2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Stability and performance analysis for input and output-constrained linear systems subject to multiplicative neglected dynamics

Giorgio Valmórbida, Sophie Tarbouriech, Germain Garcia, Jean-Marc Biannic

Abstract— **The influence of neglected dynamics appearing as multiplicative uncertainties on a closed-loop system with input and output saturations is studied. Constructive conditions based on (possibly linear) matrix inequalities, are provided in order to compute the influence of these beforehand neglected dynamics on both the estimate of the closed-loop system basin of attraction and the performance. The case of practical interest of neglected flexible modes and sensor dynamics is presented.**

I. Introduction

A controller design process is usually based on reduced models of the plant to be controlled. Such models can be obtained by intentionally neglecting some fast dynamics issued from sensors and actuators or some structural modes whose natural frequencies lie outside the controller bandwidth. However, it is hard to predict how these neglected dynamics influence the closed-loop system.

In a linear context, many results are available, most of which are based on μ-analysis [9]. In a nonlinear context, a few results have been obtained. Let us cite [2] where neglected dynamics issued from actuators were studied. In [3] and [4] the robustification or the redesign of control laws have been proposed to take into account the neglected dynamics.

Another feature of practical importance resides in the fact that most of physical control systems are subject to constraints in their domain of operation as amplitude and/or rate limitations in both actuators and sensors [6]. If the controller does not take into account these limits properly, undesired, even catastrophic behaviors may occur (see for example [10], [13]). Hence systems subject to saturating signals present numerous challenges for stability and performance analysis or the design of control laws [16].

The current paper analyzes the influence of neglected dynamics on both the estimate of the region of stability and the performance for a system subject to input and output saturations. The beforehand neglected dynamics are considered as multiplicative dynamics in the output of the system. Note that these dynamics can include flexible modes and sensor dynamics. To take into account these neglected dynamics a similar framework as the one developed in [19] is used. Then, by exploiting suitable Lyapunov functions, LMI-based conditions are proposed in order to quantify the degradation of the region of stability and the degradation of performance potentially induced by the beforehand neglected dynamics. Moreover, considering that some parameters of the neglected dynamics can be uncertain, admissible bounds on the uncertainties allowing to preserve stability and performance requirements are computed.

The paper is organized as follows. The addressed problem is formally stated in section II. Section III is dedicated to the main results of the paper, whereas computational issues are discussed in section IV. A numerical example illustrating the application of the approach for the case of a neglected dynamics composed by one flexible mode and a sensor dynamics is also presented. The paper ends with a conclusion giving some perspectives.

Notation. For any vector $x \in \Re^n$, $x \succeq 0$ means that all components of x denoted $x_{(i)}$ are non-negative. For two vectors $x, y \in \Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, for all $i = 1, ..., n$. The elements of a matrix $A \in \Re^{m\times n}$ are denoted by $A_{(i,j)}$, $i = 1, ..., m$, $j = 1, ..., n$. $A_{(i)}$ denotes the ith row of matrix A. For two symmetric matrices, A and B, $A > B$ (resp. $A \geq B$) means that $A - B$ is positive definite (resp. positive semi-definite). A' denotes the transpose of A. $Diag(x_1; \dots ; x_n)$ denotes the block-diagonal matrix obtained from vectors or matrices $x_1, ..., x_n$. Identity and null matrices are denoted respectively by I and 0. Furthermore, in the case of partitioned symmetric matrices, the symbol $\star$ denotes generically each of its symmetric blocks. For $v \in \Re^m$, $sat_{v_0}(v) : \Re^m \to \Re^m$ denotes the classical saturation function defined as $(sat_{v_0}(v))_{(i)} = sat_{v_0}(v_{(i)}) = sign(v_{(i)}) \min(v_{0(i)}, |v_{(i)}|)$, $\forall i = 1, ..., m$, where $v_{0(i)} > 0$ denotes the ith magnitude bound.

II. Problem statement

Consider the continuous-time system Σ consisting of a plant with input and output saturations

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bsat_{u_0}(u(t)) \\ y(t) &= Cx(t) \\ z(t) &= C_z x(t) \end{cases} \tag{1}$$

where $x \in \Re^n$, $u \in \Re^m$, $y \in \Re^p$. $z \in \Re^l$ is the regulated output. A dynamic output feedback controller $\mathcal{K}$ described by

$$\mathcal{K} : \begin{cases} \dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t) \\ & \quad + E_c(sat_{u_0}(y_c(t)) - y_c(t)) \\ y_c(t) &= C_c x_c(t) + D_c u_c(t) \end{cases} \tag{2}$$

with $x_c \in \Re^{n_c}$ and $u_c \in \Re^p$ has been designed by considering the connection with (1) taking $u_c = sat_{y_0}(y)$ and $u = y_c$ (see,

The first three authors are with LAAS-CNRS, Université de Toulouse, 7 Avenue du Colonel Roche 31077 Toulouse cedex 4, France. {gval-morb,tarbour,garcia@laas.fr}

G. Garcia is also with INSA, Toulouse, France.

The last author is with ONERA/DCSD, BP 74025, 31055 Toulouse Cedex, France. Jean-Marc.Biannic@cert.fr

This work is partially supported by the Programme Alban, the European Union Programme of High Level Scholarships for Latin America, scholarship no. (E06D100941BR).

for example [11]). All the matrices present in (1) and (2) are constant and have appropriate dimensions. The levels of the saturation terms are given respectively by the componentwise positive vectors $u_0 \in \Re^m$ and $y_0 \in \Re^p$.

Consider that the controller design was made neglecting some dynamics given by multiplicative uncertainties appearing on the output of the system as follows:

$$\Phi : \begin{cases} \dot{x}_{fs}(t) &= A_{fs}x_{fs}(t) + B_{fs}y(t) \\ y_{fs}(t) &= C_{fs}x_{fs}(t) \end{cases} \tag{3}$$

In this case, connection relating systems (1), (2) and (3) is then: $u = y_c$ and $u_c = sat_{y_0}(y_{fs})$. Hence, the complete closed-loop system reads:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bsat_{u_0}(y_c(t)) \\ y(t) &= Cx(t) \\ \dot{x}_c(t) &= A_c x_c(t) + B_c sat_{y_0}(y_{fs}(t)) \\ & \quad +E_c(sat_{u_0}(y_c(t)) - y_c(t)) \\ y_c(t) &= C_c x_c(t) + D_c sat_{y_0}(y_{fs}(t)) \\ \dot{x}_{fs}(t) &= A_{fs}x_{fs}(t) + B_{fs}y(t) \\ y_{fs}(t) &= C_{fs}x_{fs}(t) \\ z(t) &= C_z x(t) \end{cases} \tag{4}$$

and is depicted in Figure 1.

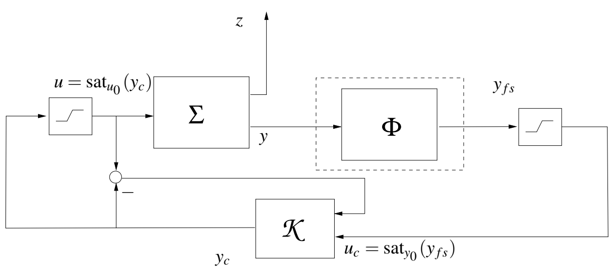

Fig. 1. Closed-loop system with multiplicative neglected dynamics and input/output saturations.

The basin of attraction of system (4), denoted $\mathcal{B}_a$, is defined as the set of all $(x, x_c, x_{fs}) \in \Re^n \times \Re^{n_c} \times \Re^{n_{fs}}$ such that for $(x(0), x_c(0), x_{fs}(0)) \in \mathcal{B}_a$ the corresponding trajectory converges asymptotically to the origin. Note, however, that the exact characterization of the basin of attraction is in general not possible. Then it is important to obtain estimates of this region. Regions of asymptotic stability can represent such estimates. Furthermore, in some practical applications one can be interested in ensuring the stability for a given set of admissible initial conditions. This set can be seen as a practical operation region for the system, or a region where the states of the system can be brought by the action of temporary disturbances.

In this work, we are first interested in evaluating the region of stability of the complete closed-loop system (4). Moreover, we are also interested in characterizing the potential degradation of the region of stability in particular directions when the neglected dynamics are taken into account. That can be used to measure the degradation of the closed-loop stability region induced by the neglected modes. In this context, the problem boils down to compare the sizes of the stability regions associated to system (1)-(2) and to the complete closed-loop system (4). Similarly, in presence of neglected dynamics, the evaluation of the potential degradation on the performances measured from the upper bound on the energy of the regulated output z.

The problem to be solved can then be summarized as follows.

Problem 1: Given the neglected dynamics (3) and a direction of interest $v \in \Re^{n+n_c+n_{fs}}$ with $v = \begin{bmatrix} \bar{v}' & 0 \end{bmatrix}$, $\bar{v} \in \Re^{n+n_c}$:

1) **Region of stability.** Characterize a region of stability for the closed-loop system (4), as large as possible.
2) **Stability degradation.** Quantify the possible degradation of the region of stability in the direction v.
3) **Performance degradation.** Quantify the possible degradation of the performance index.

Throughout the paper, Problem 1 will be addressed in two cases: matrices A_{fs}, B_{fs} and C_{fs} in (4) being known (nominal case) or being affected by parameter uncertainty (uncertain case).

III. Main Results

Let us define the augmented state vector

$$\xi = \begin{bmatrix} x' & x_c' & x_{fs}' \end{bmatrix}' = \begin{bmatrix} \bar{\xi}' & x_{fs}' \end{bmatrix}' \in \Re^{n+n_c+n_{fs}}$$

and two nonlinearities $\phi_{y_0} = sat_{y_0}(y_{fs}(t)) - y_{fs}(t)$ and $\phi_{u_0} = sat_{u_0}(y_c(t)) - y_c(t)$:

$$\begin{aligned} \phi_{y_0} &= sat_{y_0}(C_1\xi) - C_1\xi \\ \phi_{u_0} &= sat_{u_0}(C_2\xi + D_1\phi_{y_0}) - (C_2\xi + D_1\phi_{y_0}) \end{aligned}$$

with

$$\begin{aligned} C_1 &= \begin{bmatrix} 0 & 0 & C_{fs} \end{bmatrix} \\ C_2 &= \begin{bmatrix} 0 & C_c & D_c C_{fs} \end{bmatrix}; D_1 = D_c \end{aligned} \tag{5}$$

These nonlinearities are nested decentralized dead-zone functions since ϕ_{u_0} depends on ϕ_{y_0}. Hence, Lemma 1 in [17] applies.

By using the augmented state ξ above defined, the closed-loop system (4) can be written as:

$$\begin{cases} \dot{\xi}(t) &= A_1\xi(t) + B_1\phi_{y_0} + B_2\phi_{u_0} \\ y_{fs}(t) &= C_1\xi(t) \\ y_c(t) &= C_2\xi(t) + D_1\phi_{y_0} \\ z(t) &= C_3\xi(t) \end{cases} \tag{6}$$

with

$$\begin{aligned} A_1 &= \begin{bmatrix} A & BC_c & BD_cC_{fs} \\ 0 & A_c & B_cC_{fs} \\ B_{fs}C & 0 & A_{fs} \end{bmatrix} \\ B_1 &= \begin{bmatrix} BD_c \\ B_c \\ 0 \end{bmatrix}; B_2 = \begin{bmatrix} B \\ E_c \\ 0 \end{bmatrix}; C_3 = \begin{bmatrix} C_z & 0 & 0 \end{bmatrix} \end{aligned} \tag{7}$$

Suppose that we know a region of stability for the closed-loop without taking into account the neglected dynamics (3), *i.e.* for system (1)-(2), characterized as follows:

$$\mathcal{E}(W) = \{\bar{\xi} \in \Re^{n+n_c}; \bar{\xi}' W^{-1} \bar{\xi} \leq 1\}$$

where $W = W' > 0$, $W \in \Re^{(n+n_c)\times(n+n_c)}$. We also suppose that the direction of interest $\bar{v} \in \Re^{n+n_c}$ belongs to the boundary of $\mathcal{E}(W)$, *i.e.* $\bar{v}'W^{-1}\bar{v} = 1$. Moreover, for any initial condition in $\mathcal{E}(W)$, the energy of the regulated output z of system (1)-(2) satisfies:

$$\int_0^\infty z(t)'z(t)dt \leq \gamma \tag{8}$$

A. Nominal Case

Suppose that matrices A_{fs}, B_{fs} and C_{fs} of the neglected dynamics are perfectly known.

Proposition 1: Given $\bar{v} \in \Re^{n+n_c}$ and $\gamma > 0$ characterizing the region of stability and the performance of system (1)-(2). If there exist a symmetric positive definite matrix $Q \in \Re^{(n+n_c+n_{fs})\times(n+n_c+n_{fs})}$, two diagonal positive definite matrices $S_1 \in \Re^{p\times p}$, $S_2 \in \Re^{m\times m}$, two matrices, $Y_1 \in \Re^{p\times(n+n_c+n_{fs})}$, $Y_2 \in \Re^{m\times(n+n_c+n_{fs})}$, and two positive scalars β and δ satisfying:

$$M_0 = \left[\begin{array}{cccc} QA_1' + A_1Q & B_1S_1 - QC_1' + Y_1' & B_2S_2 - QC_2' + Y_2' & QC_3' \\ \star & -2S_1 & -S_1D_1' & 0 \\ \star & \star & -2S_2 & 0 \\ \star & \star & \star & -\gamma\beta I \end{array}\right] < 0 \tag{9}$$

$$\left[\begin{array}{cc} Q & Y_{1(i)}' \\ \star & y_{0(i)}^2 \end{array}\right] \geq 0, i = 1, ..., p \tag{10}$$

$$\left[\begin{array}{cc} Q & Y_{2(i)}' \\ \star & u_{0(i)}^2 \end{array}\right] \geq 0, i = 1, ..., m \tag{11}$$

$$\left[\begin{array}{cc} 1 & \delta\left[\begin{array}{cc} \bar{v}' & 0 \end{array}\right] \\ \star & Q \end{array}\right] > 0 \tag{12}$$

then:

1) System (6) is asymptotically stable for all initial conditions in the set
$$\mathcal{E}(Q) = \{\xi \in \Re^{n+n_c+n_{fs}}; \xi' Q^{-1}\xi \leq 1\} \tag{13}$$
2) The degradation of the size of the region of stability in the direction $v = \left[\begin{array}{cc} \bar{v}' & 0 \end{array}\right]'$ can be measured via the scalar δ.
3) The degradation of the performance can be measured via the scalar β.

Proof: Relations (9), (10) and (11) are obtained by considering the quadratic Lyapunov function $V(\xi) = \xi' Q^{-1}\xi$ and by invoking similar arguments like in [17] with respect to the complete closed-loop system (6). Indeed their satisfaction allows to guarantee that one gets: $\dot{V}(\xi) + \frac{1}{\beta\gamma}z'z \leq \dot{V}(\xi) + \frac{1}{\beta\gamma}z'z - 2\phi_{y_0}'S_1^{-1}(\phi_{y_0} - (Y_1Q^{-1} - C_1)\xi) - 2\phi_{u_0}'S_2^{-1}(\phi_{u_0} - (Y_2Q^{-1} - C_2)\xi + D_1\phi_{y_0}) < 0$, for any $\xi \in \mathcal{E}(Q)$. Hence item 1 readily follows.

The satisfaction of relation (12) means that $\delta\left[\begin{array}{cc} \bar{v}' & 0 \end{array}\right]' \in \mathcal{E}(Q)$. From this, given $\bar{v} \in \Re^{n+nc}$, δ corresponds to a measure of the size of the region of stability of the closed-loop system in the direction of interest $\bar{v}$ when $x_{fs} = 0$.

Similarly, a measure of the degradation on the upper bound of the regulated output energy with respect to (8) can be given by the positive scalar β. ■

Remark 1: Note that since A_{fs}, B_{fs}, C_{fs} and γ are given, relations (9), (10), (11) and (12) are LMIs in the decision variables.

Remark 2: The case where the original closed-loop system (1)-(2) is globally asymptotically stable can also be addressed. In this case, the region of stability of the closed-loop system (1)-(2) is $\Re^{n+n_c}$. In this context, when adding the neglected modes and studying the complete system (6), two scenarios are possible:

1) the property of global asymptotic stability is preserved and therefore one can measure the degradation of the performance by searching Q, S_1, S_2 and β solutions of relation (9) in which we set $Y_1 = 0$ and $Y_2 = 0$;
2) the property of global asymptotic stability is lost. As previously, the degradation of the performance is measured through the scalar β. The degradation of the region of stability in the directions $\bar{\xi}$ is directly measured by the size of the set $\left\{\bar{\xi} \in \Re^{n+n_c}; \left[\begin{array}{cc} \bar{\xi}' & 0 \end{array}\right] Q \left[\begin{array}{c} \bar{\xi} \\ 0 \end{array}\right]\right\} \leq 1$ which will be always included in the region of stability of the original system (1)-(2).

B. Uncertain Case

Let us consider the presence of uncertainty on the matrices A_{fs}, B_{fs} and C_{fs}. At this aim, two ways to represent the uncertainty are considered: norm-bounded and polytopic representations. The reader can consult, for example [5], [7], [8], [14], [15], for a description of these kinds of uncertainty. The matrices of neglected dynamics (3) are then first defined by:

$$\begin{array}{l} A_{fs} = A_{fs0} + G_1FH_1 \\ B_{fs} = B_{fs0} + G_2FH_2 \\ C_{fs} = C_{fs0} + G_3FH_3 \end{array} \tag{14}$$

with $F \in \Re^{n_i\times n_i}$ being the matrix containing all uncertain parameters.

The closed-loop system (4) reads:

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & (A_1 + \tilde{A}_1)\xi(t) + B_1\phi_{y_0} + B_2\phi_{u_0} \\ y(t) & = & (C_1 + \tilde{C}_1)\xi(t) \\ y_c(t) & = & (C_2 + \tilde{C}_2)\xi(t) + D_1\phi_{y_0} \\ z(t) & = & C_3\xi(t) \end{array}\right. \tag{15}$$

with A_1, B_1, B_2, C_1, C_2, C_3 and D_1 defined in (5) and (7) and

$$\begin{array}{rcl} \tilde{A}_1 & = & \left[\begin{array}{cc} 0_{(n+n_c)\times(n+n_c)} & B_{01}G_3FH_3 \\ G_2FH_2C_{01} & G_1FH_1 \end{array}\right]; \\ \tilde{C}_1 & = & \left[\begin{array}{cc} 0_{p\times(n+n_c)} & G_3FH_3 \end{array}\right]; \\ \tilde{C}_2 & = & \left[\begin{array}{cc} 0_{m\times(n+n_c)} & D_cG_3FH_3 \end{array}\right]. \end{array}$$

with

$$B_{01} = \begin{bmatrix} BD_c \\ B_c \end{bmatrix}; C_{01} = \begin{bmatrix} C & 0 \end{bmatrix}$$

The following result gives a solution to Problem 1 for the uncertain case.

Proposition 2: Consider $\bar{v} \in \Re^{n+n_c}$ and $\gamma > 0$ characterizing the region of stability and the performance of system (1)-(2). If there exist two diagonal positive definite matrices $Q \in \Re^{(n+n_c+n_{fs})\times(n+n_c+n_{fs})}$, $S_1 \in \Re^{p\times p}$, $S_2 \in \Re^{m\times m}$, two matrices $Y_1 \in \Re^{p\times(n+n_c+n_{fs})}$, $Y_2 \in \Re^{m\times(n+n_c+n_{fs})}$, positive scalars β, δ, ε and a diagonal matrix $\Theta \in \Re^{3n_i\times 3n_i}$ satisfying (10)-(12) and

$$\begin{bmatrix} M_0 & \varepsilon\mathcal{D} & \mathcal{E}' \\ \star & -\varepsilon I & 0 \\ \star & \star & -\Theta \end{bmatrix} < 0 \quad (16)$$

with M_0 defined in (9)

$$\mathcal{E} = \begin{bmatrix} \begin{bmatrix} H_2C_{01} & 0_{n_i\times n_{fs}} \\ 0_{n_i\times(n+n_c)} & H_3 \\ 0_{n_i\times(n+n_c)} & H_1 \end{bmatrix} Q & 0_{3n_i\times(p+m+l)} \end{bmatrix}$$

$$\mathcal{D} = \begin{bmatrix} 0_{(n+n_c)\times n_i} & B_{01}G_3 & 0_{(n+n_c)\times n_i} \\ G_2 & 0_{n_{fs}\times n_i} & G_1 \\ 0_{p\times n_i} & -G_3 & 0_{p\times n_i} \\ 0_{m\times n_i} & -D_cG_3 & 0_{m\times n_i} \\ 0_{l\times n_i} & 0_{l\times n_i} & 0_{l\times n_i} \end{bmatrix} \quad (17)$$

then:

1) System (15) is asymptotically stable in the set defined in (13), for any uncertainty satisfying $\mathcal{F}'\mathcal{F} \le \varepsilon\Theta^{-1}$, with $\mathcal{F} = \mathrm{diag}(F;F;F)$.
2) The degradation of the size of the region of stability in the direction $v = \begin{bmatrix} \bar{v}' & 0 \end{bmatrix}'$ can be measured via the scalar δ.
3) The degradation of the performance can be measured via the scalar β.

Proof: Consider A_{fs}, B_{fs} and C_{fs} given by (14). The stability of the uncertain system (15) is guaranteed if inequality below is satisfied:

$$M_0 + \mathcal{D}\mathcal{F}\mathcal{E} + \mathcal{E}'\mathcal{F}'\mathcal{D}' < 0 \quad (18)$$

with $\mathcal{E}$ and $\mathcal{D}$ given in (17) and $\mathcal{F} = \mathrm{diag}(F;F;F)$.

Using the fact that for $\varepsilon > 0$:

$$\mathcal{D}\mathcal{F}\mathcal{E} + \mathcal{E}'\mathcal{F}'\mathcal{D}' \le \varepsilon\mathcal{D}\mathcal{D}' + \frac{1}{\varepsilon}\mathcal{E}'\mathcal{F}'\mathcal{F}\mathcal{E}$$

and imposing $\frac{1}{\varepsilon}\mathcal{F}'\mathcal{F} \le \Theta^{-1}$, we have

$$M_0 + \mathcal{D}\mathcal{F}\mathcal{E} + \mathcal{E}'\mathcal{F}'\mathcal{D}' \le M_0 + \varepsilon\mathcal{D}\mathcal{D}' + \mathcal{E}'\Theta^{-1}\mathcal{E}$$

Then the satisfaction of (16) guarantees that (18) is verified. Similarly to Proposition 1, if relations (10)-(12) and (16) are satisfied, items 1 and 2 readily follow. ■

In the polytopic uncertainty case, matrices A_{fs}, B_{fs} and C_{fs} are defined as :

$$A_{fs} = \sum_{k=1}^{n_v} \eta_k A_{fsk}\ ,\ B_{fs} = \sum_{k=1}^{n_v} \eta_k B_{fsk}\ ,\ C_{fs} = \sum_{k=1}^{n_v} \eta_k C_{fsk} \quad (19)$$

where n_v is the number of vertices of the uncertainty and η belongs to the simplex

$$\mathcal{U} = \left\{ \eta \in \Re^{n_v} : \eta_k \ge 0, \sum_{k=1}^{n_v} \eta_k = 1, k = 1\dots n_v \right\}$$

Proposition 3: Given $\bar{v} \in \Re^{(n+n_c)}$ and $\gamma > 0$, if there exist a symmetric positive definite matrix $Q \in \Re^{(n+n_c+n_{fs})\times(n+n_c+n_{fs})}$, two diagonal positive definite matrices $S_1 \in \Re^{p\times p}$, $S_1 \in \Re^{m\times m}$, two matrices $Y_1 \in \Re^{p\times(n+n_c+n_{fs})}$, $Y_2 \in \Re^{m\times(n+n_c+n_{fs})}$, and two positive scalars β and δ satisfying (10)-(12) and

$$M_{0k} = \begin{bmatrix} QA'_{1k} + A_{1k}Q & B_1S_1 - QC'_{1k} + Y'_1 & B_2S_2 - QC'_{2k} + Y'_2 & QC'_3 \\ \star & -2S_1 & -S_1D'_1 & 0 \\ \star & \star & -2S_2 & 0 \\ \star & \star & \star & -\gamma\beta I \end{bmatrix} < 0, \quad k = 1,\dots,n_r$$

with

$$A_{1k} = \begin{bmatrix} A & BC_c & BD_cC_{fsk} \\ 0 & A_c & B_cC_{fsk} \\ B_{fsk}C & 0 & A_{fsk} \end{bmatrix}$$
$$C_{1k} = \begin{bmatrix} 0 & 0 & C_{fsk} \end{bmatrix}$$
$$C_{2k} = \begin{bmatrix} 0 & C_c & D_cC_{fsk} \end{bmatrix}$$

Then points 1, 2 and 3 in Proposition 1 are satisfied for the uncertain system defined by (19).

Remark 3: Propositions 1, 2 and 3 use a modified sector condition, which encompasses the classical one and allows to reduce the conservatism [18]. Other representations for the saturation terms coud also be used, in particular those based on differential inclusions framework [1], [12]. However, modifying relations (9) and (16) to account for such a representation increases the numerical complexity, leading to 2^m inequalities instead of one. When associated with the polytopic uncertain case (Proposition 3) such relations induce a number of LMIs given by $2^{(m+k)}$.

IV. Computational and Numerical Issues

Based on Proposition 1, a way to compute the degradation of the region of stability in the directions of $\bar{v} \in \Re^{n+n_c}$ consists in maximizing δ. Similarly, a way to quantify the degradation of the upper bound on the regulated output energy with respect to (8) can be done by minimizing β.

Similarly, in the uncertain case, Proposition 2 can be used to compute the maximal upper bound on the matrix of admissible uncertainty $\mathcal{F}$. To this purpose, a way consists in maximizing ε and in minimizing the elements of Θ simultaneously imposing lower bounds for degradation indices β and δ.

A. Neglected Sensor and Flexible Modes

In this section we treat the particular neglected dynamics composed by one flexible mode and a first-order mode corresponding to a sensor dynamics. This case is of practical interest because the fast dynamics of sensors and flexible dynamics whose natural frequency ω is expected to lie outside the controller bandwidth are usually neglected for control design purposes. For such dynamics, the transfer function between the true output y and the measured output y_{fs} is then given by

$$F(s) \quad = \quad \frac{\frac{1}{\tau}}{s+\frac{1}{\tau}}\left[\frac{\omega^2(1-d)}{s^2+2\zeta\omega s+\omega^2}+d\right] \qquad (20)$$

A minimal state-space representation (3) of $F(s)$ is obtained as follows:

$$\mathcal{A}_{fs} = \begin{bmatrix} -\frac{1}{\tau} & \frac{1-d}{\tau} & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & -2\zeta\omega \end{bmatrix} \qquad (21)$$

$$\mathcal{B}_{fs} = \begin{bmatrix} \frac{d}{\tau} \\ 0 \\ \omega \end{bmatrix}; \; \mathcal{C}_{fs} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad (22)$$

Remark 4: Interestingly, the matrices of the above state-space representation linearly depend on the natural frequency ω. Moreover, this property still holds in the case of multiple flexible modes possibly affecting several outputs.

One important feature in practical applications is the robustness of the controller against parametric uncertainties which usually affect the natural frequency of the flexible modes [9]. If uncertainty on the natural frequency is supposed in the form $\omega = \omega_0 + \Delta\omega$, where ω_0 corresponds to the nominal case, relation (14) is then defined by:

$$F = \Delta\omega I_3 \qquad (23)$$

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2\zeta \end{bmatrix}; \; G_2 = I_3; \; G_3 = 0 \qquad (24)$$

$$H_1 = H_3 = I_3; \; H_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (25)$$

Taking $\Theta = \theta I_3$ if conditions of Proposition 2 are satisfied, then the resulting system (15) is asymptotically stable in the set defined in (13), for any value of $\omega \in [\omega_0 - \Delta\omega, \;\; \omega_0 + \Delta\omega]$ with $\Delta\omega \leq \sqrt{\frac{\varepsilon}{\theta}}$.

Example 1

Consider the plant (1) and the controller (2) defined by matrices

$$A = \begin{bmatrix} -1 & 0.3 \\ 0 & 0.5 \end{bmatrix}; B = \begin{bmatrix} 0.5 \\ -10 \end{bmatrix}; C = \begin{bmatrix} 0.4 & 0.8 \end{bmatrix};$$
$$A_c = \begin{bmatrix} -4 & 1 \\ 0 & -8 \end{bmatrix}; B_c = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}; C_c = \begin{bmatrix} 0.18 & -1.2 \end{bmatrix};$$
$$D_c = 0.6; E_c = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}; C_z = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (26)$$

and the saturation limits $u_0 = 0.1$ and $y_0 = 0.5$.

Trying to optimize the ERA in the direction of $e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, the following matrix W and performance index γ were obtained for the nominal system:

$$W = \begin{bmatrix} 12.9716 & -2.8115 & -14.2349 & -1.7409 \\ -2.8115 & 2.1497 & 2.4734 & 0.2443 \\ -14.2349 & 2.4734 & 83.5134 & 15.4261 \\ -1.7409 & 0.2443 & 15.4261 & 3.3359 \end{bmatrix}$$

$$\gamma = 49.8825$$

and the $\bar{v}' = \begin{bmatrix} 2.6273 & 0 & 0 & 0 \end{bmatrix}$ is included in the region of attraction.

Considering a flexible dynamics with $\omega_1 = 20$, $\zeta = 0.1$, $d = 0.6$ and a sensor dynamics with $\tau = 0.001$, from expressions (21)-(22), system (3) is described by:

$$A_{fs} = \begin{bmatrix} -1000 & 400 & 0 \\ 0 & 0 & 20 \\ 0 & -20 & -4 \end{bmatrix} \qquad (27)$$

and

$$B_{fs} = \begin{bmatrix} 600 \\ 0 \\ 20 \end{bmatrix}; C_{fs} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad (28)$$

By imposing $\beta = 1$ (no degradation on the performance) and by using the conditions of Proposition 1 in order to maximize δ, one obtains

$$\delta = 0.9360$$

Figures 2 and 3 present respectively input and output time-responses for systems (1)-(2) and (6) with initial states $\bar{\xi}_0 = \begin{bmatrix} 2.4 & 0 & 0 & 0 \end{bmatrix}'$ and $\xi_0 = \begin{bmatrix} \bar{\xi}_0' & 0 & 0 & 0 \end{bmatrix}'$ belonging to the ERA. Dark solid lines correspond to the system (1)-(2) while dashed ones show the response taking into account the flexible dynamics defined by (27)-(28). Notice that the input u saturates on the time interval from $0s$ to $5s$ while output signal y saturates from $0s$ to $3s$.

Let us now consider the uncertainty on the natural frequency defining matrices F, G_i, H_i, $i = 1,2,3$ as in (23)-(25). By imposing $\beta = 1$ and $\delta > 0.7$ and by using the conditions of Proposition 2 in order to maximize the admissible upper bound on the uncertainty, one obtains:

$$\Delta\omega_1 = 0.5672$$

Example 2

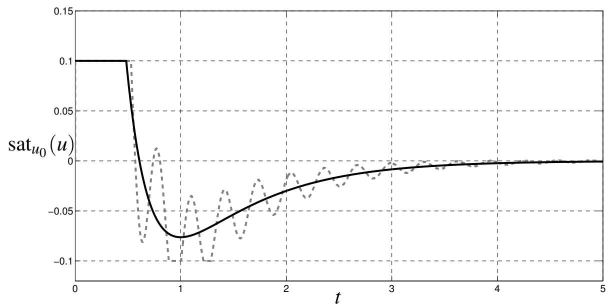

Fig. 2. System input considering the model and admissible neglected dynamics.

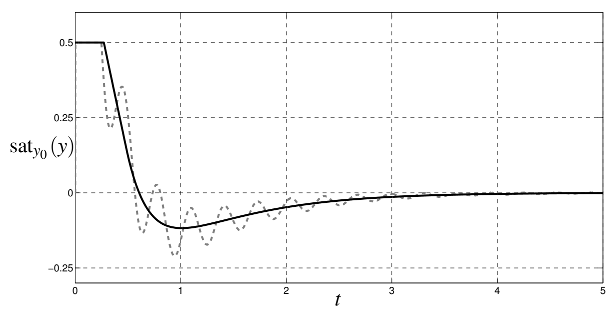

Fig. 3. Output and measured (saturated) output signal.

Consider system (1)-(2) defined with the same data as in Example 1. Now we consider that the uncertainty affecting the neglected dynamics (21)-(22) is of polytopic type. Hence, the admissible interval on the natural frequency is $\omega_1 \in [16,24]$, furthermore the other parameters are given by $\zeta = 0.1$, $\tau = 0.001$ and $d = 0.6$.

By imposing $\beta = 1$, and a lower bound $\delta > 0.7$ and by using the conditions of Proposition 3 trying to maximize δ, one obtains $\delta = 0.8571$.

V. CONCLUSION

This paper presents a framework for the analysis of the influence of neglected dynamics on a closed-loop system with input and output saturations. Such dynamics appear in a multiplicative form at the output of the system. The main result allows to characterize the influence on the size of the region of stability and on a performance index of the original system. The proposed conditions are based on quadratic Lyapunov functions and modified sector conditions. The extension to the case where uncertainties affect the matrices of the neglected dynamics are also provided. Then, the current paper can be viewed as complementary to [19] in which the influence of additive neglected dynamics was studied. It is important to note that the proposed results are based on a particular idea regarding the structure of the uncertainty affecting the closed-loop system, namely multiplicative neglected dynamics. In this sense, these results are preliminary ones. The objective in a next future could be to study how much modify them in order to cope with other kind of neglected dynamics, *i.e.* when no structure is a priory given. Such a work is under investigation, with also the objective to unify the current results with those of [19].

REFERENCES

[1] T. Alamo, A. Cepeda, and D. Limon. Improved computation of ellipsoidal invariant sets for saturated control systems. In *Proceedings of 44th Conference on Decision and Control (CDC-ECC'05)*, Sevilla, December 2005.

[2] R. W. Aldhaheri and H. K. Khalil. Effect of unmodeled actuator dynamics on output feedback stabilization of nonlinear systems. *Automatica*, 32(9):1323–1327, 1996.

[3] M. Arcak, M. Seron, J. Braslavsky, and P. Kokotovic. Robustification of backstepping against input unmodeled dynamics. *IEEE Transactions on Automatic Control*, 45(7):1358–1363, 2000.

[4] M. Arcak, A. Teel, and P. Kokotovic. Robust nonlinear control of feedforward systems with unmodeled dynamics. *Automatica*, 37:265–272, 2001.

[5] B. R. Barmish. Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *J. Optimization Theory and Applications*, 46(4):394–408, 1985.

[6] D. S. Bernstein. A plant taxonomy for designing control experiments. *IEEE Control Systems Magazine*, pages 7–14, 2001.

[7] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics, 1994.

[8] P. Dorato (Ed.). *Robust Control*. IEEE Press, 1987.

[9] G. Ferreres, J-F. Magni, and J-M. Biannic. Robustness analysis of flexible systems: Practical algorithms. *International Journal of Robust and Nonlinear Control*, 16(4):273–288, 2003.

[10] A.H. Glattfelder and W. Schaufelberger. *Control systems with input and output constraints*. Springer-Verlag, London, 2003.

[11] J.M. Gomes da Silva Jr. and S. Tarbouriech. Anti-windup design with guaranteed region of stability: an LMI-based approach. *IEEE Transactions on Automatic Control*, 50(1):106–111, 2005.

[12] T. Hu and Z. Lin. *Control Systems with Actuator Saturation: Analysis and Design*. Birkhäuser Boston, 2001.

[13] V. Kapila and K. Grigoriadis (Eds.). *Actuator saturation control*. Marcel Dekker, Inc., 2002.

[14] I. R. Petersen. Disturbance attenuation and H^∞ optimization: A design method based on the algebraic riccati equation. *IEEE Transactions on Automatic Control*, 32(5):427–429, 1987.

[15] I. R. Petersen. A stabilization algorithm for a class of uncertain systems. *System & Control Letters*, 8(4):351–357, 1987.

[16] S. Tarbouriech, G. Garcia, and A.H. Glattfelder (Eds.). *Advanced strategies in control systems with input and output constraints*. LNCIS, vol.346, Springer Verlag, 2007.

[17] S. Tarbouriech, C. Prieur, and J.M. Gomes da Silva Jr. Stability analysis and stabilization of systems presenting nested saturations. *IEEE Transactions on Automatic Control*, 51(8):1364–1371, 2006.

[18] S. Tarbouriech, C. Prieur, and J.M. Gomes da Silva Jr. Stability analysis and stabilization of systems presenting nested saturations. *IEEE Transactions on Automatic Control*, 51(8):1364–1371, 2006.

[19] S. Tarbouriech, G. Valmorbida, G. Garcia, and J.-M.Biannic. Stability and performance analysis for linear systems with actuator and sensor saturations subject to unmodeled dynamics. In *Proceedings of the American Control Conference (ACC'08)*, Seattle, USA, June 2008.