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On Decentralized Stabilization of Discrete-time Nonlinear Systems

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***Abstract*— This paper deals with stabilization of interconnected discrete-time nonlinear systems under given, arbitrary information constraints on the controller structure. As a prominent example, the considered information constraints on the controller structure include decentralized and distributed control over a given communication network. The main contribution of this paper is twofold. Firstly, we introduce the notion of structured control Lyapunov functions (CLFs) as a suitable tool for stabilizing controller synthesis under information constraints. This includes the relaxation of Lyapunov conditions at the local level. Secondly, we present a method for constructing structured CLFs and we show that the controller synthesis problem using structured CLFs can be formulated as a convex optimization problem. Possible solutions for solving this problem efficiently under several different types of information constraints are also indicated.**

I. INTRODUCTION

Over the past few years there has been a rapidly growing interest in the systems and control community in the study of networked dynamical systems. Examples of such systems include electrical power networks, formation flight of unmanned aerial vehicles, automated highways, control of communication networks and smart structures, to name just a few. The fundamental characteristics of these systems, such as coupling between local system dynamics or performance objectives, uncertainties and communication constraints, require a theory for synthesizing control laws able to cope with predefined physical and information constraints. In this context, prominent examples of constraints on the structure of control algorithms are the ones arising from *decentralized* and *distributed* implementation structures. The term decentralized is commonly used to denote a set of controllers which operate with no mutual exchange of information, while the term distributed assumes that the controllers share information over a specific communication network with a predefined and usually sparse structure.

Despite successful contributions and a long history, see e.g. [1], a general theory of feedback control under information constraints is lacking and certain cases of structured control problems have even been shown to be intractable [2]. Recently, several structured control problems with some specific characteristics have been successfully studied, such as, for example, distributed control of linear spatially invariant systems [3], control of homogeneous systems interconnected over lattices or arbitrary symmetry groups, see e.g. [4], and control of heterogenous system interconnected over an arbitrary graph [5]. For other related and recent results see, for example, [6], [7] and the references therein. In particular, in [6] the authors delineate the largest known class of structured control problems which can be formulated as convex optimization problems, while in [7] the authors introduce the notion of spatially decaying operators in the insightful study of structural properties of optimal control problems with relation to the spatial structure of the problem.

Regarding stability analysis or synthesis of stabilizing controllers for interconnected systems, a traditional and often used approach lies within the framework of dissipative dynamical systems [8], with passivity and small gain theorems as prominent examples. This approach accounts for finding appropriately defined local storage functions, corresponding supply functions and the coupling conditions which together imply stability of the overall network, see e.g. [9], [10]. The dissipativity approach is widely used as the underlying framework in many of the more recent results as well, see e.g. [11], [12] and the references therein. Alternative approaches include the usage of vector or matrix Lyapunov functions, see e.g. [13], [14] for classical results and [15] for more recent results; approaches based on Youla parametrization of stabilizing controllers [6]; or some alternative approaches based on Nyquist-like "loop gain" conditions conditions, as presented, for example, in [16].

This paper proposes a new approach to stabilization and optimal control of interconnected discrete-time nonlinear systems under given, arbitrary information constraints on the controller structure. The central ingredient of the developed results is the novel concept of a set of structured control Lyapunov functions (CLFs). While structured CLFs are still closely related to the theory of dissipative dynamical systems, they present certain different and advantageous characteristics, suited to accommodate stabilizing controller synthesis under various information constraints. A set of structured CLFs is defined as a set of positive definite functions, with each of these functions depending only of the state vector of its corresponding local system and satisfying certain coupling conditions. Although neither of these functions is required to be a CLF for its corresponding local system, it is proven that the coupling conditions guarantee a global CLF, i.e. a CLF for the overall interconnected system. Furthermore, based on the notion of structured CLFs, we show how to construct a convex optimization problem such that any of its feasible solutions provides a stabilizing control action for the interconnected system. Finally, by including an arbitrary performance criterion, we indicate how the resulting problem can be solved effectively under several different information constraints, which include decentralized control, decentral-

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ized control with global coordination and distributed control.

II. Preliminaries

A. Basic notions and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. For a set $\{x_i\}_{i\in\mathbb{Z}_{[1,N]}}$, $x_i \in \mathbb{R}^{n_i}$, $N \in \mathbb{Z}_+$, we use $\operatorname{col}(\{x_i\}_{i\in\mathbb{Z}_{[1,N]}})$, and equivalently $\operatorname{col}(x_1,\ldots,x_N)$, to denote the column vector $\left(x_1^\top,\ldots,x_n^\top\right)^\top$. We use $\operatorname{diag}(P_1,\ldots,P_N)$ to denote a block-diagonal matrix with matrices $P_1,\ldots,P_N$ on the main diagonal and zeros elsewhere. $\mathbf{0}_n$ denotes a vector in $\mathbb{R}^n$ with all elements equal to 0, while $\mathbf{1}_n$ denotes a vector in $\mathbb{R}^n$ with all elements equal to 1. For a matrix M, $\operatorname{Im}(M)$ denotes its image space, and $[M]_{ij}$ is the *ij*-th entry of M. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\operatorname{int}(\mathcal{S})$ the interior of $\mathcal{S}$. The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p := (|[x]_1|^p + \ldots + |[x]_n|^p)^{\frac{1}{p}}$ for $p \in \mathbb{Z}_{[1,\infty)}$ and $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$, where $[x]_i$, $i = 1,\ldots,n$, is the *i*-th component of x and $|\cdot|$ is the absolute value. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and it is radially unbounded, i.e. $\lim_{s\to\infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot,k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s,\cdot)$ is decreasing and $\lim_{k\to\infty} \beta(s,k) = 0$.

B. Asymptotic Lyapunov stability

Consider the discrete-time autonomous nonlinear system

$$x(k+1) \in \Phi(x(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state at the discrete-time instant k and the mapping $\Phi : \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ is an arbitrary nonlinear set-valued function. For simplicity of notation, we assume that the origin is an equilibrium in (1), i.e. $\Phi(0) = \{0\}$.

Definition II.1 We call a set $\mathcal{P} \subseteq \mathbb{R}^n$ *positively invariant (PI)* for system (1) if for all $x \in \mathcal{P}$ it holds that $\Phi(x) \subseteq \mathcal{P}$.

Definition II.2 Let $\mathbb{X}$ with $0 \in \operatorname{int}(\mathbb{X})$ be a subset of $\mathbb{R}^n$. We call system (1) asymptotically stable in $\mathbb{X}$, or shortly *AS($\mathbb{X}$)*, if there exists a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ such that, for each $x(0) \in \mathbb{X}$ it holds that all corresponding state trajectories of (1) satisfy $\|x(k)\| \leq \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{Z}_+$.

Theorem II.3 *Let $\mathbb{X}$ be a PI set for (1) with $0 \in \operatorname{int}(\mathbb{X})$. Furthermore, let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{2a}$$

$$V(x^+) - V(x) \leq -\alpha_3(\|x\|) \tag{2b}$$

for all $x \in \mathbb{X}$ and all $x^+ \in \Phi(x)$. Then system (1) is AS($\mathbb{X}$).

The proof of the above theorem is similar in nature to the proof given in [17], [18], by replacing the difference *equation* with the difference *inclusion* as in (1) and is omitted here for brevity. It is worth to point out that if $V(\cdot)$ is a continuous function, the above theorem can be recovered from Theorem 2.8 of [19], which gives sufficient conditions for robust $\mathcal{KL}$-stability of difference inclusions. We call a function $V(\cdot)$ that satisfies the hypothesis of Theorem II.3 a *Lyapunov function*.

C. CLFs for discrete-time systems

Consider the discrete-time constrained nonlinear system

$$x(k+1) = \phi(x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{3}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at the discrete-time instant k. $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary nonlinear function with $\phi(0,0) = 0$. Naturally, we assume $0 \in \operatorname{int}(\mathbb{X})$ and $0 \in \operatorname{int}(\mathbb{U})$. Next, let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$.

Definition II.4 A function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n \tag{4}$$

and for which there exists a control law, possible set-valued, $\pi : \mathbb{R}^n \hookrightarrow \mathbb{U}$ such that

$$V(\phi(x,u)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{X},\ \forall u \in \pi(x)$$

is called a *control Lyapunov function (CLF) in* $\mathbb{X}$ for the difference inclusion corresponding to system (3) in closed-loop with $u(k) \in \pi(x(k))$, $k \in \mathbb{Z}_+$. □

III. Structured CLFs

A. Network of dynamically coupled systems

Consider a directed connected graph $\mathcal{G} = (\mathcal{S}, \mathcal{E})$ with a finite number of vertices $\mathcal{S} = \{\varsigma_1,\ldots,\varsigma_N\}$ and a set of directed edges $\mathcal{E} \subseteq \{(\varsigma_i,\varsigma_j) \in \mathcal{S} \times \mathcal{S} \mid i \neq j\}$. A dynamical system is assigned to each vertex $\varsigma_i \in \mathcal{S}$, with the dynamics governed by the following equation:

$$x_i(k+1) = \phi_i(x_i(k), u_i(k), v_i(x_{\mathcal{N}_i}(k))), k \in \mathbb{Z}_+, i = \overline{1,N} \tag{5}$$

In (5), $x_i \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, $u_i \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ are the state and the control input of the i-th system, i.e. the system assigned to vertex ς_i. With each directed edge $(\varsigma_j,\varsigma_i) \in \mathcal{E}$ we associate a function $v_{ij} : \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$, which defines the interconnection signal $v_{ij}(x_j(k)), k \in \mathbb{Z}_+$, between system j and system i, i.e. $v_{ij}(\cdot)$ characterizes how the states of system j influence the dynamics of system i. We will use $\mathcal{N}_i := \{j \mid (\varsigma_j,\varsigma_i) \in \mathcal{E}\}$ to denote the set of indices corresponding to the direct neighbors of system i. The term *direct neighbor* of system i defines any system in the network whose dynamics (e.g., states or outputs) appear explicitly (via the function $v_{ij}(\cdot)$) in the state equations that govern the dynamics of system i. Clearly, if system j is a direct neighbor of system i, this does not necessarily imply the reverse. For convenience we also define $\mathcal{I} := \{1,\ldots,N\}$ as the set of vertex indices, and we define $x_{\mathcal{N}_i}(k) := \text{col}(\{x_j(k)\}_{j\in\mathcal{N}_i})$ as the vector that collects all the state vectors of the direct neighbors of system i and $v_i(x_{\mathcal{N}_i}(k)) := \text{col}(\{v_{ij}(x_j(k))\}_{j\in\mathcal{N}_i})$ as the vector that collects all the vector valued interconnection signals that "enter" system i. $\phi_i(\cdot,\cdot,\cdot)$ and $v_{ij}(\cdot)$ are arbitrary nonlinear functions that satisfy $\phi_i(0,0,0) = 0$ for all $i \in \mathcal{I}$ and $v_{ij}(0) = 0$ for all $(i,j) \in \mathcal{I}\times\mathcal{N}_i$. For all $i \in \mathcal{I}$ we assume that $\mathbb{X}_i$ and $\mathbb{U}_i$ contain the origin.

The following reasonable standing assumption is instrumental for obtaining the results presented in this paper.

Assumption III.1 The value of all interconnection signals $\{v_{ij}(x_j(k))\}_{j\in\mathcal{N}_i}$ is known at each discrete-time instant $k \in \mathbb{Z}_+$ for any system $i \in \mathcal{I}$.

Notice that Assumption III.1 does not require knowledge of any of the interconnection signals at any future time instants $k \in \mathbb{Z}_{\geq 1}$. From a technical point of view, Assumption III.1 is satisfied, e.g., if all interconnection signals $v_{ij}(x_j(k))$ are directly measurable[1] at all $k \in \mathbb{Z}_+$. Alternatively, Assumption III.1 is satisfied if all directly neighboring systems $j \in \mathcal{N}_i$ are able to communicate their local measured state $x_j(k)$ to system $i \in \mathcal{I}$. Finally, let

$$x(k+1) = \phi(x(k), u(k)) \tag{6}$$

denote the dynamics of the complete interconnected system (5) written in a compact form. In (6) $x = \text{col}(\{x_i\}_{i\in\mathcal{I}})$ and $u = \text{col}(\{u_i\}_{i\in\mathcal{I}})$ are vectors that collect all states and inputs, respectively.

B. Structured CLFs for networks of dynamically coupled systems

Next, we introduce the notion of a set of structured CLFs.

Definition III.2 A set of functions $\{V_i(\cdot)\}_{i\in\mathcal{I}}$ with $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ for all $i \in \mathcal{I}$ for which there exist $\alpha_1^i, \alpha_2^i, \alpha_3^i \in \mathcal{K}_\infty$, a set of control laws, possibly set-valued, $\{\pi_i(\cdot)\}_{i\in\mathcal{I}}$ with $\pi_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_{v_i}} \hookrightarrow \mathbb{U}_i$ for all $i \in \mathcal{I}$ and a set of functions $\{\varrho_i(\cdot)\}_{i\in\mathcal{I}}$ with $\varrho_i : \mathbb{R}^{n_i}\times\mathbb{R}^{n_{v_i}} \to \mathbb{R}$ for all $i \in \mathcal{I}$ (where n_{v_i} is the dimension of the vector $v_i(x_{\mathcal{N}_i})$) such that

$$\alpha_1^i(\|x_i\|) \leq V_i(x_i) \leq \alpha_2^i(\|x_i\|), \ \forall x_i \in \mathbb{R}^{n_i}, \ i \in \mathcal{I}, \tag{7}$$

$$\begin{aligned} &V_i(\phi(x_i, u_i, v_i(x_{\mathcal{N}_i}))) - V_i(x_i) \\ &\qquad\qquad \leq -\alpha_3(\|x_i\|) + \varrho_i(x_i, v_i(x_{\mathcal{N}_i})), \\ &\forall x_i \in \mathbb{X}_i, \forall v_i(x_{\mathcal{N}_i}), \forall u_i \in \pi_i(x_i, v_i(x_{\mathcal{N}_i})), i \in \mathcal{I}, \end{aligned} \tag{8}$$

and

$$\sum_{i\in\mathcal{I}} \varrho_i(x_i, v_i(x_{\mathcal{N}_i})) \leq 0, \tag{9}$$

is called a *set of structured control Lyapunov functions in* $\mathbb{X} := \{\text{col}(\{x_i\}_{i\in\mathcal{I}}) \mid x_i \in \mathbb{X}_i\} \subseteq \mathbb{R}^{\sum_{i\in\mathcal{I}} n_i}$ for the difference inclusion corresponding to system (6) in closed-loop with $u(k) \in \pi(x(k)) := \text{col}(\{\pi_i(x_i(k), v_i(x_{\mathcal{N}_i}(k)))\}_{i\in\mathcal{I}})$, $k \in \mathbb{Z}_+$.

In the above definition the term *structured* CLFs emphasizes the fact that each $V_i(\cdot)$ is a function of x_i only, i.e. the structural decomposition of the dynamics of the overall interconnected system (5) is reflected in the structure of the set $\{V_i(\cdot)\}_{i\in\mathcal{I}}$. Also, notice that the above definition does not impose that each function $V_i(\cdot)$ is a CLF in $\mathbb{X}_i$ for its corresponding system $i \in \mathcal{I}$. Next, based on Definition III.2, we formulate the following optimization problem.

Problem III.3 Let $\alpha_3^i \in \mathcal{K}_\infty$, $i \in \mathcal{I}$ and a set of candidate structured CLFs $\{V_i(\cdot)\}_{i\in\mathcal{I}}$ be known. At time $k \in \mathbb{Z}_+$, given the state vector $\{x_i(k)\}_{i\in\mathcal{I}}$ and each interconnection signal in the set $\{v_i(x_{\mathcal{N}_i})\}_{i\in\mathcal{I}}$, calculate a set of control actions $\{u_i(k)\}_{i\in\mathcal{I}}$ and a set of variables $\{\tau_i(k)\}_{i\in\mathcal{I}}$, with $\tau_i(k) \in \mathbb{R}$ for all $i \in \mathcal{I}$, such that:

$$u_i(k) \in \mathbb{U}_i, \ \ \phi_i(x_i(k), u_i(k), v_i(x_{\mathcal{N}_i}(k))) \in \mathbb{X}_i, \tag{10a}$$

$$\begin{aligned} &V_i(\phi_i(x_i(k), u_i(k), v_i(x_{\mathcal{N}_i}(k)))) - V_i(x_i(k)) \\ &\qquad\qquad \leq -\alpha_3^i(\|x_i(k)\|) + \tau_i(k), \end{aligned} \tag{10b}$$

$$\sum_{i\in\mathcal{I}} \tau_i(k) \leq 0. \tag{10c}$$

□

Let $\Pi(x(k)) := \{(\text{col}(\{u_i(k)\}_{i\in\mathcal{I}}), \text{col}(\{\tau_i(k)\}_{i\in\mathcal{I}})) \in \mathbb{R}^{\sum_{i\in\mathcal{I}} m_i} \times \mathbb{R}^N \mid \text{(10) holds}\}$, let $\pi(x(k)) := \{\text{col}(\{u_i(k)\}_{i\in\mathcal{I}}) \mid (\text{col}(\{u_i(k)\}_{i\in\mathcal{I}}), \text{col}(\{\tau_i(k)\}_{i\in\mathcal{I}})) \in \Pi(x(k))\}$ and let

$$\varrho(x(k)) = \text{col}(\{\varrho_i(x_i(k), v_i(\mathcal{N}_i(k)))\}_{i\in\mathcal{I}}) \tag{11}$$

Notice that $\varrho(x(k)) \in \{\text{col}(\{\tau_i(k)\}_{i\in\mathcal{I}}) \mid (\text{col}(\{u_i(k\}_{i\in\mathcal{I}}), \text{col}(\{\tau_i(k)\}_{i\in\mathcal{I}})) \in \Pi(x(k))\}$. As specified above, in Problem 10, the optimization variable $\tau_i(k)$ plays the role of the value that the function $\varrho_i(x_i(k), v_i(\mathcal{N}_i(k))$ takes at each $k \in \mathbb{Z}_+$ for each $i \in \mathcal{I}$. Furthermore, let

$$\begin{aligned} x(k+1) &\in \phi_{\text{CL}}(x(k), \pi(x(k))) \\ &:= \{\phi(x(k), u(k)) \mid u(k) \in \pi(x(k))\} \end{aligned} \tag{12}$$

denote the difference inclusion corresponding to system (6) in "closed loop" with the set of feasible control actions obtained by solving Problem III.3 at each $k \in \mathbb{Z}_+$.

Theorem III.4 *Let $\alpha_1^i, \alpha_2^i, \alpha_3^i \in \mathcal{K}_\infty$ be given and choose a set of candidate structured CLFs $\{V_i(\cdot)\}_{i\in\mathcal{I}}$ in $\mathbb{X} = \{\text{col}(\{x_i\}_{i\in\mathcal{I}}) \mid x_i \in \mathbb{X}_i\}$ for system* (6)*. Suppose that Problem III.3 is feasible for all $x \in \mathbb{X}$ and the corresponding signals $\{v_i(x_{\mathcal{N}_i})\}_{i\in\mathcal{I}}$. Then the difference inclusion*

$$x(k+1) \in \phi_{\text{CL}}(x(k), \pi(x(k))), \quad k \in \mathbb{Z}_+, \tag{13}$$

is AS($\mathbb{X}$).

To prove Theorem III.4 we make use of the following lemma.

Lemma III.5 *Let $\alpha_i \in \mathcal{K}_\infty$, $x_i \in \mathbb{R}^{n_i}$, $i \in \mathcal{I}$ and let $x := \text{col}(x_1, \ldots, x_N)$. Moreover, let $\tilde{\alpha}(s) := \min_{i\in\mathcal{I}} \alpha_i(s)$,*

[1]For example, in electrical power systems, where a dynamical system is a power generator, the interconnection signal is the generator bus voltage and line power (or current) flow in the corresponding power line, which can be directly measured.

$\underline{\alpha}(\|x\|) := \tilde{\alpha}(\frac{1}{N}\|x\|)$ *and let* $\overline{\alpha}(\|x\|) := \sum_{i=1}^{N} \alpha_i(\|x\|)$*. Then the following holds*

$$\underline{\alpha}(\|x\|) \leq \sum_{i=1}^{N} \alpha_i(\|x_i\|) \leq \overline{\alpha}(\|x\|).$$

The proof of Lemma III.5 follows from standard norm inequalities and properties of $\mathcal{K}_\infty$ functions.

Proof of Theorem III.4: Let $x(k) \in \mathbb{X}$ for some $k \in \mathbb{Z}_+$. Then, feasibility of Problem III.3 ensures that $x(k+1) \in \phi_{\mathrm{CL}}(x(k), \pi(x(k))) \subseteq \mathbb{X}$ due to constraint (10a). Hence, Problem III.3 remains feasible and thus, $\mathbb{X}$ is a PI set for system (13). Summation of inequalities (10b) over the set of indices $i \in \mathcal{I}$, together with condition (10c), yields

$$V(x(k+1)) - V(x(k)) \leq -\sum_{i\in\mathcal{I}} \alpha_3^i(\|x_i\|) \leq -\alpha_3(\|x(k)\|)), \tag{14}$$

where $V(x) := \sum_{i\in\mathcal{I}} V_i(x_i)$, $\alpha_3(\|x\|) := \tilde{\alpha}_3(\frac{1}{N}\|x\|)$ and $\tilde{\alpha}_3(s) := \min_{i\in\mathcal{I}} \alpha_3^i(s)$. The second inequality in (14) follows directly from Lemma III.5. Note that $\alpha_3 \in \mathcal{K}_\infty$. Furthermore, from inequality (7), together with Lemma III.5, we obtain

$$\begin{aligned} \alpha_1(\|x\|) \leq \sum_{i\in\mathcal{I}} \alpha_1^i(\|x_i\|) \leq V(x) &\leq \sum_{i\in\mathcal{I}} \alpha_2^i(\|x_i\|) \\ &\leq \alpha_2(\|x\|), \end{aligned} \tag{15}$$

where $\alpha_1(\|x\|) := \tilde{\alpha}_1(\frac{1}{N}\|x\|)$, $\tilde{\alpha}_1(s) := \min_{i\in\mathcal{I}} \alpha_1^i(s)$, and $\alpha_2(\|x\|) := \sum_{i=1}^{N} \alpha_2^i(\|x\|)$. Note that $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. The result now follows directly from Theorem II.3. $\square$

In other words, Theorem III.4 states that a set of structured CLFs define a control Lyapunov function for the overall system, which has *additive structure*, i.e. it is defined as a sum of "local" functions $V_i(\cdot)$ (which are not necessarily "local" CLFs).

Remark III.6 As the closed-loop difference inclusion $\phi_{\mathrm{CL}}(\cdot,\cdot)$ might be a discontinuous function of the state, due to discontinuity of $\pi(\cdot)$, it is important (see also Section II-C) to have a continuous CLF for the global system to guarantee inherent robustness. This is achieved if each function $V_i(\cdot)$ in the set of structured CLFs is a continuous function. $\square$

C. Further remarks on candidate structured CLFs

At its core, the notion of structured CLFs and Theorem III.4 is related to the stability theory of interconnected dissipative systems [8]. It is well known that if interconnected systems are dissipative with respect to *neutral* supply rates, which are suitably defined functions of the interconnecting signals, then the interconnected systems are asymptotically stable [8], see also [20]. Passivity and small gain results, see e.g. [21], represent special cases of this general stability result for interconnected systems. To make a relation with these classical results, first note that the inequality (8) can be interpreted as the standard strict dissipation inequality where $V_i(\cdot)$ is the storage function of the (controlled) i-th system, and where $\varrho_i(x_i, v_i(x_{\mathcal{N}_i}))$ represents the supply rate. The condition (9), which is the crucial condition for the stability result of Theorem III.4, can then be interpreted as corresponding to the condition of neutral supplies. However, the important difference is that the condition (9) is defined on a global level, with no reference to the interconnection graph $\mathcal{G}$. This is in contrast to the classical approach where the interconnection graph defines the stabilizing conditions of neutral supply rates, see e.g. [8] and [11] for details. The independence of condition (9) from the *interconnection graph* $\mathcal{G}$ will be instrumental for control synthesis under various information constraints defined by the *communication graph*, as it will be presented in Section IV.

Although computation of CLFs, and therefore of structured CLFs, for general nonlinear systems is an unsolved problem, there are several approaches to tackle this problem. The following lemma presents a possible approach for obtaining a set of structured infinity norm based CLFs for the case when a CLF with a diagonal structure is known for the global system. Notice that this is then a continuous CLF.

Lemma III.7 *Let* $P = \mathrm{diag}(P_1, \ldots, P_N)$*, with* $P_i \in \mathbb{R}^{l_i \times n_i}$*, have full column rank and let* $V(x) := \|Px\|_\infty$ *be a CLF for the overall system* (6)*. Then the set* $\{V_i(x_i)\}_{i\in\mathcal{I}}$ *with* $V_i(x_i) := \|P_i x_i\|_\infty$ *is a set of structured CLFs for system* (6)*.*

The proof of Lemma III.7 follows via somewhat straightforward algebraic manipulations and standard norm inequalities. Lemma III.7 illustrates that finding a control Lyapunov function for the overall system, which is characterized with a specific, block-diagonal structure of the matrix P, can be equivalent to finding a set of structured CLFs. In [22] and, more recently, in [23] techniques for computing infinity norm based CLFs for discrete-time piecewise affine (PWA) systems were presented. With appropriate modifications to include the block-diagonal structure of P, it is therefore possible to use these techniques in combination with the result of Lemma III.7 to obtain candidate structured CLFs for PWA systems, which can approximate general nonlinear systems arbitrarily well.

Finally, it is worth noticing that in many control synthesis approaches for networked systems, global stability of the controlled system is ensured through synthesis of a global Lyapunov function with an additive structure. For example, if system i is controlled by a static state feedback control law and global stability is ensured via standard dissipativity (e.g. small gain or passivity) arguments, the Lyapunov function of the overall system has an additive structure from a set of local positive definite functions. This set of local functions is then a set of structured CLFs in the sense of Definition III.2.

IV. Optimal control problem with information constraints

As it was shown in the previous section, structured CLFs can be used to compute a control action which stabilizes the overall interconnected system (6). More precisely, any feasible solution of Problem III.3 is a stabilizing control input. Furthermore, under certain mild assumptions (e.g., continuity of each $V_i(\cdot)$) even inherent robustness (or more precisely, inherent input-to-state stability) of the interconnected system can be inferred. The interested reader is referred to [17], [24] for details on input-to-state stabilization of discrete-time nonlinear systems.

Notice that to improve closed-loop performance, an additional objective that penalizes the state and input, can be included in Problem III.3. For the remainder of the article we consider the following general optimization problem for the overall interconnected system

$$\min_{\{z_i,\tau_i\}_{i\in\mathcal{I}}} \quad \sum_{i\in\mathcal{I}} J_i(z_i), \tag{16a}$$

$$\text{subject to} \quad z_i \in \mathcal{Z}_i,\ \forall i \in \mathcal{I}, \tag{16b}$$

$$h_i(z_i) \leq \tau_i,\ \ \forall i \in \mathcal{I}, \tag{16c}$$

$$\sum_{i\in\mathcal{I}} \tau_i \leq 0, \tag{16d}$$

where z_i collects all the variables local to the system i and $J_i(z_i)$ is an arbitrary convex cost function for each $i \in \mathcal{I}$. This problem includes Problem III.3, which makes use of structured CLFs to synthesize stabilizing control laws, as a particular case. For example, $z_i := \mathrm{col}(u_i)$ in the case of Problem III.3, but in general other local optimization variables can be added to z_i, as it is the case, for example, if optimized input-to-state stabilization, as defined in [24], is pursued. Naturally, at each time instant $k \in \mathbb{Z}_+$, the sets $\mathcal{Z}_i$ and the functions $h_i(\cdot)$ are in general different as they depend on the current value of the state vector $x(k)$. We are here omitting this dependance for brevity. Finally, if each $J_i(\cdot)$ is a convex function and if $\phi_i(\cdot,\cdot,\cdot)$ in (5) is affine in the control input u_i, then problem (16) is a convex optimization problem. For the remainder of the paper, we assume that (16) is a convex optimization problem.

A. Decentralized control

When the set $\{\tau_i\}_{i\in\mathcal{I}}$ is *a priory* fixed for each time instant $k \in \mathbb{Z}_+$ so that (16d) holds, the optimization problem (16) is separable in $\{z_i\}_{i\in\mathcal{I}}$. Therefore it can be solved by solving N problems independently, with each problem assigned to one local controller. However, if the set $\{\tau_i\}_{i\in\mathcal{I}}$ is *a priory* fixed, then the existence of a set of structured CLFs does not necessarily guarantee feasibility of problem (16).

Note that when the local performance criteria $J_i(\cdot)$ in (16) are replaced by $J_i(\tau_i) = \tau_i$, $\forall i \in \mathcal{I}$, the global coupling constraint (16d) can be omitted and problem (16) becomes separable in $\{z_i,\tau_i\}_{i\in\mathcal{I}}$. In other words, if no additional closed-loop performance is considered, *stabilization via structured CLFs becomes a separable optimization problem, which can readily be implemented in a completely decentralized fashion*.

B. Decentralized control with global coordination

For large-scale networks an appropriate way to achieve global optimal performance is to exploit the "almost separable" structure of problem (16) and to devise a decentralized control structure with *global* coordination among controllers. This can be achieved using the dual decomposition method, see e.g. Chapter 6 in [25], as follows.

By dualizing the coupling constraint (16d) we obtain the dual function $q(\cdot)$ as follows

$$q(\lambda) = \sum_{i\in\mathcal{I}} q_i(\lambda), \tag{17a}$$

$$q_i(\lambda) := \min_{z_i,\tau_i}\{J_i(z_i) + \lambda\tau_i \mid z_i \in \mathcal{Z},\, h_i(z_i) \leq \tau_i\}, \tag{17b}$$

where $\lambda \in \mathbb{R}$ denotes the dual variable (Lagrange multiplier). The corresponding dual problem is now given by

$$\max_{\lambda}\{q(\lambda) \mid \lambda \geq 0\}. \tag{18}$$

Note that for a fixed λ, the dual function (17) is separable in the sense that the minimization problem in (17b) involves only local variables $\{z_i, \tau_i\}$ for each $i \in \mathcal{I}$. Since (16) is a convex optimization problem, under certain mild conditions (the Slater's constraint qualification, see e.g. [26] for details) solutions of the dual problem (18) and the primal problem (16) coincide. For a given $\lambda \geq 0$, let $\{z_i^\star(\lambda), \tau_i^\star(\lambda)\}$, $i \in \mathcal{I}$, denote the corresponding minimizers in (17b). Then $g(\lambda) := \sum_{i\in\mathcal{I}} \tau_i^\star(\lambda)$ is a subgradient of the dual function at λ. For a fixed time instant k the decentralized control with global coordination is achieved using for example an iterative subgradient method, see e.g. [25], where the iterations are made between (*i*) the global coordinator which updates λ based on the knowledge of the subgradient $g(\lambda)$, and (*ii*) local optimization problems (17b) which can be solved, for a fixed λ, in a completely decentralized fashion.

C. Distributed control

To devise a distributed control scheme, suppose that among the N local controllers in the network there exists some *a priori* given communication network. This communication network is defined by specifying existence of communication links among local controllers. If there is a link between the controller at node i and the controller j, then the two controllers can exchange information *in both directions*. Now, with a given communication network we define the *communication matrix* T as follows. Let M be the total number of links in the communication network. Then $T \in \mathbb{R}^{N\times M}$ is a matrix in which the i-th row is associated with the i-th controller and each column is associated with one communication link. Each column $l \in \{1,\ldots,M\}$ has precisely two nonzero elements. Moreover, it has one element with value 1 and one element with value -1, while all the other elements in the column are zero. Suppose that the column l has a nonzero element in the i-th and j-th row. Then this means that the l-th commutation link is a link between the i-th controller and the j-th controller. Note that the communication matrix T uniquely defines the topology of the overall communication network among the controllers. Let $\tau := \mathrm{col}(\tau_1,\ldots,\tau_N) \in \mathbb{R}^N$.

Remark IV.1 For any $\tau \in \mathrm{Im}(T)$ it holds that $\mathbf{1}_N^\top \tau = 0$, since $\mathbf{1}_N^\top T = 0$. □

Let $T^+ \in \mathbb{R}^{N\times M}$ and $T^- \in \mathbb{R}^{N\times M}$ be matrices derived from T in the following way. T^+ is obtained from T by replacing all the elements of T having value 1 with 0, while T^- is obtained from T by replacing all -1 elements of T with 0. Now consider that optimization problem (16) where the global stability related constraint (16d) is replaced by the following equality constraints

$$\tau(k) = T^+ s^+(k) + T^- s^-(k), \qquad (19a)$$
$$s^+(k) = s^-(k), \qquad (19b)$$

and where $s^+ \in \mathbb{R}^M$ and $s^- \in \mathbb{R}^M$ are added as decision variables, in addition to $\{z_i, \tau_i\}_{i\in\mathcal{I}}$. Note that the parametrization of the decision variables τ via (19) ensures that for any feasible point the original constraint (16d) is necessarily satisfied. This is so since $\sum_{i\in\mathcal{I}} \tau_i = \mathbf{1}_N^\top (T^+ + T^-)s = \mathbf{1}_N^\top T s = 0$ (see Remark IV.1), where $s = s^+ = s^-$. The benefit of parametrization (19), and of using this parametrization instated of the global constraint (16d), is that it is *structured* in such a way that it reflects the topology of the communication network among the controllers and can be exploited for distributed computation.

Finally, the distributed algorithm is obtained by dualizing the global consonant (19b), i.e. a dual variable is assigned for each row in (19b). At each time instant k, the corresponding set of dual variables is iteratively updated, e.g. using a subgradient algorithm. This update can be performed in a distributed manner: each dual variable is related to one communication link (a row in (19b)), and all the information to necessary to calculate a subgradient of a dual variable is available to both controllers connected with that link. Then it is sufficient that the dual variable update is performed in one of the adjacent controllers. When the dual variables are fixed, all the remaining constraints are separable and the optimization problem can be performed in a completely decentralized fashion at local controllers.

V. Conclusions

In this paper we have introduced the notion of structured control Lypunov functions (CLFs) and we have developed several structured control algorithms for stabilization and optimal control of interconnected discrete-time nonlinear systems. Based on the notion of structured CLFs we constructed a convex optimization problem such that any of its feasible solutions provides a stabilizing control action for the interconnected system. By including an arbitrary performance criterion, we demonstrated that the resulting problem can be solved under several different information constraints, which include decentralized control, decentralized control with global coordination and distributed control.

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