synchronisation and consensus problems can be viewed from an identical point of view. Central to these problems is the graph describing the topology of the interconnections. Algebraic graph theory has been widely employed in a variety of research works dealing with such systems (see the large number of publications in this area, [1]–[5], [7]–[9], [9]–[14]), where interconnections are represented in terms of a graph. Recently, the passivity concept has been used in [11] to study the coordination of dynamical systems in a group. In [11] the difference between the output variables of individual dynamical systems in a group is controlled to belong to a defined compact set, and studied as a set stability problem. The passivity concept is then employed to design the control law. The compact set is defined as a sphere in the case of a formation of vehicles and is considered as the origin in the case of consensus problems. Ref. [12] focuses on stabilisation of formations with linear dynamics, mainly with a full order decentralised controller. Ref. [12] suggested the possibility of analysing the stability of a network of N identical sytems, N nodes of the representative graph, by simply studying the stability of a node level system with modifications according to the eigenvalues of the associated graph Laplacian. In [8], a common Lyapunov function is made use of for studying the 'pinning' of complex networks and a state feedback control structure was utilised. A particular representation of the scale-free dynamical network [4], [8], in a modified form, will be utilised in this paper.

It is probably fair to say that great strides have been made in controlling and stabilising different classes of networked systems. A decentralized state feedback control law that guarantees consensus for the closed-loop system without disturbances as well as a state-feedback controller that achieves not only consensus, but optimal $\mathscr{H}_2$ performance for disturbance attenuation are synthesised in [15]. In [16] there has been work to design centralised optimal state feedback regulators for the synchronisation problem, and the $\mathscr{L}_2$ -norm of the error dynamics is considered as a performance index of synchronisability. Decentralised state feedback control of mobile robots and formation flying is discussed in [6]. Note that most of the studies use a state feedback approach. However in general, the idea of incorporating performance aspects within the output feedback control problem has received somewhat less attention. At least part of the reason for this could be the inherent complexity in understanding and then solving the stabilisation problem of the networked system itself.

In this paper, the stabilization of a class of dynamical systems operating over a network with guaranteed upper bound on the $\mathscr{H}_2$ performance of the network is considered. Algebraic graph theoretical tools, based on the connectivity of the graph [22], are used to represent multiple dynamical systems operating over the network [7], [8]. The individual node level dynamics are represented as a combination of linear and nonlinear parts. The primary objective is to stabilize the network with certain $\mathscr{H}_2$ performance bounds on individual nodes as well as at a network level by making use of a decentralised static output feedback control strategy. References [17], [21] provide details on output feedback stabilisation. The contribution of this paper is a methodology for systematic stabilisation of networks using decentralised static output feedback control strategies, formulated as convex linear matrix inequality problems. The paper also demonstrates the possibilities of exploiting positive realness in the closed loop nodes so that the formulation can handle certain classes of nonlinearities/uncertainties satisfying sector conditions.

II. Notation

The notation in the paper is quite standard. The set of real numbers, real-valued vectors of length m, and real-valued $m \times n$ matrices are given by $\mathbb{R}$, $\mathbb{R}^m$, and $\mathbb{R}^{m\times n}$ respectively. $\mathscr{C}ol(.)$ and $\mathscr{D}iag(.)$ denote a column and diagonal matrix. The symbols $\mathscr{N}(\cdot)$ and $\mathscr{R}(\cdot)$ represent the null space and range space of a matrix respectively. For a LTI system H, its impulse response is denoted by $H(t)$. The squared

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$\mathscr{H}_2$ -norm of a causal LTI system H is equal to $\|H\|_2^2 = \int_\infty^0 \text{trace}[H(t)^{\mathrm{T}}H(t)]dt$.

For the graph $\mathscr{G}$, the adjacency matrix $\mathscr{A}(\mathscr{G}) = [a_{ij}]$, is defined by setting $a_{ij} = 1$ if i and j are adjacent nodes and $a_{ij} = 0$ otherwise. This is a symmetric matrix. The symbol $\Delta(\mathscr{G}) = [\delta_{ij}]$ represents the degree matrix, and is an $N \times N$ diagonal matrix, where δ_{ii} is the degree of the vertex i. The Laplacian of $\mathscr{G}$, $\mathscr{L}(\mathscr{G})$, is defined as the difference $\Delta(\mathscr{G}) - \mathscr{A}(\mathscr{G})$.

III. System description

A distributed dynamical system operated over a connected network, consisting of N identical dynamical elements indexed as $1, 2, ..., N$ is considered in this paper. The system is viewed as a graph $\mathscr{G}$ with N labelled vertices or nodes. Each vertex/node represents an n-dimensional dynamical system. The nodes are assumed to be coupled linearly and diffusively [1], [4], [7]. As and where there is an interconnection between any two dynamical systems, it constitutes an edge connecting those nodes. The connectivity between the systems is assumed to be provided a-priori by the Laplacian of the graph $\mathscr{L}(\mathscr{G})$, from here on denoted as $\mathscr{L}$. The dynamics of the i^{th} individual node of the graph $\mathscr{G}$ are given in equations (1) and (2):

$$\dot{x}_i = Ax_i + Bu_i - \sum_{j=1}^{N} c\mathscr{L}_{ij}\Gamma x_j + f_i(x_i) \quad (1)$$

$$y_i = Cx_i \quad (2)$$

where, $x_i \in \mathbb{R}^n$ is the n-dimensional state vector of the i^{th} node of the network. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ represent the nominal linear part of the system comprising the dynamics of the i^{th} node.

Assumption 3.1: The matrices B and C have full column and row rank; and (A, B, C) is a minimal realization of the i^{th} node of the network $\mathscr{G}$.

The real constant $c > 0$ is the coupling strength between the i^{th} and j^{th} node.

Assumption 3.2: The coupling strength is assumed to be identical for all the connections between the nodes.

As described in Section II, $\mathscr{L} \in \mathbb{R}^{N\times N}$ denotes the connectivity of the topology of the network. The matrix $\Gamma = \tau_{ij} \in \mathbb{R}^{n\times n}$ represents the local coupling configuration among the states of the nodes. All the entries of Γ are 1 or 0 and represent the existence or non-existence of coupling in the respective channels in the network.

Assumption 3.3: The local coupling matrix Γ is assumed to be identical in each node of the network

$$\Gamma = \mathscr{D}iag\ [\tau_1, \tau_2, .., \tau_i, .., \tau_n]$$

In addition, assume $\text{rank}(\Gamma) = m$, implying no coupling in $n - m$ channels. By rearrangement of the states of the dynamics of each node, it is possible to ensure without loss of generality that Γ consists of the block diagonal matrix:

$$\Gamma = \mathscr{D}iag\ [I_m, 0] \quad (3)$$

The signals $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^p$ represent the control input and the measured outputs of the i^{th} node respectively. Here it is assumed that $p \geq m$. The functions $f_i(x_i)$, represent the nonlinear parts of the dynamical system and are assumed to satisfy certain sector bounds which will be precisely defined later in the paper.

Assumption 3.4: There exists a matrix $F \in \mathbb{R}^{m\times p}$ such that the triple (A, B, FC) is minimum phase and $rank(FCB) = m$.

Assumption 3.5: Assume $\mathscr{R}(\Gamma) = \mathscr{R}(B)$

Remark 1: The restriction $rank(FCB) = m$ can be interpreted as the dynamical mapping between the control signals and the outputs, is relative degree one.

Remark 2: The synthesis of a matrix F such that (A, B, FC) is minimum phase and $rank(FCB) = m$ is discussed in [19]. Necessary but not sufficient conditions are that (A, B, C) is minimum phase and $rank(CB) = m$, see [19]. Although this is still an open problem, in special cases it can be solved explicitly. For details see [19].

IV. Decentralised network stabilisation

A. Linear static output feedback case

Before addressing the control problem associated with system (1)-(2) discussed in Section III, certain preliminary results will be developed. Consider the linear system

$$\dot{x} = Ax + Bu \quad (4)$$

$$y = Cx \quad (5)$$

Suppose Assumptions 3.1 and 3.4 introduced in Section III hold for (4)-(5). Based on Assumption 3.4, there exists a mapping $x \mapsto \tilde{T}x$, such that in the new coordinate system the triple (A, B, FC) has the following special 4-block partitioned form [19]:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \quad FC = [\ F_1 \quad 0\ ] \quad (6)$$

where $F_1 \in \mathbb{R}^{m\times m}$ is nonsingular. The matrix $A_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$ is Hurwitz since the eigenvalues of A_{22} represent the invariant zeros of the triple (A, B, FC). The specific structures in (3) and (6) will be made use of in the subsequent proofs.

Problem 4.1: Design an output feedback control law of the form $u = -\gamma Fy$, where $\gamma \in \mathbb{R}^{m\times m}$, such that the quadratic performance

$$J = \int_0^\infty (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)dt \quad (7)$$

where $Q = Q^T \in \mathbb{R}^{n\times n} \geq 0$ and $R = R^T \in \mathbb{R}^{m\times m} > 0$ associated with (4)-(5) is minimised (or at least an upper bound on J is minimised).

Proposition 4.1: (Quadratic stability criterion) Consider the system given in (4)-(5) and the performance criterion given in (7). Suppose the state space representation is in the canonical form as in (6). Then, for $\gamma = R^{-1}$, the static output feedback control law $u = -\gamma Fy$, guarantees the stability of the closed loop system $(A - B\gamma FC)$. Furthermore the $\mathscr{H}_2$ performance (7) is bounded by $trace(\hat{P})$, where $\hat{P}$ is the optimal solution obtained by minimising $trace(P)$ with respect to γ, subject to the satisfaction of the Riccati inequality and the matching condition:

$$P(A - B\gamma FC) + (A - B\gamma FC)^{\mathrm{T}}P + Q + (\gamma FC)^{\mathrm{T}}R(\gamma FC) < 0 \quad (8)$$

and

$$PB = (FC)^{\mathrm{T}} \quad (9)$$

Proof: Consider the output feedback control law $u = -\gamma F y$ and the associated closed loop system $(A - B\gamma FC)$. A matrix inequality condition associated with stability as well as the quadratic performance index in (7) (see for example [23]) can be written as

$$\Theta := P(A - B\gamma FC) + (A - B\gamma FC)^{\mathrm{T}} P + Q + (\gamma FC)^{\mathrm{T}} R(\gamma FC) < 0 \tag{10}$$

Recall that from Assumption 3.4 the system triple (A, B, FC) has the canonical form in (6). In order to satisfy (9), and commensurate with the partitioning in (6), P is chosen as

$$P := \begin{bmatrix} P_1 & 0 \\ 0 & P_3 \end{bmatrix} > 0 \tag{11}$$

If $P_1 := F_1$ then, the classical matching condition [18]–[20] in (9) is satisfied [1]. Solution methods to synthesise a (P, F) pair to satisfy (9) are discussed in [19]. Choose a quadratic Lyapunov function candidate $V(x) = x^{\mathrm{T}} P x$. It follows from Eq.(4) and (10) that

$$\dot{V}(x) \leq -x^{\mathrm{T}} Q x - x^{\mathrm{T}} (\gamma FC)^{\mathrm{T}} R(\gamma FC) x = -x^{\mathrm{T}} Q x - u^{\mathrm{T}} R u$$

which by integrating over time implies

$$V(t) - V(0) \leq -\int_0^t (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u) dt$$

The LQR type Riccati inequality (10) ensures the closed loop system matrix $(A - B\gamma FC)$ is stable and therefore $V(t) \to 0$ as $t \to \infty$. Since $V(t) \to 0$ as $t \to \infty$, $J \leq V(0)$. Substituting the matching condition $PB = (FC)^{\mathrm{T}}$ from (9) into (10) yields

$$\Theta = PA + A^{\mathrm{T}} P - 2PB\gamma B^{\mathrm{T}} P + Q + PB\gamma R\gamma B^{\mathrm{T}} P \tag{12}$$

By 'completion of the square', (12) can be written as,

$$\Theta = PA + A^{\mathrm{T}} P + Q - PBR^{-1} B^{\mathrm{T}} P + PB \underbrace{(I - \gamma R) R^{-1} (I - \gamma R)}_{\Theta_c} B^{\mathrm{T}} P \tag{13}$$

A necessary and sufficient condition for $\Theta < 0$ in (13) is

$$PA + A^{\mathrm{T}} P + Q - PBR^{-1} B^{\mathrm{T}} P < 0 \tag{14}$$

Necessity is clear from (13), since $\Theta_c \geq 0$. Sufficiency can be seen by choosing $\gamma := R^{-1}$ which makes Θ_c in (13) zero, and (13) becomes the inequality in (14).

A bound on the $\mathscr{H}_2$ performance is provided as follows: equation (14) is bilinear, but a congruence transformation

$$\tilde{P} := \begin{bmatrix} \tilde{P}_1 & 0 \\ 0 & \tilde{P}_3 \end{bmatrix} = P^{-1} \tag{15}$$

followed by a Schur complement argument can be applied to achieve (14) in the LMI form given below:

$$\tilde{\Theta} = \begin{bmatrix} A\tilde{P} + \tilde{P} A^{\mathrm{T}} - BR^{-1} B^{\mathrm{T}} & \tilde{P} Q^{1/2} \\ Q^{1/2} \tilde{P} & -I \end{bmatrix} < 0 \tag{16}$$

As a result of the optimal 'natural' choice $\gamma = R^{-1}$, which makes $\Theta_c = 0$, it can be seen that the LMI (16) has no dependency on γ. Let $\hat{P}$ represent the solution to the optimization problem and so by construction

$$J \leq \mathrm{trace}(\hat{P}^{-1})$$

Since $J \leq V(0) = x(0)\hat{P}^{-1} x(0)$, for a uniformly distributed random distribution of normalised initial conditions $x(0)$, the expectation of J equals $\mathrm{trace}(\hat{P}^{-1})$. Consequently this provides an upper bound on the $\mathscr{H}_2$ norm. This completes the proof. □

Remark 3: Formally the LMI problem associated with (16) can be posed as:
Minimise: Trace(X)
subject to:

$$\begin{bmatrix} -X & I \\ I & -\tilde{P} \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} A\tilde{P} + \tilde{P} A^{\mathrm{T}} - BR^{-1} B^{\mathrm{T}} & \tilde{P} Q^{1/2} \\ Q^{1/2} \tilde{P} & -I \end{bmatrix} < 0 \tag{18}$$

where $\tilde{P}$ has the form in (15). Inequality (17) implies $X > \tilde{P}^{-1}$ and so minimising trace(X) minimises trace($\tilde{P}^{-1}$) =trace(P). This is a convex optimisation problem [24] and can be solved using standard LMI solvers [25]. The solution to the set of LMIs provides a $\hat{P}$ which also satisfies the matching condition in (9) for an appropriately scaled F. With the optimal choice of γ and manipulation of the matching condition equality, the static output feedback control law, ensuring *$\mathscr{H}_2$ performance* is:

$$u = -\gamma F y = -\gamma F C x = -R^{-1} B^{\mathrm{T}} \hat{P}^{-1} x$$

which recovers the well known solution structure.

Remark 4: For the $\mathscr{H}_2$ performance framework which follows, it is convenient to alter the representation of the system dynamics in (4)-(5) to include a "fictitious" input signal, $v \in \mathbb{R}^n$ and a performance output w. In the transformed coordinates (6), the system can be written as:

$$\dot{x} = Ax + Bu + v \tag{19}$$

$$w = Ex \tag{20}$$

where

$$E := \begin{bmatrix} Q^{1/2} \\ R^{-1/2} FC \end{bmatrix} \tag{21}$$

Finding the controller to minimise the LQR cost J from (7) is equivalent to minimising an upper bound on the $\mathscr{H}_2$ norm of the transfer function matrix

$$G(s) := E(sI_n - (A - B\gamma FC))^{-1} I_n \tag{22}$$

B. Decentralised $\mathscr{H}_2$ performance at Network Level

This subsection tackles the problem of interest in this paper, namely, the design of decentralised static output feedback control laws $u_i = -\gamma_i y_i$, for $i = 1, \ldots, N$, for the network system in (1) - (2). The problem can be defined as follows:

Problem 4.2: Design decentralised output feedback control laws $u_i = -\gamma_i y_i$, for $i = 1, \ldots, N$, for the network system in (1) - (2), satisfying an upper bound on the quadratic performance, that entirely depends on the individual node level closed loop dynamical system performance indices:

[1] If there exists an F so that (A, B, FC) is minimum phase then (A, B, SFC) where $det(S) \neq 0$ possess the same invariant zeros and hence the choice of $P_1 = F_1$ is not additional constraint.

specifically the problem seeks to minimize $J = \sum_{i=1}^{N} J_i$, where

$$J_i = \int_0^\infty (x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i) dt \tag{23}$$

is the decoupled individual node level performance index.

The result in Proposition 4.1 does not necessarily reveal anything about the *$\mathscr{H}_2$ performance at a network level* of the N identical dynamical systems connected according to an arbitrary graph $\mathscr{G}$. Initially, a network system in the absence of nonlinearities is considered, but in the sequel, the presence of a specific class of nonlinearities in the network dynamics is addressed with no alteration to the theory developed.

Consider the network dynamics

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i - \sum_{j=1}^{N} c\mathscr{L}_{ij}\Gamma x_j && (24)\\ y_i &= Cx_i && (25) \end{aligned}$$

for $i = 1,...,N$ which is the special case of (1)-(2), when $f_i(x_i) = 0$.

Theorem 4.1: (Network quadratic performance bound) Consider the linear network $\mathscr{G}$ given in (24) and (25) together with the decentralised static output feedback control law $u_i = -\gamma_i F y_i$, for $i = 1,...,N$, each providing an $\mathscr{H}_2$ performance bound J_i at the ith decoupled node level as in Proposition 4.1. Then the $\mathscr{H}_2$ performance of the network satisfies $J \le \sum_{i=1}^{N} J_i$.

Proof:

From Lemma 4.1, consider the local decentralised static output feedback control law for the N decoupled dynamical elements given by

$$u_i = -R^{-1} F y_i, \quad i = 1,...,N \tag{26}$$

Note the optimal 'natural' choice for γ has been used in (26). Write the dynamics given in (24) as

$$\dot{x}_i = (A - BR^{-1}FC)x_i - \sum_{j=1}^{N} c\mathscr{L}_{ij}\Gamma x_j + \nu_i \tag{27}$$

where $\nu_i \in \mathbb{R}^n$ represents the fictitious input signal vector at the nodes of the network whose effect is to be minimised in an $\mathscr{H}_2$ sense. Using the expression in (27), the dynamics of the overall network can be conveniently written as

$$\dot{x} = \underbrace{\big(I_N \otimes (A - BR^{-1}FC) - c(\mathscr{L} \otimes \Gamma)\big)}_{\mathbb{A}_c} x + \nu_c \tag{28}$$

where $\nu_c = \mathscr{C}ol\{\nu_1, \nu_2, ..., \nu_N\}$ and the collective state $x = \mathscr{C}ol(x_1, x_2, ..., x_N)$. (See [27] for details and properties of the Kronecker product operator "$\otimes$".) Since the Laplacian $\mathscr{L}$ is a s.p.d matrix, by spectral decomposition (see [26]), $\mathscr{L}$ can be written as

$$\mathscr{L} = VDV^{\mathrm{T}} \tag{29}$$

where the orthogonal matrix $V \in \mathbb{R}^{N \times N}$ is formed from the eigenvectors of $\mathscr{L}$, and $D \in \mathbb{R}^{N \times N}$ is a diagonal matrix formed from the eigenvalues so that

$$D := \mathscr{D}iag(d_1, d_2, .., d_i, ..., d_N) \tag{30}$$

with the property that $d_1 \ge d_2 \ldots \ge d_N = 0$. Define a coordinate transformation $T : x \mapsto z := Tx$, where

$$T := (V^{\mathrm{T}} \otimes I_n) \tag{31}$$

and V is the orthogonal matrix obtained from the spectral decomposition in (29). The transformation matrix T is an *orthogonal transformation* since using the properties of the Kronecker product (see [27])

$$(V^{\mathrm{T}} \otimes I_n)^{\mathrm{T}} (V^{\mathrm{T}} \otimes I_n) = (V \otimes I_n)(V^{\mathrm{T}} \otimes I_n) = (VV^{\mathrm{T}} \otimes I_n) = I_{nN}$$

Now consider the $\mathscr{H}_2$ performance at a network level. Associate with each node the output

$$w_i := E_i x_i \tag{32}$$

where

$$E_i = \begin{bmatrix} Q^{1/2} \\ R^{-1/2} FC \end{bmatrix}$$

Define

$$E_c := (I_N \otimes E_i) \tag{33}$$

and write $w_c := \mathscr{C}ol(w_1, w_2, \ldots w_N)$ so that

$$w_c = E_c x \tag{34}$$

From (28) and (34) the network system can be written as

$$G(s) \simeq (\mathbb{A}_c, I_{nN}, E_c)$$

with inputs ν_c and outputs w_c. The performance sought here is to minimise the effect of ν_c on w_c in a $\mathscr{H}_2$ sense. The $\mathscr{H}_2$ norm of $G(s)$ therefore represents a measure of the overall network performance in terms of an LQR cost.

To obtain the dynamics in suitable coordinates, the transformation T in (31) is applied to the outputs as well as the states of the network level dynamics. Applying the transformation T from (31) to the network level output w_c gives

$$\begin{aligned} w_c &= (I_N \otimes E_i)(V^{\mathrm{T}} \otimes I_n)^{-1} z \\ &= (I_N \otimes E_i)(V \otimes I_n) z \\ &= (V \otimes E_i) z && (35) \end{aligned}$$

Using the properties of the Kronecker identities, the transformation $T : x \mapsto z$ in (31), applied to (28) yields

$$\dot{z} = \underbrace{\big(I_N \otimes (A - BR^{-1}FC) - c(D \otimes \Gamma)\big)}_{\tilde{A}_c} z \tag{36}$$

where D is defined in (30). To have decoupling at the output level, another orthogonal transformation T_o is applied to w_c to obtain a new signal $\tilde{w}_c$ as follows:

$$\tilde{w}_c := \underbrace{(V^{\mathrm{T}} \otimes I_{m+n})}_{T_o} w_c \tag{37}$$

and so substituting from (35)

$$\tilde{w}_c = (V^{\mathrm{T}} \otimes I_{m+n})(V \otimes E_i) z = (I_N \otimes E_i) z = E_c z \tag{38}$$

Notice that if

$$\tilde{G}(s) := T_o G(s) T^{\mathrm{T}} \tag{39}$$

then because T from (31) and T_o from (37) are orthogonal $\|\tilde{G}(s)\|_2 = \|G(s)\|_2$. A realization of $\tilde{G}(s)$ from (39) is

$$\tilde{G}(s) \simeq (\tilde{A}_c, I_{nN}, E_c) \tag{40}$$

where $\tilde{A}_c$ is defined in (36) and E_c is defined in (34). With these transformations, the dynamics of an individual node in the transformed co-ordinates can be written

$$\dot{z}_i = \underbrace{(A - BR^{-1}FC - cd_i\Gamma)}_{A_i} z_i + v_i \tag{41}$$

$$\tilde{w}_i = E_i z_i \tag{42}$$

for $i = 1 \ldots N$, where $z = \mathscr{C}ol(z_1, z_2, \ldots z_N)$. This structure follows from (36) because of the diagonal nature of D from (30). Because of the decomposition of $\tilde{A}_c$ from (36) and $\tilde{w}$ from (37) into the structure in (41)-(42), it can be easily verified that

$$\|G(s)\|_2^2 = \|\tilde{G}(s)\|_2^2 = \sum_{i=1}^{N} \|\tilde{G}_i(s)\|_2^2 \tag{43}$$

where

$$\tilde{G}_i(s) \simeq ((A - cd_i\Gamma - BR^{-1}FC), I_n, E_i) \tag{44}$$

The following argument shows the $\mathscr{H}_2$ norm of $\tilde{G}_i(s)$ is bounded by trace$(\hat{P}^{-1})$. From the definition of $G_i(s)$ in (22), there exists a block diagonal matrix $\hat{P} = \mathscr{D}iag(\hat{P}_1, \hat{P}_3)$ satisfying (16) which means that $\|G_i(s)\|_2^2 \leq \text{trace}(\hat{P}^{-1})$. Using the four block partitions from (6) and (15), inequality (16) can be written as

$$\left[\begin{array}{cc|c} A_{11}\hat{P}_1 + \hat{P}_1 A_{11}^{\mathrm{T}} - R^{-1} & \Theta_2 & \hat{P}Q^{1/2} \\ \Theta_2^{\mathrm{T}} & \Theta_3 & \\ \hline \multicolumn{2}{c|}{Q^{1/2}\hat{P}} & -I \end{array}\right] < 0 \tag{45}$$

where $\Theta_2 := A_{12}\hat{P}_3 + \hat{P}_1 A_{21}^{\mathrm{T}}$ and $\Theta_3 := A_{22}\hat{P}_3 + \hat{P}_3 A_{22}^{\mathrm{T}}$. Since $cd_i \geq 0$, where the d_i are the eigenvalues of $\mathscr{L}$, the symmetric matrix $\mathscr{D}iag(-2cd_i\hat{P}_1, 0, 0) \leq 0$. Adding $\mathscr{D}iag(-2cd_i\hat{P}_1, 0, 0)$ to both sides of (45) means

$$\left[\begin{array}{cc|c} \Theta_1 & \Theta_2 & \hat{P}Q^{1/2} \\ \Theta_2^{\mathrm{T}} & \Theta_3 & \\ \hline \multicolumn{2}{c|}{Q^{1/2}\hat{P}} & -I \end{array}\right] < 0 \tag{46}$$

where $\Theta_1 = (A_{11} - cd_i I_m)\hat{P}_1 + \hat{P}_1(A_{11} - cd_i I_m)^{\mathrm{T}} - R^{-1}$. From the structure of Γ in (3), it can be verified that

$$\begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^{\mathrm{T}} & \Theta_3 \end{bmatrix} = (A - cd_i\Gamma)\hat{P} + \hat{P}(A - cd_i\Gamma)^{\mathrm{T}} - BR^{-1}B^{\mathrm{T}} \tag{47}$$

and therefore from the definition of $\tilde{G}_i(s)$ in (44), inequality (46) implies

$$\|\tilde{G}_i\|_2^2 \leq \text{trace}(\hat{P}^{-1})$$

and consequently from (43)

$$\|G(s)\|_2^2 \leq N \times \text{trace}(\hat{P}^{-1}) \tag{48}$$

This completes the proof □

Remark 5: The control law is 'decentralised' and 'static output feedback' in nature. The $\mathscr{H}_2$ performance at the network level has no direct relevance to the topology of the graph.

V. NONLINEAR EXTENSIONS

The results discussed so far pertain to the special case of system described in (1)-(2), when $f_i(x_i) = 0$. The results will now be extended to systems represented by (1)-(2), where nonlinearities in the system satisfy an additional assumption introduced in the sequel. The dynamics of the network with nonlinearities/uncertainties can be represented as:

$$\dot{x} = (I_N \otimes (A - BR^{-1}FC))x - c(L \otimes \Gamma)x + f(x) \tag{49}$$

$$y = (I_N \otimes C)x \tag{50}$$

where $f(x) = \mathscr{C}ol(f_1(x_1), \ldots f_N(x_N))$ represents the vector of nonlinearities.

Assumption 5.1: Suppose that the nonlinearities satisfy

$$f_i(x_i) = B\xi_i(x_i) \quad i = 1 \ldots N \tag{51}$$

for some functions of the states $\xi_i(x_i)$ where

$$(Fy_i)^{\mathrm{T}}(\xi_i) \leq 0 \tag{52}$$

is satisfied for all x_i, where y_i is thought of as Cx_i. Equation (52) represents a sector condition on the nonlinearity $\xi_i(x_i)$. Define $\xi = \mathscr{C}ol(\xi_1, \ldots, \xi_N)$. Because $(A - BR^{-1}FC)$ is stable, the triple $((A - BR^{-1}FC), B, FC)$ is strictly positive real [28], with Lyapunov matrix $\hat{P}$ satisfying the constraint $\hat{P}^{-1}B = (FC)^T$. Define $\mathbb{A} := I_N \otimes A$, $\mathbb{B} := I_N \otimes B$, $\mathbb{C} := I_N \otimes C$, $\mathbb{F} := I_N \otimes F$ and $\mathbb{P} := I_N \otimes \hat{P}^{-1}$. Notice that

$$\mathbb{P}\mathbb{B} = (\mathbb{F}\mathbb{C})^{\mathrm{T}}$$

by construction, since $\hat{P}^{-1}B = (FC)^{\mathrm{T}}$. From Assumption 5.1, it follows that

$$(\mathbb{F}y)^{\mathrm{T}}\xi = \sum_{i=1}^{N}(Fy_i)^{\mathrm{T}}\xi_i \leq 0, \quad i = 1 \ldots N$$

From the corresponding algebraic Riccati equation and the definition of $\mathbb{A}_c$ in (28) and $\mathbb{P}$

$$\mathbb{P}\mathbb{A}_c + \mathbb{A}_c^{\mathrm{T}}\mathbb{P} < 0,$$

it follows that $\mathbb{V}(x) = x^{\mathrm{T}}\mathbb{P}x$ is a Lyapunov function for the *nonlinear system* in (49) written as $\dot{x} = \mathbb{A}_c x + \mathbb{B}\xi(x)$. Furthermore $N \times \text{trace}(\hat{P}^{-1})$ is still a bound on the performance index J for the *nonlinear system (49)-(50)*, or the equivalent system representation (1)-(2).

VI. NUMERICAL EXAMPLE

To demonstrate the application of the theory developed in this paper, an academic example is used. Consider an arbitrary network consisting of 10 identical dynamical systems with 8 interconnections represented as a graph $\mathscr{G}(10, 8)$. The 10 nodes of the graph represent the identical dynamical systems. The linear parts of the dynamics at individual node level in (1) - (2) are given as follows:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{53}$$

The nonlinearity $f_i(x_i) = -|x_{i3}|\sin(x_{i1})^2 y$ satisfies 5.1. The local coupling matrix $\Gamma = \mathscr{D}aig\{1, 1, 0\}$, which satisfies Assumptions 3.3 and 3.5. The coupling strength c is

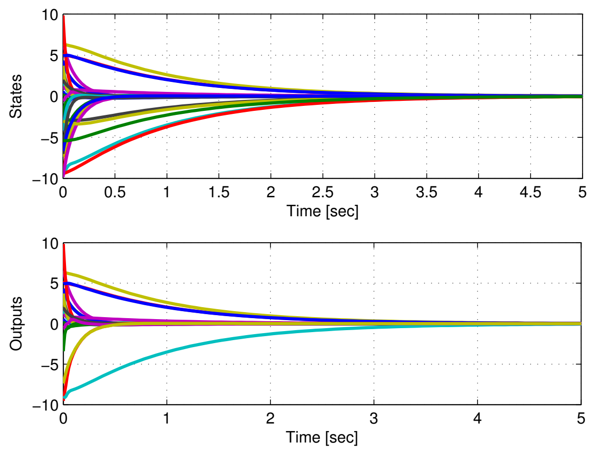

Fig. 1. Decentralised controller performance

identical (Assumption 3.2) and fixed as unity. The scaling matrix

$$F = \begin{bmatrix} 1.0598 & 9.4526 \\ 3.3409 & 1.0598 \end{bmatrix}$$

follows from Assumption 3.4 and makes the triple (A, B, FC) minimum phase. Compared with the usual state feedback policies, only output information will be utilised for stabilizing the network, which is realistic.

A locally decentralised static output feedback control law is designed following the LMI procedures described in (17)-(18). It should be noted that the decentralised static output feedback design requires only local information at the node level dynamics. Identical controllers are then used for each node of the network $\mathscr{G}$. Figure 1 shows the state and output time responses of the closed loop network dynamics with decentralised feedback. For clarity, only a 5 second interval is shown. In the decentralised feedback case, the computation of the Lyapunov matrix is at node level, no matter how large the number of nodes in the network. This is a very attractive feature of the decentralised feedback strategy.

VII. Conclusion

In this paper, the stabilisation of a class of nonlinear dynamical systems operating over a network with a guaranteed upper bound on the $\mathscr{H}_2$ performance of the network is considered. The individual node level dynamics are represented as a combination of linear and nonlinear parts, where the linear part is minimum phase and the nonlinearities/uncertainties satisfy sector bounded conditions. A transformation depending on the spectral properties of the network topology is used to achieve a suitable structure for providing the bounds on the $\mathscr{H}_2$ performance level. *Decentralised static output feedback* control is employed to stabilise a network consising of a class of dynamical systems with $\mathscr{H}_2$ performance at node level. An upper bound on $\mathscr{H}_2$ performance, relating to the performance at individual node level in the nonlinear systems and the entire network, is also provided. The controller synthesis is formulated as convex linear matrix inequality problems. This paper also demonstrates the possibilities of exploiting positive realness in the closed loop nodes so that the formulation can handle nonlinearities/uncertainties satisfying sector conditions.

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