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# State Feedback Design for Input-Saturating Nonlinear Quadratic Systems

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*Abstract*— **This paper proposes a method to design stabilizing state feedback control laws for nonlinear quadratic systems subject to input saturation. Based on a quadratic Lyapunov function and a modified sector condition, synthesis conditions in a "quasi"-LMI form are stated in a regional (local) context. An LMI-based optimization problem is then derived for computing the state feedback gains maximizing the stability region of the closed-loop system.**

## I. INTRODUCTION

One of the most challenging issues in the stability analysis of nonlinear systems is the characterization of the region of attraction (RA) of stable equilibria. In general, the exact characterization of the RA of a given stable equilibrium point is not possible [7], [10]. It is then essential to be able to obtain at least estimates of this region. At this aim, level sets of Lyapunov functions can be used to compute invariant and contractive sets defining estimates for the region of attraction [4]. For numerical reasons, usually the chosen Lyapunov function belongs to the class of quadratic ones [6], defining estimates of ellipsoidal shape.

All information about the domain of attraction of a nonlinear system is lost if the stability analysis is made on a linearized model around the equilibrium point. In order to have an approximation of the true region of attraction of a smooth nonlinear system, it is possible to consider a second order approximation, that is, a system presenting quadratic terms. The interest of considering quadratic models is also motivated by the fact that several systems can be directly described by a quadratic model, see [11], [14].

One of the objectives in the controller design for nonlinear systems can be the maximization of the size of the region of attraction of a stable equilibrium. To this aim, some nonlinear terms can be introduced in the control law to counteract nonlinearities on the system. In this scenario, actuator limitations (such as saturations) may have an impact on the region of attraction of the system and must be taken into account. Saturating inputs can alter the behavior of the closed-loop system, leading to limit cycles or even instability. See, for example, [3], [8], [9], [12].

This paper addresses the problem of controller synthesis for quadratic nonlinear systems with saturating inputs. We are interested in obtaining the largest stability domain using quadratic Lyapunov functions. The control laws investigated present quadratic terms, as the system to be controlled, which are expected to balance the influence of quadratic terms on the system. The conditions we propose are expressed through matrix inequalities that become LMIs if a scalar parameter is fixed. Optimization is made over the whole family of quadratic Lyapunov functions.

This paper is organized as follows: The studied system is presented in section II where the addressed problem is precisely stated. Section III is dedicated to develop some preliminary results useful to provide the constructive conditions for controller synthesis presented in section IV. Section V focuses on the computation and numerical issues allowing to deal with the theoretical conditions.

**Notation.** For any vector $x \in \Re^n$, $x \succeq 0$ means that all components of $x$ denoted $x_{(i)}$ are nonnegative. For two vectors $x, y \in \Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, for all $i = 1,...,n$. The elements of a matrix $A \in \Re^{m\times n}$ are denoted by $A_{(i,j)}$, $i = 1,...,m$, $j = 1,...,n$. $A_{(i)}$ denotes the $i$th row of matrix $A$. For two symmetric matrices, $A$ and $B$, $A > B$ ($A \geq B$ ) means that $A - B$ is positive definite (positive semi-definite). $A'$ denotes the transpose of $A$. $\mathrm{diag}(x_1;\ldots;x_n)$ denotes the block-diagonal matrix obtained from vectors or matrices $x_1,...,x_n$. Identity and null matrices will be denoted respectively by $I$ and 0. Furthermore, in the case of partitioned symmetric matrices, the symbol $\star$ denotes generically each of its symmetric blocks. For $v \in \Re^m$, $sat_{v_0}(v) : \Re^m \to \Re^m$ denotes the classical saturation function defined as $(sat_{v_0}(v))_{(i)} = sat_{v_0}(v_{(i)}) = sign(v_{(i)})\min(v_{0(i)}, |v_{(i)}|)$, $\forall i = 1,...,m$, where $v_{0(i)} > 0$ denotes the $i$th magnitude bound. $\mathrm{mod}_n(i)$ stands for the remainder of the division of $i$ by $n$ where $i$ and $n$ are integers.

## II. PROBLEM STATEMENT

Consider the input-saturating quadratic system

$$\dot{x}(t) \quad = \quad Ax(t) + \begin{bmatrix} x(t)'A_{q1}x(t) \\ x(t)'A_{q2}x(t) \\ \vdots \\ x(t)'A_{qn}x(t) \end{bmatrix} + Bsat_{u_0}(u(t)) \qquad (1)$$

with $x \in \Re^n$, $u \in \Re^m$, $A \in \Re^{n\times n}$, $A_{qi} \in \Re^{n\times n}$, $i = 1,\ldots,n$, $B \in \Re^{n\times m}$. Let us define matrices $\mathcal{A}_q \in \Re^{n\times n^2}$ and $\mathcal{X}(t) \in \Re^{n^2\times n}$ as follows:

$$\mathcal{A}_q = \begin{bmatrix} A_{q1(1)} & A_{q1(2)} & \cdots & A_{q1(n)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{qn(1)} & A_{qn(2)} & \cdots & A_{qn(n)} \end{bmatrix} \qquad (2)$$

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and

$$\mathcal{X}(t) = \begin{bmatrix} x(t) & 0 & \dots & 0 \\ 0 & x(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x(t) \end{bmatrix} \quad (3)$$

where $A_{qi(j)} \in \Re^{1\times n}$ denotes the $j$th row of matrix $A_{qi} \in \Re^{n\times n}$. System (1) can then be written as follows:

$$\dot{x}(t) \quad = \quad (A + \mathcal{A}_q \mathcal{X}(t))x(t) + Bsat_{u_0}(u(t)) \quad (4)$$

Throughout the paper, we consider the following nonlinear control law:

$$u(t) \quad = \quad Kx(t) + \begin{bmatrix} x(t)'K_{q1}x(t) \\ x(t)'K_{q2}x(t) \\ \vdots \\ x(t)'K_{qm}x(t) \end{bmatrix} \quad (5)$$

with $K \in \Re^{m\times n}$ and $K_{qi} \in \Re^{n\times n}$, $i = 1,\dots,m$. By using (3) and by defining the matrix $\mathcal{K}_q \in \Re^{m\times n^2}$ as:

$$\mathcal{K}_q = \begin{bmatrix} K_{q1(1)} & K_{q1(2)} & \dots & K_{q1(n)} \\ \vdots & \vdots & \ddots & \vdots \\ K_{qm(1)} & K_{qm(2)} & \dots & K_{qm(n)} \end{bmatrix} \quad (6)$$

the control law (5) reads:

$$u(t) = (K + \mathcal{K}_q \mathcal{X}(t))x(t) \quad (7)$$

and the corresponding closed-loop system is defined by:

$$\dot{x}(t) = (A + \mathcal{A}_q \mathcal{X}(t))x(t) + Bsat_{u_0}((K + \mathcal{K}_q \mathcal{X}(t))x(t)) \quad (8)$$

In the absence of saturation, the stability of system (8), and more precisely the stability of $x = 0$, is related not only to $A+BK$ but also to the state-dependent term $(\mathcal{A}_q + B\mathcal{K}_q)\mathcal{X}(t)$. It is important to point out that even if matrix $A$ is Hurwitz, the stability of the system can be studied only in a regional (local) context [2]. When saturations nonlinearities are present, the exact characterization of the basin of attraction of the origin is, in general, not possible, even for systems containing only linear terms. Thus, besides guaranteeing the stabilization of the origin, we are interested in providing an estimate of the basin of attraction for the closed-loop system (8). The problem we intend to solve can be summarized as follows.

*Problem 1:* Determine feedback matrices $K$ and $K_{qi}$, $i = 1,\dots,m$, and a region $S_0 \subseteq \Re^n$, as large as possible, such that for any initial condition $x_0 \in S_0$, the resulting trajectories of systems (8) asymptotically converge to the origin.

## III. Preliminary Results

Consider a symmetric matrix $P = P' > 0 \in \Re^{n\times n}$ and the matrix $H \in \Re^{n\times 2^n}$ whose columns correspond to the vertices of an hypercube:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & -1 & -1 & \dots & -1 \\ 1 & -1 & 1 & -1 & \dots & -1 \end{bmatrix} \quad (9)$$

Consider now matrix $V = P^{-\frac{1}{2}}H \in \Re^{n\times 2^n}$. Since no column of $V$ can be written as a convex combination of the others, they define the vertices of a polytope $\mathcal{V}$ in $\Re^n$. The $i$th face, $i = 1,\dots,2n$, of such a polytope can be defined by the convex combination of a subset of vertices of polytope $\mathcal{V}$:

$$\mathcal{F}_i = \left\{ p_{fi} \in \Re^n; p_{fi} = \sum_{j=1}^{2^{(n-1)}} \alpha_j v_{ij}, \sum_{j=1}^{2^{(n-1)}} \alpha_j = 1 \right\} \quad (10)$$

where vectors $v_{ij}$, $j = 1,\dots,2^{(n-1)}$, correspond to the vertices of the $i$th face of $\mathcal{V}$. Such vectors $v_{ij}$ are given by the columns of matrix $V_i = P^{-\frac{1}{2}}H_i$, $H_i \in \Re^{n\times 2^{(n-1)}}$ being the matrix composed by the columns of $H$ where the $\mathrm{mod}_n(i)$th, element is equal to 1 if $i \in \{1,\dots,n\}$ or equal to $-1$ for $i \in \{n+1,\dots,2n\}$. For example, in the case $n = 2$, one gets

$$H = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \quad (11)$$

and for $i = 3$, one obtains

$$H_3 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad (12)$$

Let $H_{ij}$ represent the $j$th, $j = 1,\dots,2^{(n-1)}$, column of matrix $H_i$.

*Proposition 1:* The ellipsoid $\mathcal{E}(P) = \{x \in \Re^n; x'Px \le 1\}$ is contained in the polytope $\mathcal{V}$.

*Proof:* To prove that $\mathcal{E}(P) \subset \mathcal{V}$ we show that each face of the polytope is exterior or touches the ellipsoid. Take face $\mathcal{F}_i$, it lies outside or touches $\mathcal{E}(P)$ if $p'_{fi}Pp_{fi} \ge 1$, $\forall p_{fi} \in \mathcal{F}_i$. From (10) one gets the following expression for $p'_{fi}Pp_{fi}$:

$$\begin{aligned} p'_{fi}Pp_{fi} &= (\textstyle\sum_{j=1}^{2^{(n-1)}} \alpha_j v_{ij})' P \sum_{j=1}^{2^{(n-1)}} \alpha_j v_{ij} \\ &= (\textstyle\sum_{j=1}^{2^{(n-1)}} \alpha_j H_{ij})' P^{-\frac{1}{2}} P P^{-\frac{1}{2}} \sum_{j=1}^{2^{(n-1)}} \alpha_j H_{ij} \\ &= (\textstyle\sum_{j=1}^{2^{(n-1)}} \alpha_j H_{ij})' (\sum_{j=1}^{2^{(n-1)}} \alpha_j H_{ij}) \end{aligned} \quad (13)$$

Consider the vector $\eta_i = (\sum_{j=1}^{2^{(n-1)}} \alpha_j H_{ij})$. Thanks to the structure of $H_i$, $\eta_i$ has one element, let us suppose the $\ell$th one, equal to 1 or $-1$ and the other elements are given by functions $g_h(\alpha)$, $h = 1,\dots,n$, $h \ne \ell$, linear in $\alpha$ and satisfying $|g_h(\alpha)| \le 1$. Then one gets:

$$\begin{aligned} p'_{fi}Pp_{fi} &= \eta_i'\eta_i \\ &= 1 + \textstyle\sum_{h=1}^{\ell-1} g_h(\alpha)^2 + \sum_{h=\ell+1}^{n} g_h(\alpha)^2 \\ &\ge 1 \end{aligned} \quad (14)$$

Hence, according to [5] (sec. 5.2.2) we can state that every point on the face $\mathcal{F}_i$ of the polytope is exterior to the ellipsoid $\mathcal{E}(P)$. By applying the same reasoning with each face of the polytope $\mathcal{V}$, we can conclude that all the faces of the polytope are exterior to the ellipsoid, guaranteeing then $\mathcal{E}(P) \subset \mathcal{V}$. ■

Proposition 1 states that both an ellipsoid and a polytope, containing this ellipsoid, can be parameterized using the same symmetric positive definite matrix $P$. Let us present a result on stability analysis (i.e. $u(t) = 0$).

*Proposition 2:* If there exist a matrix $P = P' > 0 \in \Re^{n\times n}$ and a positive scalar $\varepsilon$ such that the inequalities

$$\begin{bmatrix} A'P + PA & \varepsilon P\mathcal{A}_q & \mathcal{X}_i' \\ \star & -\varepsilon\tilde{P} & 0 \\ \star & \star & -\varepsilon \mathrm{I} \end{bmatrix} < 0, \quad i = 1,\dots,2^n \tag{15}$$

are satisfied with $\tilde{P} = \mathrm{diag}(P;\dots;P) \in \Re^{n^2\times n^2}$ and

$$\mathcal{X}_i = \begin{bmatrix} h_i & 0 & \dots & 0 \\ 0 & h_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_i \end{bmatrix} \tag{16}$$

$h_i$, $i$th column of $H$ as in (9), then the ellipsoid $\mathcal{E}(P) = \{x \in \Re^n \ ; \ x'Px \le 1\}$ is an estimate of the region of attraction (ERA) for the system (1) with $u(t) = 0$.

*Proof:* Consider the quadratic Lyapunov function $v(x) = x'Px$. To determine an estimate of the region of attraction we are first interested in finding the polytope $\mathcal{V}$ inside which $\dot{v}(x)$ is a negative definite function. At each vertex $x_i$, $i = 1,\dots,2^n$, of polytope $\mathcal{V}$ the derivative of $v(x)$ is given by

$$\dot{v}(x_i) = x_i'(A'P + PA + P\mathcal{A}_q\tilde{P}^{-\frac{1}{2}}\mathcal{X}_i + \mathcal{X}_i'\tilde{P}^{-\frac{1}{2}}\mathcal{A}_q'P)x_i \tag{17}$$

To guarantee that $\dot{v}(x) < 0$ at the vertices, and therefore inside $\mathcal{V}$, it suffices to verify the following inequality:

$$A'P + PA + P\mathcal{A}_q\tilde{P}^{-\frac{1}{2}}\mathcal{X}_i + \mathcal{X}_i'\tilde{P}^{-\frac{1}{2}}\mathcal{A}_q'P < 0, \ i = 1,\dots,2^n \tag{18}$$

By using

$$P\mathcal{A}_q\tilde{P}^{-\frac{1}{2}}\mathcal{X}_i + \mathcal{X}_i'\tilde{P}^{-\frac{1}{2}}\mathcal{A}_q'P \le \varepsilon P\mathcal{A}_q W^{-1}\mathcal{A}_q'P + \frac{1}{\varepsilon}\mathcal{X}_i'\tilde{P}^{-\frac{1}{2}}W\tilde{P}^{-\frac{1}{2}}\mathcal{X}_i \tag{19}$$

with $\varepsilon > 0$ and by taking $W = \tilde{P}$, we obtain

$$P\mathcal{A}_q\tilde{P}^{-\frac{1}{2}}\mathcal{X}_i + \mathcal{X}_i'\tilde{P}^{-\frac{1}{2}}\mathcal{A}_q'P \le \varepsilon P\mathcal{A}_q\tilde{P}^{-1}\mathcal{A}_q'P + \frac{1}{\varepsilon}\mathcal{X}_i'\mathcal{X}_i \tag{20}$$

Thus inequality (18) is satisfied if

$$A'P + PA + \varepsilon P\mathcal{A}_q\tilde{P}^{-1}\mathcal{A}_q'P + \frac{1}{\varepsilon}\mathcal{X}_i'\mathcal{X}_i < 0 \ , \ i = 1,\dots,2^n \tag{21}$$

which is equivalent to (15) by Schur complement.

If relation (15) is verified it follows that the time-derivative of the quadratic Lyapunov function is negative at the vertices of polytope $\mathcal{V}$. According to Proposition 1 we have $\mathcal{E}(P) \subset \mathcal{V}$. Thus, it follows $\dot{v}(x) < 0$, for all $x \in \mathcal{E}(P)$. The ellipsoid $\mathcal{E}(P)$ is therefore a region of stability for system (1) with $u(t) = 0$. This region can be considered as an estimate of the region of attraction (ERA). ■

*Remark 1:* Parameterization of polytope $\mathcal{V}$ avoids additional inequalities to guarantee the inclusion of the ellipsoid in an outer polytope as in Theorem 2 of [2].

## IV. Main results

Define the decentralized deadzone nonlinearity $\phi \in \Re^m$ as follows:

$$\phi(x(t)) = sat_{u_0}((K + \mathcal{K}_q\mathcal{X}(t))x(t)) - (K + \mathcal{K}_q\mathcal{X}(t))x(t) \tag{22}$$

From (22), the closed-loop system (8) reads:

$$\dot{x}(t) = (A + BK + (\mathcal{A}_q + B\mathcal{K}_q)\mathcal{X}(t))x(t) + B\phi(x(t)) \tag{23}$$

Let us now propose a result in the control design context by using the same tools as in Proposition 2.

*Proposition 3:* If there exist $Q = Q' > 0 \in \Re^{n\times n}$, $L \in \Re^{m\times n}$, $\mathcal{L}_q \in \Re^{m\times n^2}$, $Y \in \Re^{m\times n}$, a diagonal matrix $S_1 > 0 \in \Re^{m\times m}$ and a positive scalar $\varepsilon$ such that the following inequalities hold:

$$\begin{bmatrix} QA' + AQ + L'B' + BL & BS_1 + Y' - L' & \varepsilon(\mathcal{A}_q\tilde{Q} + B\mathcal{L}_q) & Q\mathcal{X}_i' \\ \star & -2S_1 & -\varepsilon\mathcal{L}_q & 0 \\ \star & \star & -\varepsilon\tilde{Q} & 0 \\ \star & \star & \star & -\varepsilon I \end{bmatrix} < 0 \quad i = 1,\dots,2^n \tag{24}$$

$$\begin{bmatrix} Q & Y_{(j)}' \\ \star & u_{0(j)}^2 \end{bmatrix} \ge 0 \ , \ j = 1,\dots,m \tag{25}$$

with $\tilde{Q} = \mathrm{diag}(Q;\dots;Q) \in \Re^{n^2\times n^2}$ then matrix gains $K = LQ^{-1}$ and $\mathcal{K}_q = \mathcal{L}_q\tilde{Q}^{-1}$, stabilize the closed-loop system (8) for every initial condition belonging to $\mathcal{E}(Q^{-1}) = \{x \in \Re^n \ ; \ x'Q^{-1}x \le 1\}$.

*Proof:* Considering the Lyapunov quadratic function $v(x) = x'Px$, its time-derivative along the trajectories of system (23) is given by:

$$\dot{v}(x) = \begin{bmatrix} x' & \phi' \end{bmatrix} \begin{bmatrix} R_1 & PB \\ \star & 0 \end{bmatrix} \begin{bmatrix} x \\ \phi \end{bmatrix} \tag{26}$$

with

$$\begin{aligned} R_1 = {} & (A + BK)'P + P(A + BK) \\ & + P(\mathcal{A}_q + B\mathcal{K}_q)\mathcal{X} + \mathcal{X}'(\mathcal{A}_q + B\mathcal{K}_q)'P \end{aligned}$$

Using Lemma 1 from [13] we can verify that

$$-2\phi'S_1^{-1}(\phi + (K + \mathcal{K}_q\mathcal{X})x - YPx) \ge 0 \tag{27}$$

with $S_1$ a positive diagonal matrix, provided that $x \in S(u_0) = \{x \in \Re^n; -u_0 \preceq YPx \preceq u_0\}$. Relation (25) guarantees that the ellipsoid $\mathcal{E}(Q^{-1})$ is included in $S(u_0)$, by noting $P = Q^{-1}$. Hence, for any $x \in \mathcal{E}(Q^{-1})$ one gets $\dot{v}(x) \leq \dot{v}(x) - 2\phi' S_1^{-1}(\phi + (K + \mathcal{K}_q \mathcal{X})x - YPx)$ which allows to obtain the expression

$$\dot{v}(x) \leq \begin{bmatrix} x'P & \phi' S_1 \end{bmatrix} \mathcal{M}(x) \begin{bmatrix} Px \\ S_1 \phi \end{bmatrix} \tag{28}$$

where matrix $\mathcal{M}(x)$ is defined by setting $Q = P^{-1}$ and $L = KQ$ as:

$$\mathcal{M}(x) = \begin{bmatrix} R_2 & BS_1 + Y' - L - Q(\mathcal{K}_q \mathcal{X})' \\ \star & -2S_1 \end{bmatrix} \tag{29}$$

with

$$\begin{aligned} R_2 = {} & Q(A + BK)' + (A + BK)Q \\ & + (\mathcal{A}_q + B\mathcal{K}_q)\mathcal{X}Q + Q\mathcal{X}'(\mathcal{A}_q + B\mathcal{K}_q)' \end{aligned}$$

To have $\dot{v}(x) < 0$ inside the polytope $\mathcal{V}$ it suffices to verify that at its vertices $x_i$, $i = 1, \ldots, 2^n$, we have $\mathcal{M}(x_i) < 0$. Knowing that $\mathcal{X} = \tilde{Q}^{\frac{1}{2}} \mathcal{X}_i$ at $x_i$ ($\mathcal{X}_i$ as in (16)) we obtain:

$$\mathcal{M}(x_i) = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_1' \tag{30}$$

with

$$\mathcal{M}_0 = \begin{bmatrix} AQ + BL + QA' + L'B' & BS_1 + Y' - L' \\ \star & -2S_1 \end{bmatrix}$$

$$\mathcal{M}_1 = \begin{bmatrix} \mathcal{A}_q + B\mathcal{K}_q \\ -\mathcal{K}_q \end{bmatrix} \tilde{Q}^{\frac{1}{2}} \begin{bmatrix} \mathcal{X}_i Q & 0 \end{bmatrix}$$

Term $\mathcal{M}_1$ verifies

$$\begin{aligned} \mathcal{M}_1 + \mathcal{M}_1' \leq {} & \varepsilon \begin{bmatrix} \mathcal{A}_q + B\mathcal{K}_q \\ -\mathcal{K}_q \end{bmatrix} \tilde{Q} \begin{bmatrix} \mathcal{A}_q' + \mathcal{K}_q' B' & -\mathcal{K}_q' \end{bmatrix} \\ & + \varepsilon^{-1} \begin{bmatrix} Q\mathcal{X}_i' \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}_i Q & 0 \end{bmatrix} \end{aligned} \tag{31}$$

Then if the set of inequalities

$$\begin{aligned} \mathcal{M}_0 + {} & \varepsilon \begin{bmatrix} \mathcal{A}_q + B\mathcal{K}_q \\ -\mathcal{K}_q \end{bmatrix} \tilde{Q} \begin{bmatrix} \mathcal{A}_q' + \mathcal{K}_q' B' & -\mathcal{K}_q' \end{bmatrix} \\ & + \varepsilon^{-1} \begin{bmatrix} Q\mathcal{X}_i' \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}_i Q & 0 \end{bmatrix} < 0 \ , \ i = 1, \ldots, 2^n \end{aligned} \tag{32}$$

is satisfied, we have $\mathcal{M}(x) < 0$, hence $\dot{v}(x) < 0$ inside the polytope $\mathcal{V}$. If we take $\mathcal{L}_q = \mathcal{K}_q Q$, the set of inequalities (32) becomes equivalent to (24). That allows us to conclude that $\dot{v}(x) \leq \dot{v}(x) - 2\phi' S_1^{-1}(\phi + (K + \mathcal{K}_q \mathcal{X})x - YPx) < 0$ for any $x \in \mathcal{E}(Q^{-1}) \subset \mathcal{V}$ (recall that by definition the ellipsoid $\mathcal{E}(Q^{-1})$ is included in the polytope $\mathcal{V}$). Thus the ellipsoid $\mathcal{E}(Q^{-1})$ is a region of stability for the closed-loop system (8).

■

*Remark 2:* A particular situation for control design proposed here is when the controller contains only linear terms. In this case, it suffices to take $\mathcal{L}_q = 0$ in Proposition 3.

*Remark 3:* The approach used in [2] could also be used to provide a solution to Problem 1. The main difference would consist of fixing a priori a polytope defined as

$$\begin{aligned} \mathcal{P} & = \operatorname{co}\{x_i \in \Re^n; i = 1, \ldots, n_v\} \\ & = \left\{ x \in \Re^n; a_{(k)}' x \leq 1, k = 1 \ldots, n_v \right\} \end{aligned} \tag{33}$$

and to verify that the closed-loop trajectories of system (8) converge to the origin for all initial condition belonging an ellipsoid contained in such a polytope. Hence, by using the framework developed in [2], Proposition 3 is modified as follows:

- Relation (24) is changed by

$$\begin{bmatrix} T_1 & BS_1 + Y' - L' - \frac{1}{\varepsilon} \begin{bmatrix} x_i' L_{q1} \\ \vdots \\ x_i' L_{qm} \end{bmatrix}' \\ \star & -2S_1 \end{bmatrix} < 0 \quad i = 1, \ldots, n_v \tag{34}$$

with

$$\begin{aligned} T_1 = {} & QA' + AQ + L'B' + BL + \frac{1}{\varepsilon} \begin{bmatrix} x_i' A_{q1} \\ \vdots \\ x_i' A_{qn} \end{bmatrix} Q \\ & + \frac{1}{\varepsilon} Q \begin{bmatrix} x_i' A_{q1} \\ \vdots \\ x_i' A_{qn} \end{bmatrix}' + \frac{1}{\varepsilon} B \begin{bmatrix} x_i' L_{q1} \\ \vdots \\ x_i' L_{qm} \end{bmatrix} + \frac{1}{\varepsilon} \begin{bmatrix} x_i' L_{q1} \\ \vdots \\ x_i' L_{qm} \end{bmatrix}' B' \end{aligned}$$

- Relation (25) is kept unchanged.
- A relation ensuring that the ellipsoid $\mathcal{E}(Q^{-1})$ is included in the chosen polytope has to be added

$$\begin{bmatrix} Q & \varepsilon Q a_{(k)}' \\ \star & 1 \end{bmatrix} \geq 0 \ k = 1, \ldots, q \tag{35}$$

*Remark 4:* In Proposition 3, a certain conservatism can be introduced by the upper bounds issued from inequalities (31). To obtain the maximal ERA in the form of an ellipsoid it could be possible to combine the results of Proposition 3 and those based on [2] discussed in Remark 3. Indeed, Proposition 3 (or Proposition 2 in the stability analysis context) could be used to find the initial polytope from which the framework issued from [2] could be used.

*Remark 5:* Other tools could be used do deal with the saturation term, for example those issued from LDI representation (Linear Differential Inclusion) [8], [12]. In this case, a slightly modified set of inequalities (24) (or (34)) could be exhibited, increasing the number of inequalities to be verified (from $2^n$ to $2^{(n+m)}$). Such a fact lead the numerical complexity to grow with the order of the system.

## V. Computational and Numerical Issues

If $\varepsilon$ in (15) or (24) is fixed, the inequalities become LMIs. Notice also that from relations (20) and (31), the inequalities (15) and (24) are convex on such a parameter if the Lyapunov matrix is fixed.

The solution of the following problem gives an ellipsoid yielding an estimate of the region of attraction:

$$\begin{array}{c} \max\limits_{Q,L,\mathcal{L}_q,Y,\varepsilon} \mathrm{Trace}(Q) \\ \text{s.t.}(24),(25) \end{array} \tag{36}$$

*Remark 6:* The optimization results are performed through a line search on the parameter $\varepsilon$.

Some conditions can be added to bound the elements of matrices $L$ (or $\mathcal{L}_q$) and therefore the elements of $K$ (or $\mathcal{K}_q$). Similarly we can impose some constraints to limit the conditioning number of matrix $Q$.

*Remark 7:* The quadratic terms on this control law can be interpreted as a counteraction to the influence of the quadratic terms of the system. Considering that the states of the system are available, the implementation of the quadratic state feedback is not more difficult than the linear state feedback presented in [1]. In the case $m = n$ and non-singular $B$ matrix, it is possible for the closed-loop system without saturation to eliminate completely the quadratic terms by choosing $K_{qi} = -B^{-1}A_{qi}$. In the saturating case even with an invertible matrix $B$, it is not possible to eliminate the quadratic terms of the system using quadratic terms on the control law. In this case, the saturation nonlinearity is the major constraint for the maximization of the ERA. In the general case ($m \neq n$) it is hard to predict which nonlinearity is more critical for the optimization of the ERA.

*Remark 8:* The choice of $A_{qi}$ $i = 1,\ldots,n$ in (1) is not unique and one should expect different numerical results for different $A_{qi}$ representing the same system. For the numerical examples below, matrices $A_{qi}$, $i = 1,\ldots,n$, were chosen to be symmetric.

**Example 1** Consider system (4) with $u(t) = 0$ defined by matrices

$$A = \begin{bmatrix} -1.1 & 0.7 \\ 0.1 & -.6 \end{bmatrix}; \quad \mathcal{A}_q = \begin{bmatrix} 0.32 & -0.05 & -0.05 & -0.03 \\ 0 & -0.01 & -0.01 & 0.1 \end{bmatrix} \tag{37}$$

By applying Proposition 2, we obtain

$$P = \begin{bmatrix} 0.2212 & 0 \\ 0 & 0.5086 \end{bmatrix} \tag{38}$$

which yields

$$V = \begin{bmatrix} 2.1262 & 2.1262 & -2.1262 & -2.1262 \\ 1.4022 & -1.4022 & 1.4022 & -1.4022 \end{bmatrix} \tag{39}$$

Figure 1 depicts the trajectories for different initial conditions, the obtained ERA in the form of an ellipsoid and the associated polytope whose vertices are defined by the columns of $V$.

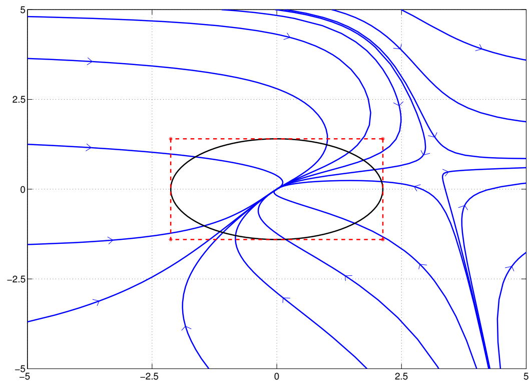


Fig. 1. Phase portrait of system defined by (37), region of stability and associated polytope.

**Example 2** Consider now matrices

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1.5 \end{bmatrix}; \quad \mathcal{A}_q = \begin{bmatrix} -0.283 & -0.118 & -0.118 & -0.31 \\ 0.041 & 0.42 & 0.42 & -0.225 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{40}$$

To measure the impact of the quadratic term of control law (5) on the ERA of system (1) defined with (40) we compare the value of $\mathrm{Trace}(P)$ obtained using conditions from Proposition 2 resulting in a control law having gains $K$ and $\mathcal{K}_q$ and the one obtained through a modified version, based on Remark 2, resulting only in a gain $K$. Two values of the input magnitude bound $u_0$ were tested. For $u_0 = 1$ the following gains were obtained:

- only gain $K$:

$$K = \begin{bmatrix} 3.8877 & -21.7333 \end{bmatrix} \tag{41}$$

- gains $K$ and $\mathcal{K}_q$:

$$K = \begin{bmatrix} 3.7689 & -21.5745 \end{bmatrix}; \quad \mathcal{K}_q = \begin{bmatrix} -0.0428 & -0.4313 & -0.4313 & 0.2300 \end{bmatrix} \tag{42}$$

and for $u_0 = 7$:

- only gain $K$:

$$K = \begin{bmatrix} 0.6094 & -3.2350 \end{bmatrix} \tag{43}$$

- gains $K$ and $\mathcal{K}_q$:

$$K = \begin{bmatrix} 0.1719 & -2.1961 \end{bmatrix}; \quad \mathcal{K}_q = \begin{bmatrix} -0.0455 & -0.4393 & -0.4393 & 0.2317 \end{bmatrix} \tag{44}$$

The values of $\mathrm{Trace}(P)$ are given in the table I. Notice that in the case $u_0 = 1$ the quadratic gain does not improve the criteria $\mathrm{Trace}(P)$. However for $u_0 = 8$ results

| $u_0$ | gain $K$ | gains $K$, $\mathcal{K}_q$ |
|---|---|---|
| 1 | 3.4353 | 3.4353 |
| 7 | 0.5120 | 0.3497 |

TABLE I

VALUES OF TRACE($P$) FOR DIFFERENT CONTROL STRUCTURES AND MAGNITUDE BOUNDS.

show that probably the quadratic term $\mathcal{A}_q$ is more critical than the saturation for the definition of the region of attraction.

A comparison between the controller synthesis based on Proposition 2 and the one based on matrix inequalities of Remark 3 is performed. It is important to outline that ideas of Remark 3 were adapted from [2] where a framework is set to compute the biggest polytope of a given shape contained in the region of attraction of a quadratic system. For the example below (dimension $n = 2$) a square is chosen to be such a polytope.

A set of systems, defined by matrices $A$ and $B$ in (40) and randomly generated matrices $\mathcal{A}_q$, was tested to analyze the influence of the quadratic terms. Tests were performed imposing bounds for $\mathcal{A}_{q(i,j)}$ as $-a_{q0} \leq \mathcal{A}_{q(i,j)} \leq a_{q0}$. Two values for $u_0$ and four values of $a_{q0}$ are then considered: $u_0 = \{\ 1;\ \ 8\ \}$; $a_{q0} = \{\ 0.15;\ \ 0.30;\ \ 0.45;\ \ 0.60\ \}$. Figures 2-3 show the results for 50 cases of each pair $\{a_{q0},\ u_0\}$. Dark-blue bars correspond to the cases where Trace($P$) obtained using Remark 3 is smaller and light-blue ones correspond to the cases where Proposition 2 gives a smaller value for Trace($P$).

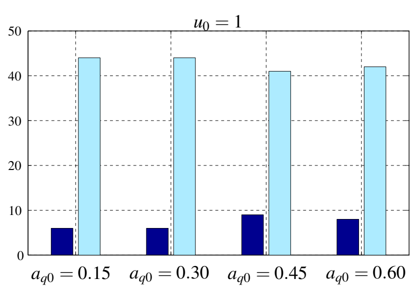


Fig. 2. Results for $u_0 = 1$

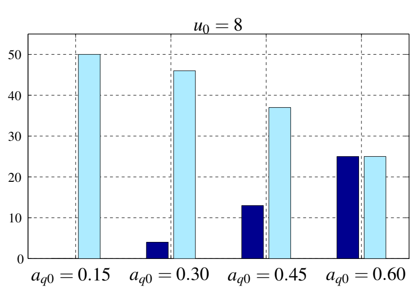


Fig. 3. Results for $u_0 = 8$

For most of the tested cases, the method proposed in Proposition 2 outperforms the one presented in Remark 3.

## VI. CONCLUSION

Control design for nonlinear quadratic systems with saturating input are presented. The control law investigated contains a quadratic term. Such control law aimed at enlarging the estimate of the region of attraction of the closed-loop saturated system. The proposed conditions are based on the use of a quadratic Lyapunov function and a modified sector condition. More complex Lyapunov functions, for example, piecewise quadratic Lyapunov functions could be used to extend the results presented in this paper.

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