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A Scalable Robust Stability Criterion for Systems with Heterogeneous LTI Components

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Abstract— **A scalable robust stability criterion for networked interconnected systems with heterogeneous linear time-invariant components is presented in this paper. The criterion involves only the properties of individual components and the spectrum of the interconnection matrix, which can be verified with relatively low computational effort, and more importantly maintains scalability of the analysis. Moreover, if the components are single-input-single-output (SISO), the criterion has an appealing graphical interpretation which resembles the classical Nyquist criterion.**

I. Introduction

Many large-scale network interconnected systems have structure that can be explored in analysis and design. It is desirable to find conditions that ensure system-wide robust stability based on easily verifiable conditions on the local dynamics and the interconnection structure. The resulting analysis and design tools will then scale gracefully as the network size grows. Several such results have appeared recently for linear homogeneous systems in e.g. [1], [2] and for systems with heterogeneous dynamics in e.g. [3], [4], [5], [6], [7].

In this paper we consider a class of heterogeneous linear time-invariant dynamical systems interconnected over a network interconnection matrix that can be unitarily diagonalized. The resulting stability criterion involves only the eigenvalue distribution of the interconnection matrix and the dynamics of individual subsystems. This decentralization feature allows the stability criterion to be checked with relatively low computational effort, and more importantly, maintains the scalability of the network. Under a further assumption that all subsystems are single-input-single-output (SISO), the stability criterion has a graphical interpretation which resembles the classical Nyquist criterion: the convex combination of frequency responses of the individual subsystems should avoid a polyhedral region (referred to as the "instability region") that may be constructed from the characteristics of the network. More specifically, suppose that the eigenvalues of the interconnection matrix are contained in a given closed convex polytope. Then the convex duality theory can be used to derive a systematic procedure for constructing the instability region from the vertices of this polytope. Moreover, to justify the term "instability region" we show that if the stability criterion is violated, a destabilizing interconnection matrix with eigenvalues residing in the given convex polytope can be constructed using the solution of a linear program.

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The result presented here is an extension of our previous work in [8], which relaxes a certain condition imposed on the eigenvalue distribution of the interconnection matrix. More specifically, the result presented in [8] requires the eigenvalues of the interconnection matrix to be within a convex region generated by *intersection of circles which have the origin on their boundaries*. This condition is sometimes too restrictive and makes the result less applicable in practice. The generalization done in this paper relaxes this restriction and allows one to consider interconnection matrices with eigenvalues within *any convex polytope*. As such, the result presented in this paper is applicable to *any* interconnection matrix that satisfies the unitary diagonalization assumption.

Due to the length limitation, the proofs of our results are omitted. They can be found in the report [9].

A. Notation and Preliminaries

In the paper we consider the following functional spaces

$\mathcal{A}^{m\times m}$: The space of transfer functions obtained as the Laplace transform of the impulse response functions

$$h(t) = h_c(t)\theta(t) + h_0\delta(t)$$

where $h_c \in \mathbf{L}_1^{m\times m}[0,\infty)$, $h_0 \in \mathbf{R}^{m\times m}$, $\theta(\cdot)$ and $\delta(\cdot)$ denote the unit step function and the Dirac delta function, respectively.

$S_{\mathcal{A}}^{m\times m}$: The space of transfer functions obtained as the (double sided) Laplace transform of the impulse response functions of the form

$$h(t) = h_c(t) + h_0\delta(t)$$

where $h_c(t) = h_c(-t)^T \in \mathbf{L}_1^{m\times m}(-\infty,\infty)$ and $h_0 = h_0^T \in \mathbf{R}^{m\times m}$. Any $H(s)$ from $S_{\mathcal{A}}^{m\times m}$ satisfies $H(s) = H(-s)^T$ in its domain of definition, which includes the imaginary axis.

$S_{\mathbf{C}}^{m\times m}$: The set of matrices $\{M \in \mathbf{C}^{m\times m} : M = M^*\}$.

Any $H(s) \in \mathcal{A}^{m\times m}$ defines a bounded linear operator on $\mathbf{L}_2[0,\infty)$, which can be defined by a convolution integral in the time domain and as a multiplication operator in the frequency domain.

Any $\Upsilon \in S_{\mathcal{A}}^{m\times m}$ defines a bounded self-adjoint linear operator on $\mathbf{L}_2(-\infty,\infty)$. It is called positive semi-definite if and only if (iff) $\langle v, \Upsilon v\rangle \geq 0$, for all $v \in \mathbf{L}_2(-\infty,\infty)$ and (strictly) positive definite iff there exists some $\epsilon > 0$ such that $\langle v, \Upsilon v\rangle \geq \epsilon\|v\|^2$, for all $v \in \mathbf{L}_2(-\infty,\infty)$. It can be shown that an operator $\Upsilon \in S_{\mathcal{A}}^{m\times m}$ is positive semi-definite if $\Upsilon(j\omega) \geq 0,\ \forall\omega \in \mathbf{R}\cup\{\infty\}$ and (strictly) positive definite if $\Upsilon(j\omega) > 0,\ \forall\omega \in \mathbf{R}\cup\{\infty\}$, which are denoted as $\Upsilon \geq 0$ and $\Upsilon > 0$, respectively. Here we used that functions in $S_{\mathcal{A}}^{m\times m}$ (and $\mathcal{A}^{m\times m}$) are continuous on the extended imaginary axis. Negative semi-definiteness and (strict) negative definiteness are defined with opposite inequalities and are denoted as $\Upsilon \leq 0$ and $\Upsilon < 0$, respectively.

We use $\otimes$ to denote the Kronecker product. The direct sum of Δ_k, $k = 1,\cdots,n$, is defined as

$$\oplus_{k=1}^{n}\Delta_k = \mathrm{diag}(\Delta_1,\ldots,\Delta_n).$$

Given a matrix Γ, $\mathrm{eig}(\Gamma)$ denotes the spectrum of Γ. Given a set Λ, the notations $\mathrm{co}(\Lambda)$ and $\mathrm{cone}(\Lambda)$ (sometimes, the parentheses are dropped for simplicity) denote the convex hull of Λ and the convex cone generated by the elements of Λ, respectively. For a complex number λ, the conjugate of λ is denoted by $\bar{\lambda}$.

II. A Scalable Stability Result

Let us consider the system

$$\begin{aligned} u &= \Gamma y \\ y &= Hu + e, \end{aligned} \tag{1}$$

where $H = \oplus_{k=1}^{n}H_k$ with each $H_k \in \mathcal{A}^{m\times m}$. The feedback interconnection is modeled as $\Gamma = \hat{\Gamma}\otimes I_m$, where $\hat{\Gamma} \in \mathcal{A}^{n\times n}$, and the disturbance $e \in \mathbf{L}_2[0,\infty)$. Stability of this system means that there exists a constant $c > 0$ such that $\|y\| + \|u\| \leq c\|e\|$ for all $e \in \mathbf{L}_2[0,\infty)$.

The system can be viewed as a set of linear dynamics $\{H_k,\ k = 1,\ldots,n\}$ interconnected over a graph described by $\hat{\Gamma}$. The k,l component of $\hat{\Gamma}$ represents the gain from node l to node k. We allow local feedback; i.e., $\gamma_{k,k}$ is allowed to be non-zero. By allowing $\hat{\Gamma} \in \mathcal{A}^{n\times n}$, the components of $\hat{\Gamma}$ may sometimes be used to model communication channels with transmission delays and bandwidth limitations. We say that $\hat{\Gamma}$ is normal if[1] $\hat{\Gamma}(j\omega)^*\hat{\Gamma}(j\omega) = \hat{\Gamma}(j\omega)\hat{\Gamma}(j\omega)^*$, $\forall\omega \in \mathbf{R}\cup\{\infty\}$. The system class includes linearized models for Internets congestion control, vehicle platoon models, and consensus problems.

The next result shows that the stability of (1) can be characterized using only the individual transfer functions H_k and the spectrum of $\hat{\Gamma}(j\omega)$. The result is proven in the same way as the corresponding result in [8] and the result is also related to earlier contributions in [4], [5].

Proposition 1: Consider the system (1), where $H = \oplus_{k=1}^{n}H_k$ with $H_k \in \mathcal{A}^{m\times m}$ and $\Gamma = \hat{\Gamma}\otimes I_m$, where $\hat{\Gamma} \in \mathcal{A}^{n\times n}$ is normal. The system is stable if there exists $\Pi \in S_{\mathcal{A}}^{2m\times 2m}$ of the form

$$\Pi = \begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12}^* & \Pi_{22}\end{bmatrix}$$

with $\Pi_{11} \in S_{\mathcal{A}}^{m\times m}$ and $\Pi_{22} \in S_{\mathcal{A}}^{m\times m}$ satisfying

(i) $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$,

(ii) $|\lambda|^2\Pi_{11} + \bar{\lambda}\Pi_{12} + \lambda\Pi_{12}^* + \Pi_{22} \leq 0,\ \forall\lambda \in \mathrm{eig}(\hat{\Gamma})$

such that

(iii) $\Pi_{11} + \Pi_{12}H_k + H_k^*\Pi_{12}^* + H_k^*\Pi_{22}H_k > 0,\ \forall\ k = 1,\cdots,n$.

Remark 1: The condition $\Pi_{11} \geq 0$ in (i) implies that (ii) is equivalent to

$$|\lambda|^2\Pi_{11} + \bar{\lambda}\Pi_{12} + \lambda\Pi_{12}^* + \Pi_{22} \leq 0,\quad \forall\ \lambda \in \mathrm{co}\left(\mathrm{eig}(\hat{\Gamma})\right).$$

This provides a robustness interpretation of our results that will be further elaborated in Section IV. Note that in many applications the eigenvalues of Γ are easy to compute or estimate. This is, for example, the case in circulant and circular networks, in consensus problems involving the graph Laplacian [10] and in Internet congestion control [6], [3].

In the next section, we will show that the stability criterion also has a simple and illuminating graphical representation when all H_k's are single-input-single-output (SISO).

III. Duality and Graphical Representation

The next proposition provides a dual condition that holds whenever the main stability condition (iii) in Proposition 1 fails to hold. The dual condition will be used to derive simple graphical tests for stability.

Proposition 2: Let $H_k \in \mathcal{A}^{m\times m}$, for $k = 1,\cdots,n$, and let $\Pi_\Lambda \subset S_{\mathbf{C}}^{2m\times 2m}$ be a closed (in the topology defined by the Frobenius norm) convex cone. Then either of the following two statements holds

(a) There exists $\Pi \in S_{\mathcal{A}}^{2m\times 2m}$ such that

(i) $\Pi(j\omega) \in \Pi_\Lambda,\ \forall\omega \in \mathbf{R}\cup\{\infty\}$;

(ii) $\Pi_{11} + \Pi_{12}H_k + H_k^*\Pi_{12}^* + H_k^*\Pi_{22}H_k > 0,\ \forall\ k = 1,\ldots,n$.

(b) There exists an $\omega \in \mathbf{R}\cup\{\infty\}$ and a nonzero tuple $Z \in \mathcal{Z} := \{(Z_1,\ldots,Z_n) : Z_k \in S_{\mathbf{C}}^{m\times m},\ Z_k \geq 0\}$, such that

$$\sum_{k=1}^{n}\begin{bmatrix} I\\ H_k(j\omega)\end{bmatrix} Z_k \begin{bmatrix} I\\ H_k(j\omega)\end{bmatrix}^* \in \Pi_\Lambda^{\ominus},$$

where $\Pi_\Lambda^{\ominus} := \left\{W \in S_{\mathbf{C}}^{2m\times 2m} : \mathrm{tr}(\Pi W) \leq 0, \forall\Pi \in \Pi_\Lambda\right\}$ is the polar cone of Π_Λ.

Let us now restrict the attention to the case where the H_k's are SISO. Suppose $\Lambda \subseteq \mathbf{C}$ is a closed convex polytope and suppose we are given the following spectral characterization of the interconnection matrix: $\mathrm{eig}(\Gamma)(j\omega) \in \Lambda$ for all $\omega \in \mathbf{R}\cup\{\infty\}$. The stability conditions in Proposition 1 lead us to consider the following convex cone of multipliers

$$\begin{aligned}\Pi_\Lambda = \{\Pi \in S_{\mathbf{C}}^{2\times 2} : \Pi_{11} \geq 0;\ \Pi_{22} \leq 0;\\ |\lambda|^2\Pi_{11} + \bar{\lambda}\Pi_{12} + \lambda\Pi_{12}^* + \Pi_{22} \leq 0, \forall\lambda \in \Lambda\}.\end{aligned} \tag{2}$$

We have the following simple characterization of the polar cone $\Pi_\Lambda^{\ominus}$ when Λ is a convex polytope.

Lemma 1: Consider Π_Λ as defined in (2) and suppose that $\Lambda = \mathrm{co}\{\lambda_1,\ldots,\lambda_n\}$. Then the polar cone of Π_Λ is characterized as follows

$$\Pi_\Lambda^{\ominus} = \mathrm{cone}\left\{W_k \in S_{\mathbf{C}}^{2\times 2} : k = 1,\ldots,n+2\right\},$$

[1] For the sake of simplifying the notations, we often suppress the dependence on ω and write $\hat{\Gamma}^*\hat{\Gamma} = \hat{\Gamma}\hat{\Gamma}^*$.

where $W_k = v_k v_k^*$ for $k = 1, \ldots, n+1$, $W_{n+2} = -v_{n+2}v_{n+2}^*$, and

$$v_k = \begin{bmatrix}\lambda_k \\ 1\end{bmatrix},\ k = 1, \ldots, n,\ v_{n+1} = \begin{bmatrix}0 \\ 1\end{bmatrix},\ v_{n+2} = \begin{bmatrix}1 \\ 0\end{bmatrix}.$$

A. Graphical Representation of Stability Condition

From Proposition 2 and Lemma 1 it follows that the stability condition is violated if and only if there exists an $\omega_0 \in \mathbf{R} \cup \{\infty\}$, and a nonzero n-tuple $z = (z_1, \ldots, z_n) \geq 0$, such that

$$\sum_{k=1}^{n} z_k \begin{bmatrix}1 \\ H_k(j\omega_0)\end{bmatrix} \begin{bmatrix}1 \\ H_k(j\omega_0)\end{bmatrix}^* = \sum_{k=1}^{n+2} \psi_k W_k \qquad (3)$$

for some $\psi_k \geq 0$. This gives rise to the equation system

$$\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} \psi_k |\lambda_k|^2 - \psi_{n+2} \qquad (4)$$

$$\sum_{k=1}^{n} z_k H_k(j\omega_0) = \sum_{k=1}^{n} \psi_k \bar{\lambda}_k \qquad (5)$$

$$\sum_{k=1}^{n} z_k |H_k(j\omega_0)|^2 = \sum_{k=1}^{n} \psi_k + \psi_{n+1}. \qquad (6)$$

Since the equation system can be multiplied by a positive scalar without changing any of the required conditions, it is possible to normalize the coefficients such that $\sum_{k=1}^{n} z_k = 1$. Let us introduce the Nyquist polytope of the system matrices $\mathcal{N}[H_1, \ldots, H_n](\omega)$, parameterized by ω, to be

$$\mathrm{co}\{(\mathrm{Re}\, H_k(j\omega), \mathrm{Im}\, H_k(j\omega), |H_k(j\omega)|^2) : k = 1, \ldots, n\} \quad (7)$$

and the instability region

$$\Omega = \left\{ \sum_{k=1}^{n} \psi_k (\mathrm{Re}\,\lambda_k, -\mathrm{Im}\,\lambda_k, 1) + (0, 0, \psi_{n+1}) : \psi_k \geq 0,\ k = 1, \ldots, n+1; \sum_{k=1}^{n} \psi_k |\lambda_k|^2 \geq 1 \right\}. \qquad (8)$$

We have the following graphical characterization of robust stability of system (1).

Proposition 3: Consider system (1), where $H = \oplus_{k=1}^{n} H_k$, each $H_k \in \mathcal{A}$, and $\Gamma \in \mathcal{A}^{n\times n}$ is normal. The system is stable if

(i) $\mathrm{eig}(\Gamma(j\omega)) \in \Lambda := \mathrm{co}\{\lambda_1, \ldots, \lambda_n\}$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$,
(ii) $\mathcal{N}[H_1, \ldots, H_n](\omega) \bigcap \Omega = \emptyset$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$.

Figure 1 illustrates the graphical test of the stability criterion in the case of two subsystems. Note that the two three-dimensional Nyquist curves $C_k := \{(\mathrm{Re}\, H_k(j\omega), \mathrm{Im}\, H_k(j\omega), |H_k(\omega)|^2) : \omega \in \mathbf{R} \cup \{\infty\}\}$, $k = 1, 2$, reside on a parabolic surface. For stability, the Nyquist polyhedron $\mathcal{N}[H_1, H_2](\omega)$ must avoid the polyhedron region Ω. Note that $\mathcal{N}[H_1, H_2](\omega)$, illustrated by the dashed blue lines in the figure, resides inside the parabolic surface.

It is easier to construct the region Ω using parameters $\theta_k := \psi_k |\lambda_k|^2$, $k = 1, \cdots, n$ and $\theta_{n+1} = \psi_{n+1}$. To this end, we first note that, any zero λ_k may be removed from (8) without changing the set Ω. Hence, without loss of generality,

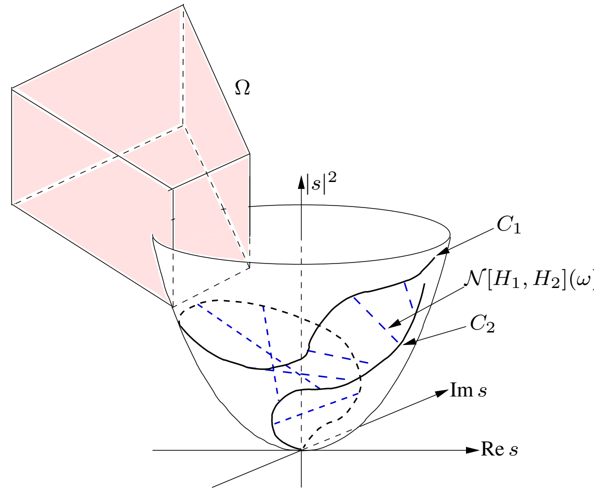

Fig. 1. Illustration of three-dimensional graphical test for robust stability in the case of two subsystems. Note that the three-dimensional Nyquist curves C_1 and C_2 evolve on a parabolic surface. For stability, the Nyquist polyhedron $\mathcal{N}[H_1, H_2](\omega)$ (illustrated by the dashed blue lines) must avoid the polyhedron region Ω.

we may assume all λ_k, $k = 1, \cdots, n$, are nonzero. Then it is easy to verify that the instability region Ω can be equivalently expressed as

$$\Omega = \left\{ \alpha \cdot \mathrm{co}\left\{ \left(\mathrm{Re}\,\frac{1}{\lambda_k}, \mathrm{Im}\,\frac{1}{\lambda_k}, \frac{1}{|\lambda_k|^2} \right) : k = 1, \cdots, n \right\} + (0, 0, \theta_{n+1}) : \alpha \geq 1, \theta_{n+1} \geq 0 \right\}. \qquad (9)$$

To illustrate the implication of this graphical criterion, we start with the special case where statement (ii) of Proposition 3 simplifies to

$$\mathrm{co}\{H_1(j\omega), \ldots, H_n(j\omega)\} \bigcap \Omega_0 = \emptyset,\ \forall \omega \in \mathbf{R} \cup \{\infty\}, \qquad (10)$$

where

$$\Omega_0 := \left\{ \alpha \cdot \mathrm{co}\left\{ \frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n} \right\} : \alpha \geq 1 \right\} \qquad (11)$$

is the projection of Ω to the complex plane. This case appears when the members of the set Π_Λ defined in (2) are further restricted to have only zero Π_{22}. Then the parameter ψ_{n+1} becomes a free variable, and therefore equation (6) is always satisfied by some ψ_{n+1}. Thus equation (6) can be disregarded and statement (ii) in Proposition 3 reduces to (10). Note that this simplified stability test always is a sufficient condition for stability but it may be conservative or even useless as we will see in Example 1 and Example 2 below. The reason why restricting Π_{22} to be zero generally leads to a conservative criterion is somewhat revealed in the next proposition, which shows that any closed convex polytope in the complex plane that contains the origin can be exactly characterized using quadratic forms with $\Pi_{22} \leq 0$.

Proposition 4: Suppose $0 \in \Lambda = \mathrm{co}\{\lambda_1, \ldots, \lambda_n\}$. Let

$$\Lambda^{\mathrm{cl}} = \bigcap_{\Pi \in \Pi_\Lambda} \left\{ \lambda \in \mathbf{C} : \Pi_{11}|\lambda|^2 + 2\mathrm{Re}\,\Pi_{12}\bar{\lambda} + \Pi_{22} \leq 0 \right\}.$$

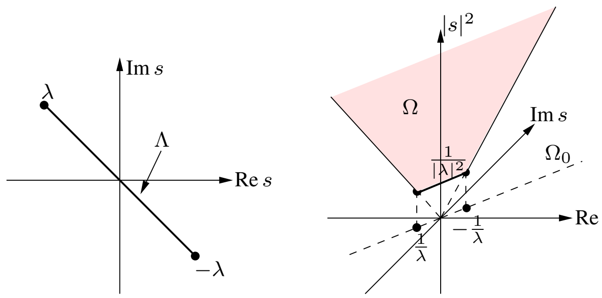

Fig. 2. Figure in the left-hand side: the set Λ where eigenvalues of Γ belongs. Figure in the right-hand side: the corresponding Ω.

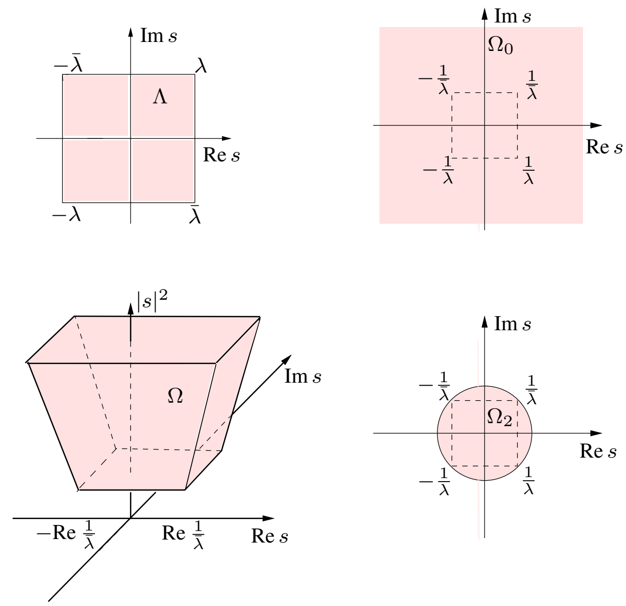

Fig. 3. (Left upper) the set Λ where eigenvalues of Γ belongs. (Left lower) the polyhedron Ω which $\mathcal{N}[H_1(j\omega), \cdots, H_n(j\omega)](\omega)$ must avoid for stability of the interconnected system. (Right upper) the projection of Ω on to the Re s-Im s plane, i.e., the corresponding Ω_0. (Right lower) The region Ω_2 for the two-dimensional stability test.

Then $\Lambda = \Lambda^{\rm cl}$.

Example 1: Suppose $\Lambda = {\rm co}\{\lambda, -\lambda\}$, $\lambda \neq 0$. In this case, it is easy to verify that the set Ω is equivalent to

$$\left\{\alpha \cdot {\rm co}\left\{\left({\rm Re}\,\frac{1}{\lambda}, {\rm Im}\,\frac{1}{\lambda}, \frac{1}{|\lambda|^2}\right), \left(-{\rm Re}\,\frac{1}{\lambda}, -{\rm Im}\,\frac{1}{\lambda}, \frac{1}{|\lambda|^2}\right)\right\} : \alpha \geq 1\right\}.$$

Figure 2 illustrates the region Ω. The set Ω_0 corresponds to the dashed line through the points $-1/\lambda$ and $1/\lambda$. For this example, the simplified condition in (10) is conservative. One may have ${\rm co}\{H_1(j\omega), \cdots, H_n(j\omega)\}$ intersect the line segment ${\rm co}\{-1/\lambda, 1/\lambda\}$ while ${\rm co}\{|H_1(j\omega)|^2, \cdots, |H_n(j\omega)|^2\}$ is below $1/|\lambda|^2$. In this case, condition (10) fails but the stability condition still holds.

Example 2: Suppose $\Lambda = {\rm co}\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\}$, where ${\rm Re}\,\lambda > 0$ and ${\rm Im}\,\lambda > 0$. In this case

$$\Omega = \left\{\alpha \cdot {\rm co}\left\{\left({\rm Re}\,\frac{1}{\lambda}, {\rm Im}\,\frac{1}{\lambda}, \frac{1}{|\lambda|^2}\right), \left({\rm Re}\,\frac{1}{\bar{\lambda}}, {\rm Im}\,\frac{1}{\bar{\lambda}}, \frac{1}{|\bar{\lambda}|^2}\right), \left(-{\rm Re}\,\frac{1}{\lambda}, -{\rm Im}\,\frac{1}{\lambda}, \frac{1}{|\lambda|^2}\right), \left(-{\rm Re}\,\frac{1}{\bar{\lambda}}, -{\rm Im}\,\frac{1}{\bar{\lambda}}, \frac{1}{|\bar{\lambda}|^2}\right)\right\} + (0,0,\psi) : \alpha \geq 1, \psi \geq 0\right\}.$$

Figure 3 illustrates the region Ω. In this example the simplified stability test in (10) is useless since Ω_0 covers the whole complex plane, while the stability condition in Proposition 3 only requires that the Nyquist polytope $\mathcal{N}[H_1, \ldots, H_n]$ stays outside the set Ω. Although it may appear that the stability region; i.e., the space $\mathcal{N}[H_1, \ldots, H_n]$ can stay, is large, one can show that ${\rm co}\{H_1(\omega), \ldots, H_n(\omega)\} \in \Omega_2 := \{s \in \mathbf{C} : |s| \leq 1/|\lambda|\}$ is a necessary and sufficient for the criterion in Proposition 3 to hold. This simplifies the stability test.

B. Inverse Nyquist Polytope

It may sometimes be easier to visualize the stability test in Proposition 3 if we consider the following alternative normalization

$$\sum_{k=1}^{n} z_k |H_k(j\omega)|^2 = 1.$$

Introducing the new variables $\widehat{z}_k = z_k |H_k(j\omega)|^2$, we may replace statement (ii) in Proposition 3 by the following equivalent condition

$$\widehat{\mathcal{N}}[H_1, \ldots, H_n](\omega) \bigcap \widehat{\Omega} = \emptyset,$$

where $\widehat{\mathcal{N}}[H_1, \ldots, H_n](\omega)$, referred to as "the inverse Nyquist polytope", is defined as

$${\rm co}\left\{\left({\rm Re}\,\frac{1}{H_k(j\omega)}, {\rm Im}\,\frac{1}{H_k(j\omega)}, \frac{1}{|H_k(j\omega)|^2}\right) : k = 1, .., n\right\} \tag{12}$$

and the instability region to avoid is

$$\widehat{\Omega} = \left\{\sum_{k=1}^{n} \psi_k ({\rm Re}\,\lambda_k, {\rm Im}\,\lambda_k, |\lambda_k|^2) - (0, 0, \psi_{n+2}) : \psi_k \geq 0;\ \sum_{k=1}^{n} \psi_k \leq 1; \psi_{n+2} \geq 0\right\}.$$

This graphical test involving the inverse Nyquist polytope is illustrated in Figure 4. Note that this change of variable is frequency dependent. Furthermore, if $H_k(j\omega_0) = 0$ for some ω_0, then we should set $\hat{z}_k$ to zero. The instability regions corresponding to $\Lambda = {\rm co}\{\lambda, -\lambda\}$ and $\Lambda = {\rm co}\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\}$ are illustrated in Figure 5.

Example 3: To further illustrate the graphical test, let us consider a system consisting of three SISO subsystems $H_1(s)$, $H_2(s)$, and $H_3(s)$ which are circularly connected. In this case, the eigenvalues of the interconnection matrix Γ are $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $\lambda_3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

For the case where $H_1(s) = \frac{{\rm e}^{-0.8s}}{s^2+s+1.5}$, $H_2(s) = \frac{{\rm e}^{-0.5s}}{s+1.25}$, $H_3(s) = -\frac{{\rm e}^{-s}}{s+1}$, one can show that the system is robustly stable by applying the inverse Nyquist criterion. The left-hand-side figure in Figure 6 illustrates the graphical test.

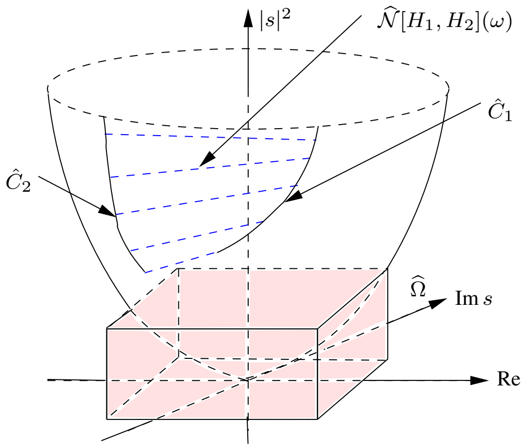

Fig. 4. Illustration of three-dimensional inverse Nyquist test for robust stability in the case of two subsystems. Note that the three-dimensional inverse Nyquist curves $\hat{C}_1$ and $\hat{C}_2$ evolve on a parabolic surface. For stability, the Nyquist polyhedron $\widehat{\mathcal{N}}[H_1, H_2](\omega)$ (illustrated by the dashed blue lines) must avoid the polyhedron region $\widehat{\Omega}$.

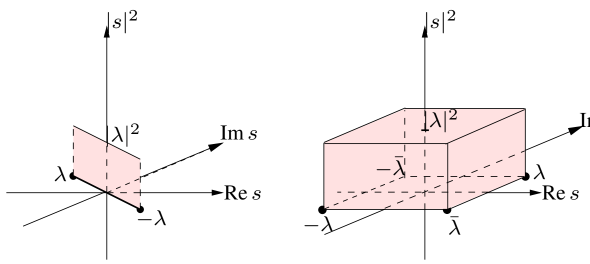

Fig. 5. (Left) The instability regions corresponding to $\Lambda = \text{co}\{\lambda, -\lambda\}$. (Right) The instability regions corresponding to $\Lambda = \text{co}\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\}$.

One can see that the inverse Nyquist polytope avoids the instability region, which is illustrated by the blue region.

On the other hand, if we lower the damping ratio of $H_1(s)$ by 20%; i.e., let $H_1(s) = \frac{\mathrm{e}^{-0.8s}}{s^2+0.8s+1.5}$. Then the system is no longer robustly stable. The corresponding inverse Nyquist polytope intersects the instability region, as illustrated in the right-hand-side figure of Figure 6. This means, as we will see in Section IV, that one may find a destabilizing interconnection matrix whose eigenvalues are in the set $\text{co}\{\lambda_1, \lambda_2, \lambda_3\}$.

IV. A Robust versus Non-Robust Formulation

It can be shown that if there exists an ω such that $\mathcal{N}[H_1, \cdots, H_n](\omega) \bigcap \Omega \neq \emptyset$, then a systematic algorithm can be applied to construct a simple destabilizing network interconnection matrix with eigenvalues in the specified convex polytope. Hence the term "instability region" for Ω is justified. The construction of the destabilizing network interconnection matrix is rather involved and can be found in [9]. The following necessary and sufficient condition for robust stability of heterogeneous interconnected systems is a consequence of the claim stated above.

Proposition 5: Let $H = \oplus_{k=1}^{n} H_k$, where each $H_k \in \mathcal{A}$ is a stable LTI system. Let $\lambda_1, \cdots, \lambda_m \in \mathbf{C}$ and $\Lambda =$

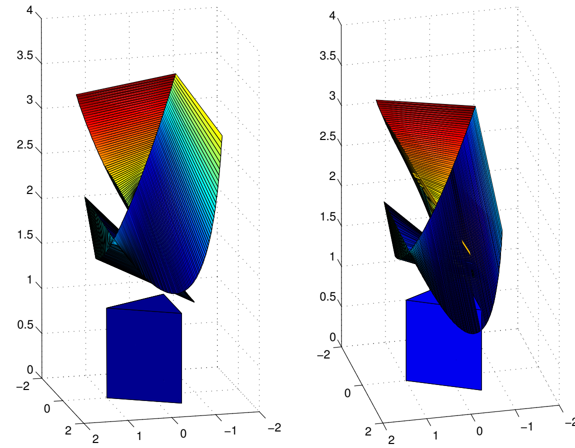

Fig. 6. The inverse Nyquist criterion applied to a circularly connected system. Left-hand-side: the inverse Nyquist polytope avoids the instability region and thus the system is stable. Right-hand-side: the inverse Nyquist polytope intersects the instability region and thus the system is not robustly stable.

$\text{co}\{\lambda_1, \cdots, \lambda_m\}$ which includes 0. Then the interconnection of H and Γ is input-output stable for any normal interconnection matrix Γ which satisfies $\text{eig}(\Gamma) \in \Lambda$ if and only if $\mathcal{N}[H_1, \ldots, H_n](\omega) \bigcap \Omega = \emptyset$ for all $\omega \in \mathbf{R} \cup \{\infty\}$, where $\mathcal{N}[H_1, \ldots, H_n](\omega)$ and Ω are defined in (7) and (8), respectively.

The condition $\Pi_{11} \geq 0$ may be omitted if additional assumptions on the dynamics are introduced. The robustness interpretation discussed in Remark 1 and Proposition 5 is then lost but instead we obtain a less restrictive stability condition. We will here only give a brief illustration of the consequences of removing the constraint Π_{11}. We refer to [9] for further details and to Section V for an example.

The following result is analogous to Proposition 4.

Proposition 6: Let $\Lambda = \{\lambda_1, \cdots, \lambda_n\}$,

$$\Pi_{\Lambda,e} = \left\{ \Pi \in S_{\mathbf{C}}^{2\times 2} : \Pi_{22} \leq 0; \begin{bmatrix}\lambda\\1\end{bmatrix}^* \Pi \begin{bmatrix}\lambda\\1\end{bmatrix} \leq 0, \forall \lambda \in \Lambda \right\}, \tag{13}$$

and define

$$\Lambda_e^{\text{cl}} = \left\{ \lambda : \begin{bmatrix}\lambda\\1\end{bmatrix}^* \Pi \begin{bmatrix}\lambda\\1\end{bmatrix} \leq 0,\ \forall \Pi \in \Pi_\Lambda \right\}.$$

Then $\Lambda_e^{\text{cl}} = \{0, \lambda_1, \cdots, \lambda_n\}$.

For $\Pi_{\Lambda,e}$ in (13), one may derive graphical tests for testing the corresponding statement (a) of Proposition 2. The tests are identical to those presented in Section III, except that the regions the Nyquist polytope/Inverse Nyquist polytope must avoid become smaller. Specifically, without the constraint $\Pi_{11} \geq 0$, the new Ω set becomes

$$\Omega_e = \text{co}\left\{ \left(\text{Re}\,\frac{1}{\lambda_k}, \text{Im}\,\frac{1}{\lambda_k}, \frac{1}{|\lambda_k|^2} \right) : \forall k \right\} + (0, 0, \mathbf{R}_+) \tag{14}$$

where $\mathbf{R}_+$ denotes the set of nonnegative real numbers. There is also the additional assumption that there must exist a stable homogeneous interconnection; i.e. a transfer function $H_0 \in \text{co}\{H_1, \ldots, H_n\}$ such that the interconnection of $H_0 \otimes I_n$ and Γ is stable.

In the next section we use this new characterization to derive a stability result for heterogeneous consensus networks.

V. Heterogeneous Consensus Networks

Consider the system equation

$$y = H(\Gamma y + r) \tag{15}$$

where $H = \oplus_{k=1}^n H_k$ with $H_k(s) = \frac{1}{s}(1 + \Delta_k(s))$, where Δ_k is a stable perturbation of the dynamics. The disturbance r represents the effect of the initial condition and $\Gamma \in \mathbf{R}^{n\times n}$ is a normal matrix satisfying

$$\Gamma \mathbf{1} = 0, \quad \text{where } \mathbf{1} = \frac{1}{n}\begin{bmatrix}1 & \ldots & 1\end{bmatrix}^T,$$

and $\text{eig}(\Gamma) \in \text{co}\{0, \lambda_2, \ldots, \lambda_n\}$, where we assume $\text{Re}\,\lambda_n \leq \ldots \leq \text{Re}\,\lambda_2 < 0$. Given the structure of H and Γ, the output of (15) can at best converge to a steady state solution y^0 satisfying

$$0 = (I + \Delta(0))\Gamma)y^0$$

where $\Delta = \oplus_{k=1}^n \Delta_k$. Provided that $\Delta_k(0) \neq -1$, it follows that the steady state solution must lie in the subspace spanned by the eigenvector $\mathbf{1}$; that is,

$$\lim_{t\to\infty} y(t) \in \text{span}\{\mathbf{1}\}. \tag{16}$$

This means that all outputs converge to the same value, which is also referred to as "*the outputs of the system reach consensus*". In the case where all $\Delta_k = 0$, it is well known that the consensus is reached and that the rate of exponential decay to consensus is equal to $\eta = -\text{Re}\,\lambda_2$. See, for example [10]. The goal here is to provide conditions under which the perturbed system is guaranteed to reach consensus with a prescribed rate of convergence.

To this end, we first note that Γ has a spectral decomposition

$$\Gamma = U\text{diag}(0, \lambda_2, \ldots, \lambda_n)U^* \tag{17}$$

where $U = \begin{bmatrix}\mathbf{1} & V\end{bmatrix}$, $\mathbf{1} \perp V$, and $V^*V = I_{n-1}$. We further assume that $\text{Re}\,\lambda_k \leq -\eta$, $\forall k = 2, \ldots, n$ for some $\eta > 0$. Finally, consider the "instability" region

$$\Omega_e = (0, 0, \mathbf{R}_+) + \text{co}\left\{\left(\text{Re}\,\frac{1}{\lambda_k}, \text{Im}\,\frac{1}{\lambda_k}, \frac{1}{|\lambda_k|^2}\right) : k \geq 2\right\}.$$

The next result shows that the number η provides an upper bound on the maximum possible rate of exponential decay to consensus.

Theorem 1: Consider the system in (15), where $H = \oplus_{k=1}^n H_k$, and $H_k(s) = \frac{1}{s}(1 + \Delta_k(s))$. Suppose Γ can be decomposed as in (17), where $\text{Re}\,\lambda_k \leq -\eta$, $k = 2, \ldots, n$, for some $\eta > 0$. Suppose in addition that there exists α, $0 < \alpha < \eta$, such that

(*i*) $\Delta_k(s - \alpha) \in \mathcal{A}$, $k = 1, \ldots, n$,

(*ii*) $\mathcal{N}[\check{H}_0, \check{H}_1, \ldots, \check{H}_n](\omega) \bigcap \Omega_e = \emptyset$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$, where $\check{H}_0(s) := \frac{1}{s-\alpha}$, $\check{H}_k(s) = \frac{1}{s-\alpha}(1 + \Delta_k(s - \alpha))$.

Then the outputs of the system satisfy $e^{\alpha t}y(t) \to \text{span}\{\mathbf{1}\}$ as $t \to \infty$ for any input r which satisfies $e^{\alpha t}r(t) \in \mathbf{L}_2[0, \infty)$.

Remark 2: The same conclusion holds if we replace (ii) in the theorem statement by the following two-dimensional stability criterion:

$$\text{co}\{\check{H}_0, \check{H}_1, \ldots, \check{H}_n\}(j\omega) \bigcap \Omega_{e,0} = \emptyset, \ \forall \omega \in \mathbf{R} \cup \{\infty\},$$

where $\Omega_{e,0} = \text{co}\left\{\frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right\}$.

Remark 3: Heterogeneous consensus networks have previously been considered in e.g. [6], [7], [11] and [11] considers integral quadratic constraints as the basis of the derivation of the results, which is related to our approach. Our result is to our knowledge the first to explicitly consider rate of convergence bounds of heterogeneous consensus networks.

VI. Concluding Remark

A scalable robust stability criterion for interconnected systems with heterogeneous linear time-invariant components is presented. The criterion has an appealing feature that only the properties of dynamics of individual components and the spectrum of the interconnection matrix are involved. The criterion has an illustrative graphical representation if we further assume that the individual components are SISO. It can be shown that violation of the criterion allows explicit construction of an admissible interconnection matrix which results in an unstable interconnected system.

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