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Flexible Control Lyapunov Functions

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Abstract*— A central tool in systems theory for synthesizing control laws that achieve stability are control Lyapunov functions (CLFs). Classically, a CLF enforces that the resulting closed-loop state trajectory is contained within a cone with a fixed, predefined shape, and which is centered at and converges to a desired converging point. However, such a requirement often proves to be overconservative. In this paper we propose a novel idea that improves the design of CLFs in terms of flexibility, i.e. the CLF is permitted to be locally non-monotone along the closed-loop trajectory. The focus is on the design of optimization problems that allow certain parameters that define a cone associated with a standard CLF to be decision variables. In this way non-monotonicity of the CLF is explicitly linked with a decision variable that can be optimized on-line. Conservativeness is significantly reduced compared to classical CLFs, which makes *flexible CLFs* more suitable for stabilization of constrained discrete-time nonlinear systems and real-time control.

I. INTRODUCTION

One of the interesting problems in nonlinear control systems is the synthesis of control laws that achieve stability [1], [2]. Control Lyapunov functions (CLFs) [3], [4] represent a powerful tool for providing a solution to this problem. The classical approach is based on the *off-line* design of an explicit feedback law that renders the derivative of the CLF negative. An alternative to this approach is to construct an optimization problem to be solved *on-line*, such that any of its feasible solutions renders the derivative of a candidate CLF negative. This method can be traced back to the early results presented in [5], followed by the more recent articles [6], [7], where synthesis of CLFs is performed in a receding horizon fashion.

All the above works mainly deal with the continuous-time case, while conditions under which these results can be extended to sampled-data nonlinear systems using their approximate discrete-time models can be found in [8]. An important article on control Lyapunov functions for discrete-time systems is [9]. Therein, classical continuous-time results regarding existence of CLFs are reproduced for the discrete-time case. A significant relaxation in the *off-line* design of CLFs for discrete-time systems was presented in [10], where parameter dependent quadratic CLFs are introduced. Also, interesting approaches to the off-line construction of Lyapunov functions for stability analysis were recently presented in [11], [12] and [13].

Despite the popularity of CLFs within systems theory, there is still a significant gap in the application of CLFs in real-time control. The main reason for this is conservativeness of the sufficient conditions for Lyapunov asymptotic stability which are employed by most off-line and on-line methods for constructing CLFs. To illustrate this consider the graphical depiction in Figure 1. Classically, a CLF enforces that the resulting closed-loop state trajectory is contained within a cone with a fixed, predefined shape, which is centered at and converges to a desired converging point. Typical examples of relevant classes of systems for which classical CLFs are overconservative are linear and nonlinear chains of integrators with bounded inputs and state constraints [14] and discontinuous nonlinear and hybrid systems [15]. Furthermore, in many real-life control problems classical CLFs prove to be overconservative. For example, consider the control of a simple electric circuit, such as the Buck-Boost DC-DC converter. At start-up, to drive the output voltage to the reference very fast, the inductor current must rise and stay far away (e.g., 5[A]) from its corresponding steady-state value (e.g., 0.01[A]) for quite some time. Another typical and very relevant real-life example is control of position and speed in mechatronic devices, such as electromagnetic actuators. For a given position reference, the speed must increase very fast at start-up and then return to its steady state value, which is equal to zero. In both cases enforcing a classical CLF design is obviously conservative.

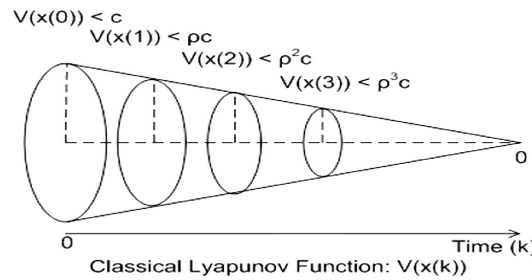

Fig. 1. A graphical illustration of classical CLFs ($\rho \in [0, 1)$, $c \in \mathbb{R}_{>0}$).

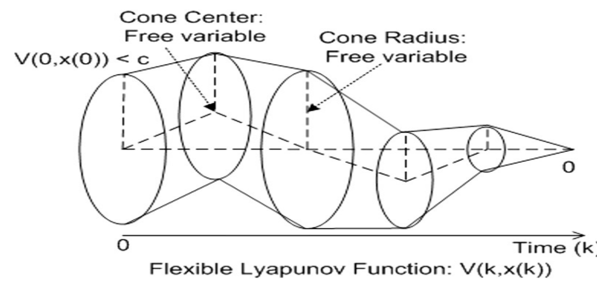

Fig. 2. A graphical illustration of flexible CLFs ($c \in \mathbb{R}_{>0}$).

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Motivated by such examples, in this paper we propose a methodology that reduces the conservatism of CLF design for discrete-time nonlinear systems. Rather than searching for a global CLF (i.e. on the whole admissible state-space), we focus on relaxing CLF-type conditions for a predetermined local CLF through on-line optimization problems. This approach makes it possible to derive a trajectory-dependent CLF (i.e. the stabilization conditions are only imposed for each measured state, along the closed-loop trajectory generated on-line), which is flexible (i.e. it can be locally non-monotone). A unique distinguishing feature of the idea presented in this paper is the explicit link between non-monotonicity of the CLF and a decision variable that can be optimized on-line. Besides the theoretical appeal of the proposed approach, we indicate that for nonlinear systems affine in control and CLFs based on infinity norms, the developed optimization problems can be formulated as a single linear or quadratic program, which is also attractive for real-time control.

II. Preliminaries

A. Basic notions and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\text{int}(\mathcal{S})$ the interior and by $\text{cl}(\mathcal{S})$ the closure of $\mathcal{S}$. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. For a vector $x \in \mathbb{R}^n$ let $\|x\|$ denote an arbitrary p-norm and let $[x]_i$, $i = 1, \ldots, n$ denote the i-th component of x. Let $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$, where $|\cdot|$ denotes the absolute value. For a matrix $Z \in \mathbb{R}^{m\times n}$ let $\|Z\| := \sup_{x\neq 0} \frac{\|Zx\|}{\|x\|}$ denote its corresponding induced matrix norm. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ ($\varphi \in \mathcal{K}_\infty$) if $\varphi \in \mathcal{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

B. Lyapunov asymptotic stability for difference inclusions

Consider the discrete-time autonomous nonlinear system

$$x(k+1) \in \Phi(x(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state at the discrete-time instant k and the mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an arbitrary nonlinear set-valued function. For simplicity of notation, we assume that the origin is an equilibrium in (1), i.e. $\Phi(0) = \{0\}$.

Definition II.1 We call a set $\mathcal{P} \subseteq \mathbb{R}^n$ *positively invariant (PI)* for system (1) if for all $x \in \mathcal{P}$ it holds that $\Phi(x) \subseteq \mathcal{P}$.

Definition II.2 *(i)* System (1) is *Lyapunov stable* if for any $\varepsilon > 0$ $\exists\delta(\varepsilon) > 0$ such that for all corresponding state trajectories of (1) it holds that $\|x(0)\| \leq \delta(\varepsilon) \Rightarrow \|x(k)\| \leq \varepsilon$ for all $k \in \mathbb{Z}_+$. *(ii)* Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ be a subset of $\mathbb{R}^n$. We call system (1) *attractive in* $\mathbb{X}$ if for each $x(0) \in \mathbb{X}$ it holds that all corresponding state trajectories of (1) satisfy $\lim_{k\to\infty} \|x(k)\| = 0$. *(iii)* We call system (1) *AS($\mathbb{X}$)* if it is Lyapunov stable and attractive in $\mathbb{X}$.

Theorem II.3 *Let $\mathbb{X}$ be a PI set for (1) with $0 \in \text{int}(\mathbb{X})$. Furthermore, let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{[0,1)}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{2a}$$

$$V(x^+) \leq \rho V(x) \tag{2b}$$

for all $x \in \mathbb{X}$ and all $x^+ \in \Phi(x)$. Then system (1) is AS($\mathbb{X}$).

The proof of the above theorem is similar in nature to the proofs given in [16], [17], by replacing the difference *equation* with the difference *inclusion* as in (1) and is omitted here for brevity. We call a function $V(\cdot)$ that satisfies the hypothesis of Theorem II.3 a *Lyapunov function*.

C. CLFs for discrete-time systems

Consider the discrete-time constrained nonlinear system

$$x(k+1) = \phi(x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{3}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at time instant k. $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function with $\phi(0, 0) = 0$. We assume that $\mathbb{X}$ and $\mathbb{U}$ are bounded sets with $0 \in \text{int}(\mathbb{X})$ and $0 \in \text{int}(\mathbb{U})$.

Definition II.4 A function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n \tag{4}$$

and for which there exists a control law, possible set-valued, $\pi : \mathbb{R}^n \rightrightarrows \mathbb{U}$ such that

$$V(\phi(x, u)) \leq \rho V(x), \quad \forall x \in \mathbb{X},\ \forall u \in \pi(x)$$

is called a *control Lyapunov function (CLF) in* $\mathbb{X}$ for the difference inclusion corresponding to system (3) in closed-loop with $u(k) \in \pi(x(k))$, $k \in \mathbb{Z}_+$. □

Problem II.5 Choose a candidate CLF $V(\cdot)$ for system (3). At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and calculate a control action $u(k)$ that satisfies:

$$u(k) \in \mathbb{U},\ \phi(x(k), u(k)) \in \mathbb{X}, \tag{5a}$$

$$V(\phi(x(k), u(k))) \leq \rho V(x(k)). \tag{5b}$$

Let $\pi(x(k)) := \{u(k) \in \mathbb{R}^m \mid (5) \text{ holds}\}$ and let $\phi_{\text{cl}}(x, \pi(x)) := \{\phi(x, u) \mid u \in \pi(x)\}$.

Proposition II.6 *Let a CLF $V(\cdot)$ in $\mathbb{X}$ be given for system (3). Suppose that Problem II.5 is feasible for all states x in $\mathbb{X}$. Then the difference inclusion*

$$x(k+1) \in \phi_{\text{cl}}(x(k), \pi(x(k))), \quad k \in \mathbb{Z}_+, \tag{6}$$

is AS($\mathbb{X}$).

The above result, which follows directly from Theorem II.3, establishes that feasible solutions of Problem II.5 are stabilizing feedback laws. However, feasibility of the inequalities (5) at all time instants, which also requires finding a CLF in $\mathbb{X}$ for the nonlinear system (3), often proves to be a too conservative requirement.

III. Flexible Control Lyapunov Functions

In this section we will propose a solution for relaxing classical CLFs. We assume that a time-invariant CLF is known for system (3) only in a subset of the state-space $\mathbb{X}$, i.e. in $\Omega \subseteq \mathbb{X}$ with $0 \in \mathrm{int}(\Omega)$, and we focus on relaxing inequality (5b) on-line, for each measured state. This will result in a flexible CLF that is trajectory-dependent.

Consider the following inequality corresponding to (5b):

$$V(x(k+1)) \leq \rho V(x(k)) + \lambda(k),\ k \in \mathbb{Z}_+, \qquad (7)$$

where $\lambda(k)$ is an additional decision variable which allows the radius of the sublevel set $\{x(k+1) \in \mathbb{X} \mid V(x(k+1)) \leq \rho V(x(k)) + \lambda(k)\}$ to be flexible, i.e. it can increase if (5b) is too conservative (see Figure 2 for a graphical illustration). Based on inequality (7) we can formulate the following optimization problem. Let $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ and $J : \mathbb{R} \to \mathbb{R}_+$ be a function such that $\alpha_3(|\lambda|) \leq J(\lambda) \leq \alpha_4(|\lambda|)$ for all $\lambda \in \mathbb{R}$. Let $\Omega \subseteq \mathbb{X}$ with $0 \in \mathrm{int}(\Omega)$ be a set with non-empty interior where $V(\cdot)$ is a CLF for system (3). For example, Ω can be taken as a the region of validity of a linearized model, for which a CLF function can be computed efficiently.

Problem III.1 Choose a candidate CLF $V(\cdot)$ for system (3). At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and minimize the cost $J(\lambda(k))$ over $u(k), \lambda(k)$ subject to

$$u(k) \in \mathbb{U},\ \phi(x(k), u(k)) \in \mathbb{X},\ \lambda(k) \geq 0, \qquad (8a)$$
$$V(\phi(x(k), u(k))) \leq \rho V(x(k)) + \lambda(k). \qquad (8b)$$

Let $\bar{\pi}(x(k)) := \{u(k) \in \mathbb{R}^m \mid \exists \lambda(k) \in \mathbb{R} \text{ s.t. (8) holds}\}$ and let $\phi_{\mathrm{cl}}(x, \bar{\pi}(x)) := \{\phi(x, u) \mid u \in \bar{\pi}(x)\}$. Let $\lambda^*(k)$ denote the optimum of Problem III.1 for all $k \in \mathbb{Z}_+$.

Theorem III.2 *Let a CLF $V(\cdot)$ in $\Omega \subseteq \mathbb{X}$ (with $0 \in \mathrm{int}(\Omega)$) be known for system (3) and suppose that Problem III.1 is feasible for all states x in $\mathbb{X}$. If $\lim_{k\to\infty} \lambda^*(k) = 0$, the difference inclusion*

$$x(k+1) \in \phi_{cl}(x(k), \bar{\pi}(x(k))), \quad k \in \mathbb{Z}_+, \qquad (9)$$

is AS($\mathbb{X}$).

Proof: Let $x(k) \in \mathbb{X}$ for some $k \in \mathbb{Z}_+$. Then, feasibility of Problem III.1 ensures that $x(k+1) \in \phi_{\mathrm{cl}}(x(k), \bar{\pi}(x(k))) \subseteq \mathbb{X}$ due to constraint (8a). Hence, Problem III.1 remains feasible and thus, $\mathbb{X}$ is a PI set for system (9). Then, by applying the inequality (8b) repetitively and using the inequality (4) (the upper bound) it can be shown (see [17], Chapter 2) that there exists a $\mathcal{K}$-function σ such that

$$V(x(k+1)) \leq \rho^{k+1} \alpha_2(\|x(0)\|) + \sigma(\|\lambda^*_{[k]}\|), \qquad (10)$$

where $\lambda^*_{[k]} := \{\lambda^*(l)\}_{l \in \mathbb{Z}_{[0,k]}}$. Exploiting the lower bound in inequality (4) and the fact that $\alpha_1^{-1} \in \mathcal{K}_\infty$ it results that there exists a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\|\lambda^*_{[k-1]}\|),\ \forall k \in \mathbb{Z}_{\geq 1}.$$

As $\lim_{k\to\infty} \lambda^*(k) = 0$ by the hypothesis, it follows that the closed-loop system (9) is "converging-input converging-state", as shown in [16], with $\lambda^*(k)$ as input. Hence, $\lim_{k\to\infty} \|x(k)\| = 0$ and thus, system (9) is attractive in $\mathbb{X}$. This further implies that the closed-loop state trajectory $x(k)$ reaches the set Ω in finite time. Hence, there exists a $j(x(0)) \in \mathbb{Z}_+$ such that $\lambda(j) = 0$ is a feasible solution of Problem III.1, by definition of a CLF in Ω (see Definition II.4). Then, due to minimization of the cost $J(\lambda)$ and constraint (8b), we have that solving Problem III.1 yields $\lambda^*(k) = 0$ for all $k \in \mathbb{Z}_{\geq j}$. Hence, applying inequality (8b) repetitively starting with time instant $j \in \mathbb{Z}_{\geq 1}$ yields:

$$\begin{aligned} V(x(k+1+j)) &\leq \rho^{k+1} \alpha_2(\|x(j)\|) + \sigma(\|\lambda^*_{[j,k]}\|) \\ &= \rho^{k+1} \alpha_2(\|x(j)\|), \end{aligned}$$

for some $\sigma \in \mathcal{K}$ (note that $\lambda^*_{[j,k]} := \{\lambda^*(l)\}_{l \in \mathbb{Z}_{[j,k]}}$). Combining the above local property with the lower bound in (4) and since for any $\varepsilon > 0$ we can choose a $\delta \in (0, \varepsilon)$ such that $\alpha_2(\delta) < \alpha_1(\varepsilon)$, it follows that (9) is Lyapunov stable (see [17], Chapter 2 for more details). Hence, system (9) is AS($\mathbb{X}$). □

Remark III.3 The condition $\lim_{k\to\infty} \lambda^*(k) = 0$ ensures finite time convergence to the set Ω, where $V(\cdot)$ is a classical CLF. However, in contrast with the terminal constraint set method employed in model predictive control (MPC) (see, e.g., [17]), it does not impose a fixed number of discrete-time instants for reaching Ω. Thus, the size of the set Ω is no longer influencing the set of states for which Problem III.1 is feasible. This turns out to be a crucial relaxation in terms of computing a candidate local CLF off-line, as Ω can be arbitrarily small. □

Next, we provide a non-conservative solution for guaranteeing that $\lim_{k\to\infty} \lambda^*(k) = 0$. By non-conservative we mean that a non-monotone evolution of $\lambda^*(k)$ should be allowed, while $\lambda^*(k) \to 0$ as $k \to \infty$.

Lemma III.4 *Let $M \in \mathbb{Z}_{\geq 1}$ be a fixed constant to be chosen a priori and let $\mu \in \mathbb{R}_{[0,1)}$. If*

$$0 \leq \lambda(k) \leq \max_{i \in [1,M]} \mu^i \lambda^*(k-i), \quad \forall k \in \mathbb{Z}_{\geq M} \qquad (11)$$

then $\lim_{k\to\infty} \lambda(k) = 0$.

The proof of Lemma III.4 follows by somewhat straightforward algebraic manipulations and is omitted for brevity.

Notice that constraint (11) allows a non-monotone evolution of $\lambda(k)$ for the first M sampling instant and furthermore, as long as $\lambda^*(k-i) > 0$ for at least one $i \in \mathbb{Z}_{[1,M]}$. By augmenting Problem III.1 with the constraint (11) on $\lambda(k)$ enables flexibility of the candidate CLF $V(\cdot)$ along the closed-loop state trajectory via constraint (8b), while still guaranteeing Lyapunov asymptotic stability. In contrast, if no constraint on $\lambda(k)$ is added to Problem III.1, although $\lambda(k)$ can freely vary in time, which is obviously much less conservative and can still lead to convergent trajectories, only practical stability can be guaranteed a priori.

Another possibility to further relax Problem III.1 is to optimize the parameters $\rho, \mu \in \mathbb{R}_{[0,1)}$ on-line, for every $x(k)$, as they control the rate of convergence, see, e.g., [12].

Remark III.5 The role of inequality (11) is to ensure that the closed-loop trajectory reaches the set $\Omega \subseteq \mathbb{X}$ in finite time by imposing a monotonically decreasing *upper bound* on $\lambda(k)$. A different way to guarantee the same property is to design an artificial higher dimensional system and define the upper bound on $\lambda(k)$ as an output of this system. Then $\lim_{k\to\infty} \lambda^*(k) = 0$ can be ensured through partial stability [18] of the artificial system, without imposing a monotonically decreasing upper bound. □

Remark III.6 A different possibility for relaxing (5b) is given by the following inequality:

$$V(x(k+1) - e(k)) \leq \rho V(x(k) - e(k)), \quad k \in \mathbb{Z}_+, \quad (12)$$

where $e(k)$ is an additional decision variable which allows the center of the sublevel set $\{x(k+1) \in \mathbb{X} \mid V(x(k+1) - e(k)) \leq \rho V(x(k) - e(k))\}$ to be flexible, i.e. it can be non-zero if (5b) is too conservative, moving the cone symmetry axis away from zero (see Figure 2 for an illustration). Inequality (12) is particularly suitable when regulation to a set rather than to a point is of concern. In this case $e(k)$ can be constrained in the set of interest, which results in asymptotic stabilization with respect to a set. □

Remark III.7 The requirement that a local standard CLF must be predetermined can be eliminated by synthesizing the function $V(\cdot)$ on-line simultaneously with the computation of $u(k)$ and $\lambda^*(k)$. This leads to a *time-variant* trajectory-dependent CLF, as opposed to a time-invariant one, which can still be computed efficiently by introducing a suitable parameterization, as shown in [19]. □

So far we have assumed that Problem III.1 is feasible for all $x \in \mathbb{X}$, which is usually difficult to verify or even not the case. This is because feasibility of Problem III.1 is in general only guaranteed for states within the *maximal controlled invariant set* contained in $\mathbb{X}$, for system (3), which is not necessarily identical to $\mathbb{X}$. A simple but numerically complex way to establish a domain of feasibility is to solve Problem III.1 explicitly off-line via multiparametric programming, whenever it can be formulated as a linear or quadratic (mixed-integer) program. Next, we present two solutions for guaranteeing recursive feasibility of Problem III.1. The first solution is based on establishing an easily verifiable sufficient condition under which a particular sublevel set of $V(\cdot)$ contained in $\mathbb{X}$, rather than Ω, is a PI set for the closed-loop system.

Lemma III.8 *Given a CLF $V(\cdot)$ in $\Omega \subseteq \mathbb{X}$ for system (3) and a $\Delta \in \mathbb{R}_{>0}$, let $\mathcal{V}_\Delta := \{x \in \mathbb{R}^n \mid V(x) \leq \Delta\}$. Let*

$$\bar{\lambda} := \sup_{x\in \mathrm{cl}(\mathbb{X}), u \in \mathrm{cl}(\mathbb{U})} \{V(\phi(x,u)) - \rho V(x)\}.$$

Then, for any $\Delta \in \mathbb{R}_{>0}$ such that $\mathcal{V}_\Delta \subseteq \mathbb{X}$, if the above supremum is a maximum[1] *and $\bar{\lambda} \leq (1-\rho)\Delta$, Problem III.1 is feasible for all $x \in \mathcal{V}_\Delta$ and remains feasible for all closed-loop trajectories that start in $\mathcal{V}_\Delta$.*

Corollary III.9 *Let a CLF $V(\cdot)$ in Ω for system (3) be given. Then, for any $\Delta \in \mathbb{R}_{>0}$ such that $\mathcal{V}_\Delta \subseteq \Omega$, Problem III.1 is feasible for all $x \in \mathcal{V}_\Delta$ and remains feasible for all resulting closed-loop trajectories that start in $\mathcal{V}_\Delta$.*

The second solution combines a terminal inequality constraint on the state trajectory with an equality constraint on the trajectory of $\lambda(k)$. Let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a function with $h(0) = 0$. For any $N \in \mathbb{Z}_{\geq 1}$ let $\lambda_{[N-1]}(k) := (\lambda(0,k), \ldots, \lambda(N-1,k)) \in \mathbb{R}^N$ and let $\mathbf{u}_{[N-1]}(k) := (u(0,k), \ldots, u(N-1,k)) \in \mathbb{U}^N$. Finally, let $x(i+1,k) := \phi(x(i,k), u(i,k))$ for $i = 0, \ldots, N-1$ and $x(0,k) := x(k)$.

Problem III.10 Choose a candidate CLF $V(\cdot)$ for system (3), a $\rho \in \mathbb{R}_{[0,1)}$ and a $\Delta \in \mathbb{R}_{>0}$. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and minimize the cost $J(\lambda_{[N-1]}(k)) := \sum_{i=0}^{N-1} |\lambda(i,k)|$ over $u(i,k), \lambda(i,k)$ subject to the constraints

$$\begin{aligned}
&u(i,k) \in \mathbb{U},\ \phi(x(i,k), u(i,k)) \in \mathbb{X},\ \lambda(i,k) \geq 0, \\
&\text{for all } i = 0, \ldots, N-1, && (13a)\\
&V(\phi(x(i,k), u(i,k))) \leq \rho V(x(i,k)) + \lambda(i,k), \\
&\text{for all } i = 0, \ldots, N-1, && (13b)\\
&V(x(N,k)) \leq \Delta. && (13c)
\end{aligned}$$

Let $\Pi(x(k)) := \{\mathbf{u}_{N-1}(k) \in \{\mathbb{R}^m\}^N \mid \exists \lambda_{[N-1]}(k) \in \mathbb{R}^N \text{ s.t. (13) holds}\}$ and let $\tilde{\pi}(x(k)) := \{u(0,k) \mid \mathbf{u}_{[N-1]}(k) \in \Pi(x(k))\}$. Define $\phi_{\mathrm{cl}}(x, \tilde{\pi}(x)) := \{\phi(x,u) \mid u \in \tilde{\pi}(x)\}$.

Theorem III.11 *Let $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ be such that $\alpha_3(|\lambda(0,k)|) \leq J(\lambda_{[N-1]}(k)) \leq \alpha_4(|\lambda(0,k)|)$. Choose $\Delta \in \mathbb{R}_{>0}$ such that $V(\cdot)$ is a CLF for system (3) in closed-loop with $u(k) = h(x(k))$, $k \in \mathbb{Z}_+$, in the set $\mathcal{V}_\Delta = \{x \in \mathbb{R}^n \mid V(x) \leq \Delta\} \subseteq \mathbb{X}$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of states for which Problem III.10 is feasible. Then:*

(i) *$\mathbb{X}_f(N)$ is a PI set for the difference inclusion*

$$x(k+1) \in \phi_{\mathrm{cl}}(x(k), \tilde{\pi}(x(k))), \quad k \in \mathbb{Z}_+. \quad (14)$$

Moreover, $\mathcal{V}_\Delta \subseteq \mathbb{X}_f(N)$.

(ii) *The difference inclusion (14) is AS($\mathbb{X}_f(N)$).*

The proof of Theorem III.11 follows by combining the reasoning used in the proof of Theorem III.2 with the shifted sequence technique employed in the standard terminal constraint MPC stability proof, see, e.g., [17].

[1] Boundedness of $\phi(\cdot,\cdot)$ and $V(\cdot)$ on bounded sets is a sufficient condition for this hypothesis, as $V(\cdot)$ is lower and upper bounded by a $\mathcal{K}_\infty$ function.

IV. IMPLEMENTATION ISSUES

The controller synthesis methodology developed in this paper is generally applicable to discrete-time systems, at the price of solving an optimization problem on-line. In this section we present some ingredients for implementing Problem III.1 as a single linear or quadratic program. Firstly, suppose there exist functions $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ such that:

$$x(k+1) = \phi(x(k), u(k)) = f(x(k)) + g(x(k))u(k). \quad (15)$$

Also, assume that the sets $\mathbb{X}$ and $\mathbb{U}$ are polyhedra. Secondly, we restrict our attention to CLFs defined using the ∞-norm, i.e. $V(x) := \|Px\|_\infty$, where $P \in \mathbb{R}^{p \times n}$ is a full-column rank matrix, to be computed off-line. The interested reader is referred to [17] for techniques to compute local CLFs based on infinity norms. These methods apply to linear and piecewise affine (PWA) systems. Note that $V(x)$ defined above satisfies the inequality (4) for $\alpha_1(s) := \frac{s_p}{\sqrt{p}} s$ ($s_p > 0$ is the smallest singular value of P) and for $\alpha_2(s) := \|P\|_\infty s$. Consider now Problem III.1, possibly augmented with (11). Since at discrete-time instant k the measured state $x(k)$ and the previously computed $\lambda^*(k-i)$, $i \in \mathbb{Z}_{[1,M]}$ are known, it is clear that the constraints (8a) and (11) are linear inequalities in $u(k)$ and $\lambda(k)$. Next, letting $V(x) = \|Px\|_\infty$, (8b) becomes:

$$\|P(f(x(k)) + g(x(k))u(k))\|_\infty \leq \rho\|Px(k)\|_\infty + \lambda(k),$$

where $x(k)$, P and $\rho \in \mathbb{R}_{[0,1)}$ are known. By the definition of the infinity norm, for (8b) to be satisfied it is necessary and sufficient to require that:

$$\pm[P(f(x(k)) + g(x(k))u(k))]_i \leq \rho\|Px(k)\|_\infty + \lambda(k)$$

for all $i \in \mathbb{Z}_{[1,p]}$, which yields $2p$ linear inequalities in $u(k)$ and $\lambda(k)$. Therefore, if $J(\lambda(k)) := \|\Gamma\lambda(k)\|_\infty$, $\Gamma \in \mathbb{R}_+$, or if $J(\lambda(k))$ is linear or quadratic in $\lambda(k)$, then Problem III.1 can be formulated as a single linear or quadratic program, respectively. Furthermore, for any $N \in \mathbb{Z}_{\geq 1}$, Problem III.10 can be implemented as a single LP or QP for linear systems and as a mixed integer LP or QP for PWA systems.

Remark IV.1 In Problem III.1 a cost of the form $J(x(k), u(k), \lambda(k)) := \|Q_1 x(k+1)\|_\infty + \|Qx\|_\infty + \|Ru(k)\|_\infty + \|\Gamma\lambda(k)\|_\infty$, $Q_1, Q \in \mathbb{R}^{q \times n}$, $R \in \mathbb{R}^{r \times m}$ can be specified to provide a way for selecting a feasible control action on-line. Then Problem III.1 can still be formulated as a single LP. This is because the minimization of $J(x(k), u(k), \lambda(k))$ can be reformulated as minimizing $\varepsilon_1 + \varepsilon_2 + \|\Gamma\lambda_k\|_\infty$ subject to $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, $\pm[Ru(k)]_j \leq \varepsilon_1$ and $\pm[Q_1(f(x(k)) + g(x(k))u(k))]_i + \|Q(x(k))\|_\infty \leq \varepsilon_2$ for all $j \in \mathbb{Z}_{[1,r]}$ and $i \in \mathbb{Z}_{[1,q]}$. □

Remark IV.2 The number of linear inequalities needed to specify the inequalities (8) depends linearly on the number of rows of P, the state dimension and the input dimension. This means that Problem III.1 can be rendered numerically efficient even for high dimensional systems or for fast nonlinear systems, as modern linear program solvers (OSL from IBM or Cplex) can handle up to 16 million constraints. This brings a wide range of applications in large scale systems, mechatronics, power electronics, aeronautics and robotics within reach. □

V. ACADEMIC EXAMPLE

In this section we illustrate the effectiveness of flexible CLFs, employed via Problem III.1. For a practical application of flexible CLFs we refer the interested reader to [20], which deals with control of electromagnetic actuators. Now consider the nonlinear system (15) with $x(k) \in \mathbb{X} = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 5\}$, $u(k) \in \mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 1\}$ and

$$f(x) = \begin{pmatrix} [x]_1 + 0.7[x]_2 + ([x]_2)^2 \\ [x]_2 \end{pmatrix},$$
$$g(x) = \begin{pmatrix} 0.245 + \sin([x]_2) \\ 0.7 \end{pmatrix}.$$

The technique of [17] was used to compute the weight $P \in \mathbb{R}^{2 \times 2}$ of the local CLF $V(x) = \|Px\|_\infty$ for $\rho = 0.8$, yielding

$$P = \begin{bmatrix} 2.6851 & 0.7203 \\ 0.2083 & 4.0458 \end{bmatrix}.$$

A rough approximation of the region where $V(x) = \|Px\|_\infty$ is a CLF for the considered nonlinear system is given by the set $\Omega := \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$. In (11) we have set $\mu = 0.94$ and $M = 1$. The cost $J(x(k), u(k), \lambda(k)) := \|Q_1 x(k+1)\|_\infty + \|Qx(k)\|_\infty + |Ru(k)| + |\lambda(k)|$, where

$$Q_1 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \ Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ R = 0.4,$$

was used to improve performance. The optimization problem corresponding to Problem III.1 was formulated as a single LP utilizing the techniques presented in Section IV. The resulting LP has 4 optimization variables and 12 constraints (13 if (11) is added). During the simulations, the worst case computational time required by the CPU over 4000 runs was 2 milliseconds, which clearly shows the potential of the proposed algorithm for controlling fast nonlinear systems.

Figure 3 presents closed-loop simulation results in terms of state trajectories (upper plot), $\lambda^*(k)$ (middle plot) and control input history (lower plot) as follows: *solid lines* depict the results obtained by solving Problem III.1 with constraint (11) for initial state $x(0) = [3, -1]^\top$ and *dashed lines* depict the results obtained by solving Problem III.1 for initial state $x(0) = [5, -1]^\top$. For the first simulation (solid lines) the optimization problem was feasible at all times and $\lambda^*(k)$ converges monotonically to zero, as imposed by (11). Problem II.5, which implements the classical CLF approach, is not feasible for the same initial state, which clearly illustrates the decrease of conservativeness. In fact, Problem II.5 is only feasible for states in Ω. This can also be observed from the solid line trajectories in Figure 3, i.e. at time instant $k = 18$, when $\lambda^*(k) = 0$, both states are inside Ω (its boundaries are represented by *dotted lines* in the upper plot of Figure 3). Hence, locally, Problem III.1 based on a

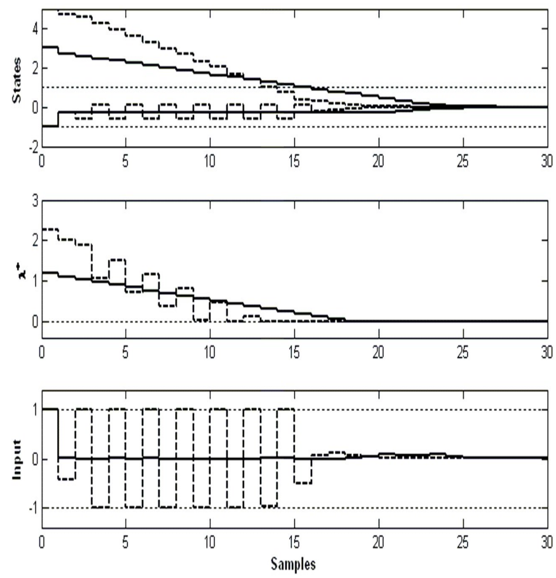

Fig. 3. Closed-loop simulation results.

flexible CLF recovers Problem II.5 based on a classical CLF, while Problem III.1 remains feasible for a much wider set of initial conditions.

The feasible region of Problem III.1 can be further extended by either increasing M or by discarding the constraint (11). To illustrate this, in the second simulation we considered the initial state $x(0) = [5, -1]^\top$, which is on the border of the state constraints set. For this initial condition neither Problem II.5 nor Problem III.1 with (11) and $M = 1$ is feasible. However, by setting $M = 4$ in (11) or by removing (11), Problem III.1 becomes feasible and the controller successfully stabilizes the system. Again, one can observe that $\lambda^*(k) = 0$ and remains zero thereafter at time instant $k = 13$, which coincides with the time instant at which both states reach the set Ω, as guaranteed by Corollary III.9.

VI. Conclusions

In this paper we proposed a novel methodology for reducing the conservatism of CLFs for discrete-time constrained nonlinear systems. Rather than imposing a CLF globally (i.e. on the whole state-space), we focused on relaxing CLF-type conditions for a predetermined local CLF through on-line optimization problems. This approach made it possible to derive a trajectory-dependent CLF, which is flexible (i.e. it can be locally non-monotone when required). The non-monotonicity of the CLF was explicitly linked with a decision variable that can be optimized on-line. Moreover, we indicated that for nonlinear systems affine in control and CLFs based on infinity norms, the developed optimization problems can be formulated as a single LP or QP, which is attractive for real-time control.

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