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Modeling Multi-agent Systems with Hybrid Interacting Dynamics

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Abstract— **We present a new framework for describing multi-agent systems with hybrid interacting dynamics, where the interaction between agents occurs at both the continuous and discrete levels. We define multi-agent systems as Interconnected Hybrid Systems, recast fundamental hybrid concepts such as hybrid execution and reachability in this new interconnected hybrid systems framework, and prove a necessary and sufficient condition for the existence and uniqueness of the interconnected hybrid executions, extending previous work on hybrid systems. We provide conditions on each agent's hybrid model that guarantee the multi-agent system's existence and uniqueness property. Finally, we provide an example that shows how to apply the existence and uniqueness conditions in the design of the agents' dynamics.**

Index Terms— **Interconnected hybrid systems, reachability, execution, existence and uniqueness.**

I. INTRODUCTION

In most of the work reported on cooperative systems, individual models for cooperating agents are described by purely continuous dynamics [2], [4], [5], [8], [9], [12]. There are few exceptions, where discrete event system theory is applied [3]. However new communication network paradigms [6], [10] had motivated the need for studying multi-agent systems where the interaction between agents happen at both continuous and discrete levels.

We envision an Internet in which functions (e.g. routing) are not fixed to physical nodes, but are instead implemented by software agents that are free to migrate from node to node, depending on resources (e.g. connectivity, bandwidth) that they may have to compete for [10]. This approach gives rise to a new type of multi-agent system where agent dynamics are composed by discrete states that represent the location of the agent in the network and its operating mode, and by continuous states that represent the amount of resources that the agent is receiving from the network. The node dynamics are also hybrid. The discrete states represent changes in the agents hosted by the node, while continuous states represent the resource availability in the node. The continuous dynamics of agents and nodes evolve according the agents requirements affecting the availability of resources in the nodes. Agents may also jump to different locations looking for more resources. These jumps affect the continuous evolution of other agents and nodes, and cause discrete jumps at the nodes reflecting the new agent distribution. A pictorial example of this situation is depicted in Figure 1.

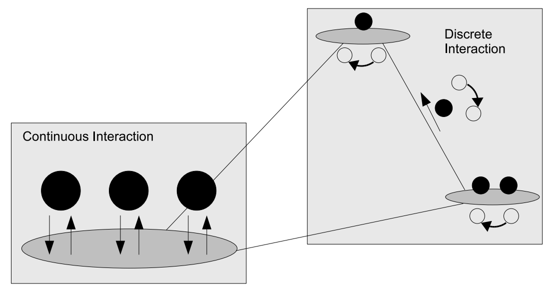

Fig. 1. Example of dynamical behavior of agents and nodes. Agents are modeled as hybrid automata. Each mode in an automaton corresponds to a possible location of an agent in the network. Each transition between modes represents a change of location made by an agent. The dynamics of the nodes are also modeled as hybrid systems. Each discrete mode represents a set of agents residing at a node. The agents on the left are located on a node. Therefore discrete states of agents and nodes are fixed while their continuous dynamics interact. The agent on the right is moving between nodes, so a discrete transition occurs.

It is not clear how to capture the operation of such a system with existing hybrid frameworks. The interactions between the hybrid systems that model agents and nodes happen at both the continuous and discrete levels. The continuous and discrete dynamics of the agents depend on both the continuous and discrete states of the nodes and viceversa. A single hybrid model to study this type of system would result on a centralized model where it would be difficult to decouple individual agent's descriptions. Instead, we attempt to capture this interaction with a new class of systems: the interconnected hybrid systems. Such systems are not mere parallel compositions [13], or products of the component hybrid subsystems. The existence and evolution of an individual subsystem can be meaningless if isolated. Moreover, interactions are not limited to common or uncommon events. In our case, the hybrid state in one of the systems modifies the execution in another one. Therefore we formally define the interconnected hybrid system such that the continuous evolution in one agent depends on the continuous states of agents that are connected to it, and similarly the discrete dynamics depend on continuous and discrete dynamics of neighboring agents. This definition also includes a description of the connectivity of the multi-agent system in each agent's hybrid state. We then recast concepts

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from hybrid systems theory into the new framework, and provide a necessary and sufficient condition for the existence and uniqueness of the interconnected hybrid execution in terms of each agent's hybrid model, extending some of the concepts in [7]. We use an application example to illustrate how to use the existence and uniqueness condition for designing the dynamics of agents locally, while guaranteeing the existence and uniqueness of the execution of the multi-agent system globally. A preliminary version of this work has appeared in [11].

We define the Interconnected Hybrid System in Section II. and the interconnected hybrid execution in Section III. Section IV states the necessary and sufficient conditions for its existence and uniqueness while Section V provides an application example. Section VI outlines our conclusions.

II. Interconnected Hybrid Systems

We consider a multi-agent system with individual hybrid dynamics. The agents in the system interact at both the continuous and discrete level through the continuous control input and the discrete transition guards respectively. We index the agents in the system by the set I. A hybrid system is denoted $\mathbf{H}_i$, for all $i \in I$.

We use the following notation: ν_k denotes dependence of ν on k. ν_{q_k} denotes dependence of ν on q_k which depends on k. ν^n, denotes the n^{th} element of a sequence in ν, $\nu(t)$ denotes the value of ν at time t, and with some abuse of notation, ν_0 marks an initial condition. $\{\nu_k\}_{k\in K}$ denotes a collection of ν_k indexed by the set K. Similarly $(\nu_k)_{k\in K}$ denotes the vector $(\nu_1, \nu_2, \dots, \nu_{|K|})$ indexed by the set K.

Let O_i is the set of operating states and D_i is the set of connectivity states of $\mathbf{H}_i$. Each $o_i \in O_i$ represents a different operating condition of $\mathbf{H}_i$. Each $d_i \in D_i$, represents different connectivity conditions. Let Q_i be the set of discrete states of $\mathbf{H}_i$, such that $Q_i = O_i \times D_i$, where $(o_i, d_i) \in Q_i$ is denoted as q_i. Each q_i has an associated set $V(q_i) \subseteq I \ \forall q_i \in Q_i$, which stores the indexes of the systems that are connected to $\mathbf{H}_i$, i.e., if $j \in V(q_i)$ then $\mathbf{H}_j$ is connected to $\mathbf{H}_i$. Note that $V(q) = V(q')$ for all $q = (o, d), q' = (o', d') \in Q_i$ that satisfy $d = d'$.

Let $\Sigma_i = \{\Sigma_{q_i}\}_{q_i\in Q_i}$ be a collection of continuous dynamical systems Σ_{q_i} indexed by the set Q_i. Each continuous system is a tuple $\Sigma_{q_i} = (X_{q_i}, f_{q_i}, U_{q_i}, \mathbb{R}^+)$ where X_{q_i} is the continuous state space, f_{q_i} the continuous dynamics, U_{q_i} the set of continuous controls, and $\mathbb{R}^+ = [0, \infty)$ the time set. Also let $X_i = \bigcup_{q_i\in Q_i} X_{q_i}$ be the continuous state space over all the discrete states of agent i.

Let $\mathbf{S}_i = \{S_{q_i}\}_{q_i\in Q_i}$ be the set of discrete transition labels of $\mathbf{H}_i$. Symbol $s_{q_i} \in S_{q_i}$ determines the discrete state after a transition from $q_i \in Q_i$ in system $\mathbf{H}_i$. We consider only state based (autonomous) transition in this paper.

Let $\mathbf{G}_i = \{G_{q_i}\}_{q_i\in Q_i}$ be the set of guard conditions for $\mathbf{H}_i$. G_{q_i} is a map that determines when a transition is possible from $q_i \in Q_i$. Let $\mathbf{Z}_i = \{Z_{q_i}\}_{q_i\in Q_i}$ be the set of transition maps of $\mathbf{H}_i$, where $Z_{q_i} : G_{q_i} \times S_{q_i} \to X_i$ determines the continuous state of $\mathbf{H}_i$ after a transition label in S_{q_i} from a hybrid state in G_{q_i}.

The hybrid state space of agent i is $H_i = Q_i \times X_i$, the continuous state space of the agents that are connected to i is $X_{V(q_i)} = \prod_{j\in V(q_i)} X_j$, and the hybrid state space of the agents that are connected to i is $H_{V(q_i)} = \prod_{j\in V(q_i)} H_j$.

Definition 1 (Interconnected Hybrid System) *An* Interconnected Hybrid System *(IHS) is a set $\mathbf{H}^* = \{\mathbf{H}_i\}_{i\in I}$ of Controlled Hybrid Dynamical Systems [1] $\mathbf{H}_i$ indexed by the set I, where $\mathbf{H}_i = [Q_i, \Sigma_i, \mathbf{G}_i, \mathbf{Z}_i, \mathbf{S}_i]$ such that*

- *The continuous control inputs in U_{q_i} are the continuous states of the systems that are connected to $\mathbf{H}_i$. Therefore $U_{q_i} = X_{q_i} \times X_{V(q_i)}$.*
- *A guard condition for a discrete transition of agent i is a function $G_{q_i} : S_{q_i} \to X_{q_i} \times H_{V(q_i)}$. G_{q_i} specifies when a transition is possible as a function of the continuous state of agent i and the hybrid states of agents connected to i. $G_{q_i}(s) = G^L_{q_i}(s) \times G^R_{q_i}(s)$ where $G^L_{q_i}(s) \subseteq X_{q_i}$ denotes the local condition of $G_{q_i}(s)$ i.e. the condition on the continuous state of agent i, and $G^R_{q_i}(s) \subseteq H_{V(q_i)}$ denotes the remote condition of $G_{q_i}(s)$ i.e. the condition on the hybrid states of agent connected to i.*

The discrete state space of the IHS $\mathbf{H}^*$ is $Q_I = \prod_{i\in I} Q_i$, its continuous state space is $X_I = \prod_{i\in I} X_i$, and its hybrid state space is $H_I = \prod_{i\in I} H_i$. The state of the IHS is denoted as $\vec{h} = (\vec{q}, \vec{x}_{\vec{q}})$ where $\vec{q} = (q_i)_{i\in I} \in Q_I$, and $\vec{x}_{\vec{q}} = (x_{q_i})_{i\in I} \in X_I$.

Definition 1 presents a hybrid analog to the standard multi-agent setting [2], [4], [5], [8], [9], [12] where each agent uses the states of its neighbors to update its own evolution. The discrete states of the systems are divided into operating states, used to describe modes of operation of each individual agent in the system, and connectivity states, which describe the possible configurations for information exchange between agents in the system.

Interactions between the continuous dynamics of the agents occur through their continuous control inputs. The continuous control inputs of agent $i \in I$ in the IHS are functions of the continuous states of the agents that are directly connected to agent $i \in I$. Interactions between the discrete dynamics of the agents occur at their discrete transition guards. The transition guards of agent $i \in I$ set conditions on the hybrid states of the agents that are connected to agent $i \in I$. A graphical example of an IHS is shown in Figure 2. The following is an standing assumption for the rest of this paper.

Assumption 1 *The sets of discrete states Q_i are finite for all $i \in I$. The continuous state space $X_i \subseteq \mathbb{R}^d$ for all $i \in I$, where d is an integer. The vector fields $f_{q_i}(x_{q_i}, u_{q_i}, t)$ are globally Lipschitz continuous on both x_{q_i} and u_{q_i} with Lipschitz constants $L^x_{q_i}$ and $L^u_{q_i}$ for all $q_i \in Q_i$ for all $i \in I$.*

III. Interconnected Hybrid Execution

In this section we introduce the Interconnected Hybrid Execution (IHE) based on the concept of hybrid execution

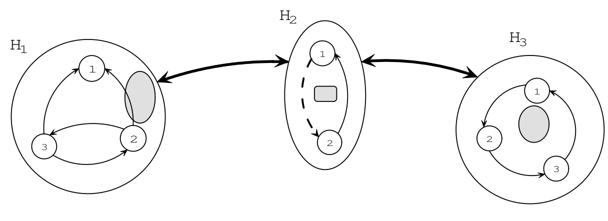

Fig. 2. Graphical representation of an IHS. Three hybrid systems interconnected in line topology with bidirectional connectivity. The area inside ovals and circles represent the continuous state space of the agents. The graphs inside the continuous spaces represent the automata for each system. The gray area within the state space represents the guard for the discrete transition indicated with segmented line in $\mathbf{H}_2$. This transition depends on the continuous states of the agents connected to $\mathbf{H}_2$

in [7]. The IHE is the analog of the state evolution of a continuous multi-agent dynamical system, and captures the system's hybrid behavior over time.

An *Interconnected Hybrid Time Trajectory (IHTT)* is a sequence $\tau = \{\tau^0, \tau^1, \dots, \tau^n, \dots, \tau^N\}$, where τ^n for $n = 0, \dots, N$ is the time at which there is at least one system $\mathbf{H}_i \in \mathbf{H}^*$ that makes a discrete transition from q_i^n to q_i^{n+1}, such that the Interconnected Hybrid System $\mathbf{H}^*$ makes a discrete transition from $\vec{q}^n$ to $\vec{q}^{n+1}$.Two consecutive elements in the IHTT satisfy $\tau^n \leq \tau^{n+1}$ for all $n = 0, 1, \dots, N-1$. τ is infinite if $N = \infty$ and finite otherwise.

The IHTT is used to encode timing information for the continuous and discrete dynamics of the IHS $\mathbf{H}^*$. The IHTT stores the times when at least on one of the agents executes a discrete transition. The IHTT also specifies time intervals between two consecutive elements in the sequence where uninterrupted continuous evolution takes place.

Let the elements of the collection $(\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ be:

- τ is an interconnected hybrid time trajectory.
- $\mathbf{q} = \{\vec{q}^0, \vec{q}^1, \dots, \vec{q}^n, \dots, \vec{q}^N\}$ is a sequence of vectors of discrete locations $\vec{q}^n = (q_i^n)_{i \in I}$ where q_i^n is the discrete mode of system $\mathbf{H}_i$ at the n^{th} step in the execution.
- $\mathbf{s} = \{\vec{s}^0, \vec{s}^1, \dots, \vec{s}^n, \dots, \vec{s}^N\}$ is a sequence of vectors of switching labels $\vec{s}^n = (s^n_{q_i})_{i \in I}$ where $s^n_{q_i}$ is the discrete transition that occurs on system $\mathbf{H}_i$ at n^{th} step in the execution.
- $\mathbf{x} = \{\vec{x}^0, \vec{x}^1, \dots, \vec{x}^n, \dots, \vec{x}^N\}$ is a sequence of continuous evolution $\vec{x}^n = (x_{q_i^n})_{i \in I}$ where $x_{q_i^n}$ is a differentiable map $x_{q_i^n} : [\tau^{n-1}, \tau^n) \to X_{q_i^n}$.
- $\mathbf{u} = \{\vec{u}^0, \vec{u}^1, \dots, \vec{u}^n, \dots, \vec{u}^N\}$ is a sequence of continuous control inputs $\vec{u}^n = \left(u_{q_i^n}\right)_{i \in I}$ where $u_{q_i^n}$ is a continuous map $u_{q_i^n} : [\tau^{n-1}, \tau^n) \to U_{q_i^n}$.

In order to simplify notation, we use the following convention: Unless otherwise noted, if we use an interconnected hybrid state $\vec{h}$, an interconnected discrete state $\vec{q}$, and/or a discrete state q_i in the same sentence/paragraph it implies that the discrete state q_i is a component of an interconnected discrete state $\vec{q}$, which is a component of an interconnected hybrid state $\vec{h}$. We will follow that convention for continuous states, transition labels and continuous inputs as well.

We say that $\vec{h}(t)$ satisfies the discrete transition guard $G_{q_i}(s)$, if the local component of $\vec{h}(t)$ satisfies the local part of $G_{q_i}(s)$ - $x_{q_i}(t) \in G^L_{q_i}(s)$, and the remote component of $\vec{h}(t)$ satisfies the remote part of $G_{q_i}(s)$ - $(h_j)_{j \in V(q_i)}(t) \in G^R_{q_i}(s)$. We say that $\vec{h}(t)$ satisfies interconnected discrete transition guard $G_{\vec{q}}(\vec{s})$, if $\vec{h}(t)$ satisfies $G_{q_i}(s_{q_i})$ for all $i \in I$, where $\vec{q} = (q_i)_{i \in I}$ and $\vec{s} = (s_{q_i})_{i \in I}$.

We say that $\vec{x}_{\vec{q}'}$ is in the interconnected transition map $Z_{\vec{q}}(\vec{h}, \vec{s})$, if each component of $\vec{x}_{\vec{q}'}$ satisfies the transition map of each component of $\vec{s}$, i.e. $x_{q_i'} \in Z_{q_i}(\vec{h}, s_{q_i})$ for all $i \in I$, such that $\vec{q}' = (q_i')_{i \in I}$, $\vec{q} = (q_i)_{i \in I}$, $\vec{x}_{\vec{q}'} = (x_{q_i'})_{i \in I}$, and $\vec{s} = (s_{q_i})_{i \in I}$.

Definition 2 (Interconnected Hybrid Execution) *An* Interconnected Hybrid Execution *(IHE) with initial conditions $\vec{h}_0$ is a collection $\chi(\vec{h}_0) = (\tau, \mathbf{q}, \mathbf{s}, \mathbf{x}, \mathbf{u})$ where:*

- *Initial Condition: $\vec{h}_0 = (\vec{q}^0, \vec{x}^0(0))$ is an initial condition of $\mathbf{H}^*$.*
- *Continuous Dynamics: $\dot{\vec{x}}^n = \vec{f}_{\vec{q}^n}(\vec{x}^n, \vec{u}^n, t)$ for all $t \in [\tau^{n-1}, \tau^n)$, for all $n \in \{1, 2, \dots, N\}$, where $\vec{f}_{\vec{q}^n}(\cdot) = \left(f_{q_i^n}(\cdot)\right)_{i \in I}$ is the vector field of all agents in the IHS $\mathbf{H}^*$.*
- *Discrete Dynamics: The following conditions hold for all $n \in \{0, 1, 2, \dots, N-1\}$:*
  - *The discrete state after a transition $\vec{q}^{n+1}$ is equal to the corresponding discrete transition label $\vec{s}^n$.*
  - *The hybrid state before a transition $\vec{h}^n(\tau^n)$ satisfies the corresponding transition guard $G_{\vec{q}^n}(\vec{s}^n)$.*
  - *The continuous state after a transition $\vec{x}_{\vec{q}^{n+1}}(\tau^{n+1})$ is in the corresponding transition map $Z_{\vec{q}^n}(\vec{h}^n, \vec{s}^n)$.*

The IHE provides the information about the continuous and discrete states and inputs of the system on time. The conditions imposed in Definition 2 are required for the execution to to satisfy the dynamics of $\mathbf{H}^*$. Therefore an IHE starts at a valid initial condition. The continuous evolution between two times in the IHTT satisfies the continuous dynamics of all agent, and the discrete transitions have valid transition guards and maps.

IV. Existence and Uniqueness of the IHE

We provide conditions for the existence and uniqueness of an infinite IHE. The conditions are stated as a function of each agent in the system. Therefore the existence and uniqueness of the IHE can be guaranteed by the specification of local design variables in each agents dynamics.

Let $\chi^S(\vec{h}_0)$ denote the set of all IHEs with initial condition $\vec{h}_0$, and similarly $\chi^F(\vec{h}_0)$ denotes the set of all finite IHEs, $\chi^\infty(\vec{h}_0)$ denotes the set of all infinite IHEs, and $\chi^M(\vec{h}_0)$ denotes the set of all maximal IHEs. Init denotes the set of all initial conditions.

We say that a finite IHE $\chi(\vec{h}_0) \in \chi^F(\vec{h}_0)$ maps $\vec{h}_0$ to $\vec{h}$ if its IHTT $\tau = \{\tau^0, \tau^1, \dots, \tau^N\}$ and $\vec{h} = (\vec{q}^N, \vec{x}^N(\tau^N))$. The interconnected hybrid state $\vec{h}$ is reachable from initial condition $\vec{h}_0$ - denoted $\vec{h} \in Reach(\vec{h}_0)$ if there exists a finite IHE $\chi(\vec{h}_0) \in \chi^F(\vec{h}_0)$ that maps $\vec{h}_0$ to $\vec{h}$. The set of states

$\vec{h}$ that can be reached from any initial condition is called *Interconnected Reachable Set*:

$$Reach_{\mathbf{H}^*} = \bigcup_{\vec{h}_0 \in \text{Init}} Reach(\vec{h}_0)$$

Let $\psi(q_i, t)$ denote the continuous flow of $f_{q_i}(x_{q_i}, u_{q_i}, t)$. We define the *Set of Blocked Continuous Evolution* as the set that specifies what states in the system require a discrete transition for the system to continue its evolution:

$$\begin{aligned} Out_{\mathbf{H}^*} \;=\; & \{\vec{h} \in H_I; \forall \epsilon > 0, \exists t \in [0, \epsilon) \text{ and } \exists i \in I, \\ & \text{s.t. } \psi(q_i, t) \notin X_{q_i}\} \end{aligned}$$

We say that $\mathbf{H}^*$ is deterministic if given $\vec{h}_0$, $\chi^M(\vec{h}_0)$ contains at most one element. The following result provides the necessary and sufficient conditions for existence of an infinite execution assuming the system is deterministic.

Lemma 1 (Deterministic existence) *Suppose $\mathbf{H}^*$ is deterministic. Then given an initial condition $\vec{h}_0$, $\chi^\infty(\vec{h}_0)$ is nonempty (an infinite execution exists) if and only if for all $\vec{h} \in Reach_{\mathbf{H}^*} \bigcap Out_{\mathbf{H}^*}$ there exist a $\vec{s} \in S_{\vec{q}}$ such that $\vec{h}$ satisfies $G_{\vec{q}}(\vec{s})$, where $\vec{q}$ is the discrete state of $\vec{h}$, and $S_{\vec{q}} = \prod_{i \in I} S_{q_i}$ such that $\vec{q} = (q_i)_{i \in I}$.*

Proof: (Sketch) ($\Rightarrow$) Suppose all conditions in Lemma 1 hold except that there is a $\vec{h}$ whose continuous evolution is blocked but does not satisfy any discrete transition guard. Since $\vec{h}$ is reachable there is a finite execution $\chi(\vec{h}_0)$ that maps the initial condition to $\vec{h}$. This finite execution is maximal because it can not be extended through continuous or discrete dynamics. However, the infinite execution assumed to exist is also maximal. Thus the system is not deterministic. Contradiction.

($\Leftarrow$) Suppose all conditions in Lemma 1 hold except that there is a $\vec{h}_0$ for which no infinite execution exists. Then it is possible to find a finite, maximal execution $\chi(\vec{h}_0)$ that maps $\vec{h}_0$ to $\vec{h}$. However Lemma 1 implies that $\chi(\vec{h}_0)$ can be extended either through continuous evolution or by a discrete transition. Then $\chi(\vec{h}_0)$ is not maximal. Contradiction. ■

Note that the conditions in Lemma 1 require that whenever the system gets into an state where continuous evolution is blocked, it is guaranteed that a discrete transition from that state exists. In the following we state the necessary and sufficient conditions for an IHS to be deterministic.

Definition 3 (Forced Transition Condition) *$\vec{h} \in Reach_{\mathbf{H}^*}$ satisfies the Forced Transition (FT) condition if the following condition holds: If there exists a transition label $\vec{s} \in S_{\vec{q}}$ such that $\vec{h}$ satisfies the corresponding transition guard $G_{\vec{q}}(\vec{s})$, then $\vec{h} \in Out_{\mathbf{H}^*}$.*

Definition 4 (Disjoint Transition Guard Condition) *$\vec{h} \in Reach_{\mathbf{H}^*}$ satisfies the Disjoint Transition Guard (DTG) condition if the following condition holds: If there exist two discrete transition labels $\vec{s}, \vec{s}' \in S_{\vec{q}}$ such that $\vec{s} \neq \vec{s}'$, then $\vec{h}$ satisfies at most one of the discrete transition guards $G_{\vec{q}}(\vec{s})$ or $G_{\vec{q}}(\vec{s}')$.*

Definition 5 (Singleton Transition Map Condition) *$\vec{h} \in Reach_{\mathbf{H}^*}$ satisfies the Singleton Transition Map (STM) condition if the following condition holds: If there exists a discrete transition $\vec{s} \in S_{\vec{q}}$ such that $\vec{h}$ satisfies the transition guard $G_{\vec{q}}(\vec{s})$, then the transition map $Z_{\vec{q}}(\vec{h}, \vec{s})$ contains at most one element.*

Lemma 2 (Determinism) *Given an initial condition $\vec{h}_0$, $\chi^M(\vec{h}_0)$ contains at most one element if and only if for all $\vec{h} \in Reach_{\mathbf{H}^*}$ the Forced Transition, the Disjoint Transition Guard, and the Singleton Transition Map conditions hold.*

Proof: (Sketch) ($\Leftarrow$) Suppose there are two different maximal executions $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$ starting at $\vec{h}_0$, but all FT, DTG, and SMT conditions hold. Since $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$ start at the same initial condition, there is a finite execution $\chi(\vec{h}_0)$ that is the maximal prefix of both of them. If $\vec{h}$ is the state obtained from $\chi(\vec{h}_0)$ then either:

1) Both $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$ evolve continuously from $\vec{h}$. The Lipschitz continuous dynamics imply that $\chi(\vec{h}_0)$ can be extended on continuous evolution and still be a prefix of both $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$. Then $\chi(\vec{h}_0)$ is not the maximal prefix of $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$. Contradiction.
2) $\tilde{\chi}(\vec{h}_0)$ evolves continuously from $\vec{h}$, while $\check{\chi}(\vec{h}_0)$ executes a discrete transition (or viceversa). The former implies (Lipschitz) that continuous evolution of $\vec{h}$ is possible. The latter implies (FT condition) that $\vec{h}$ has its continuous evolution blocked. Contradiction.
3) Both $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$ execute a discrete transition from $\vec{h}$. DTG and STM conditions imply that these transitions are the same, and the state of the system after the transition is identical. Then $\chi(h_0)$ can be extended by a discrete transition and still be a prefix of both $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$. Thus $\chi(\vec{h}_0)$ is not the maximal prefix of $\tilde{\chi}(\vec{h}_0)$ and $\check{\chi}(\vec{h}_0)$. Contradiction.

($\Rightarrow$) Suppose there is only one maximal execution but at least one of the FT, DTG, or STM conditions is violated for $\vec{h}$. Since $\vec{h}$ is reachable there is a finite execution $\chi(\vec{h}_0)$ from $\vec{h}_0$ to $\vec{h}$. If the FT condition is violated then both continuous evolution and a discrete transition are possible. Then there are two maximal executions starting at $\vec{h}_0$. Contradiction.

If the DTG condition does not hold, $\chi(\vec{h}_0)$ can be extended on two different discrete transitions creating two different maximal executions starting at $\vec{h}_0$. Contradiction. If the STM condition does not hold, a discrete transition may lead to two different continuous states, then $\chi(\vec{h}_0)$ can be extended into two different continuous evolutions, creating two different maximal executions starting at $\vec{h}_0$. Contradiction. ■

Lemma 2 rules out any possibility of the system taking more than one path at the same time: If a discrete transition is possible then continuous evolution is blocked and *vice versa* (FT condition). If there exist two possible transitions then only one of the corresponding transition guards may be satisfied (DTG condition). And a discrete transition may only have one possible destination point (STM condition). Combining Lemmas 1 and 2 we obtain the following result.

Theorem 1 (Existence and Uniqueness) *Given an initial condition $\vec{h}_0$, $\chi^\infty(\vec{h}_0)$ contains exactly one element if and only if the conditions of Lemmas 1 and 2 hold.*

Theorem 1 states the necessary and sufficient conditions for the existence and uniqueness of an infinite IHE in terms of the global model of the IHS. The result that follows uses these conditions to check existence and uniqueness of the IHE based on the individual agent dynamics. Let the set of blocked continuous evolution of system $i \in I$ be:

$$Out_i = \{\vec{h} \in H_I; \forall \epsilon > 0, \exists t \in [0, \epsilon) \text{ s.t. } \psi(q_i, t) \notin X_{q_i}\}.$$

Corollary 1 (Locally Specified Existence and Uniqueness) *Given an initial condition $\vec{h}_0$, $\chi^\infty(\vec{h}_0)$ contains exactly one element if and only if all the following conditions hold for all agents $i \in I$, and for all $\vec{h} \in Reach_{\mathbf{H}^*}$:*

1) *The interconnected hybrid state $\vec{h} \in Out_i$ if and only if there is a $s \in S_{q_i}$ such that $\vec{h}$ satisfies $G_{q_i}(s)$.*
2) *If there exists two discrete transitions $s, s' \in S_{q_i}$ such that $s \neq s'$ then their corresponding transition guards are disjoint $G_{q_i}(s) \bigcap G_{q_i}(s') = \emptyset$.*
3) *If there is a $s \in S_{q_i}$ such that $\vec{h}$ satisfies $G_{q_i}(s)$ then $Z_{q_i}(\vec{h}, s)$ is a singleton.*

Proof: (Sketch) Condition 1 and Out_i satisfied for all $i \in I$ are equivalent to Lemma 1 and Definition 3 (FT). Condition 2 is equivalent to Definition 4 (DTG). Condition 3 is equivalent to Definition 5 (STM). ■

According to Corollary 1, existence and uniqueness of the IHE follows if every agent is designed such that its discrete transitions are forced, two different discrete transitions are impossible, and a discrete transition maps the continuous state of the agent to a single location.

V. Application Example

We consider an example taken from [10], and use Corollary 1 to design the general structure of the dynamics of agents in the system. This example is motivated by an architecture for the future Internet [6] that addresses several issues the Internet is facing due to its size explosion and the addition of mobile devices to its operation. This architecture abstracts the functional components of the network from the hardware that is used to implement them. This is achieved by enabling software agents that implement these functions (e.g. routing, computing), while considering the hardware as a resource (e.g. connectivity, memory) to be used by agents to implement their functions (Fig. 3). Software agents in the network are capable of requesting resources to the nodes they occupy, as well as migrating between nodes in the network seeking better resources to implement their functions. Hardware nodes distribute their resources according to agents requests, and can host multiple agents at the same time. The network may suffer changes in the resources available in the nodes, and the presence of nodes and links.

Following [10], the dynamics of each software agent and hardware node are described by a hybrid system. I_n denotes the set of hardware nodes, while I_a denotes the set of software agents, and $I = I_n \bigcup I_a$. H_i with $i \in I_n$ denotes the hybrid system that describes node i, while H_k with $k \in I_a$ denotes the hybrid system that describes agent k.

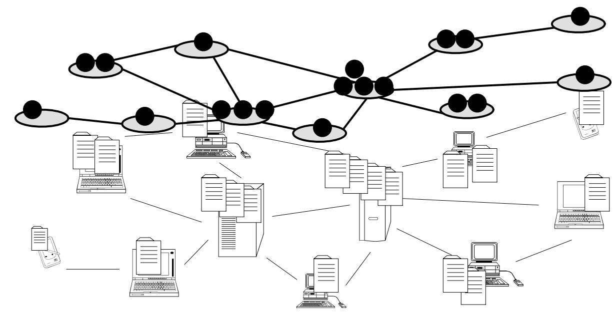

Fig. 3. Network example: Each platform in the network can run several processes concurrently. The network is abstracted as a graph and the processes as agents (black circles) that can move among the nodes.

In this example we consider a special case of the system studied in [10]. The network in this example is fixed. Changes in the network generate event dynamics that are beyond the scope of this paper. The dynamical description of agents and nodes according to [10] is the following:

The set of discrete modes of node i is $Q_i = O_i \times D_i$, where O_i (operating state) is a singleton because we assume a fixed network, and D_i (connectivity state) has one element $d \in D_i$ for each possible combination of agents that may occupy node i. The set of discrete modes of agent k denoted $Q_k = O_k \times D_k$, where O_k is a singleton because agents have a single operating condition, and there is one element $d \in D_k$ for each possible location of agent k in the network. Note that the discrete states of the nodes are affected by the states of the agents and viceversa.

The continuous dynamics of node i and agent k, Σ_i and Σ_k respectively, are designed using resource allocation theory [14]. We do not include a detailed description of Σ_i and Σ_k here because of space constraints. Interested readers are referred to [10]. However we note that the continuous dynamics of agent k have the state of the node it occupies as control input, while the continuous dynamics of node i have the states of all agents located in it as control inputs.

Transitions in the nodes are influenced by the agents' discrete states: Transition labels in node i, $S_{q_i} \in \mathbf{S}_i$ reflect changes in the agents occupying node i. Therefore, if agent k leaves from or arrives to node i, a discrete transition in the node's dynamics $s \in S_{q_i}$ occurs to update the connectivity state $d \in D_i$.

Transition $s \in S_{q_i}$ may be enabled according to the discrete dynamics of the agents: A node's transition guard $G_{q_i}(s)$ is satisfied if the neighborhood of the destination state q_i' is equal to the set of agents that are connected to node i. In this form, node i changes its discrete state to reflect the new set of agents it hosts as soon as an agent arrives/leaves that node. A graphical example of this is shown in the discrete interaction of Figure 1. The discrete transition maps $Z_{q_i}(\vec{h}, s)$ leave the continuous states unchanged for all $\vec{h} \in H_I$ and all $s \in S_{q_i}$.

Transitions in the agents are influenced by the nodes' continuous states: Transition labels in agent k, $S_{q_k} \in \mathbf{S}_k$ reflect the ability of agent k to change its location. Therefore, if agent k is located at node i, and migrates to node i', a discrete transition $s \in S_{q_k}$ occurs to update $d \in D_k$.

The transition guards $G_{q_k} \in \mathbf{G}_k$ are designed such that agent find a node with the best resources available in the network. Assume the discrete state of agent k is q_k, which corresponds to agent k located at node i. Given $s \in S_{q_k}$ let the destination state of transition s be q'_k, which corresponds to agent k located at node i', then $\vec{h}$ satisfies $G_{q_k}(s)$ if

$$\mathrm{mig}(x_{i'}, i') < \mathrm{nmig}(x_i, i) \quad \text{and} \tag{1a}$$

$$\mathrm{mig}(x_{i'}, i') < \mathrm{mig}(x_j, j), \quad \forall j \neq i'; j \in V(q_k) \tag{1b}$$

where $\mathrm{mig}(\alpha, \beta)$ and $\mathrm{nmig}(\alpha, \beta)$ are monotonically increasing functions of α for all $\beta \in I$. Function $\mathrm{mig}(\alpha, \beta)$ measures the benefit of migrating to node β, whose current continuous state is α. Similarly $\mathrm{nmig}(\alpha, \beta)$ measures the benefit of staying at node β, whose current continuous state is α. Therefore, since the states x_i, $x_{i'}$, and x_j are inversely related to the availability of resources in nodes i, i', and j respectively[1], the transition guard that allows agent k to migrate from node i to node i' is satisfied when the benefit of migrating to node i' is better than staying at node i, and when the benefit of migrating to node i' is better than migrating to any other node j that is reachable by agent k in one transition. This is illustrated in Figure 4.

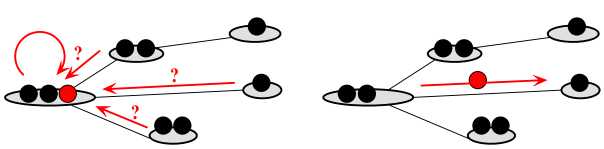

Fig. 4. Agent decision process. The agent compares the benefit of staying in its current location with that of migrating to different nodes in its neighborhood (Arrows with ?-marks; left). Then the agent migrates to (or stays at) the node that offers the best benefit (arrow with agent; right).

Finally the discrete transition map $Z_{q_k}(\vec{h}, s)$ sets the continuous state after the transition $x_{q'_k} = 0$ for all $\vec{h} \in H_I$ and all $s \in S_{q_i}$. This implies that when agent k arrives to a new node, it starts with no resources.

We use Corollary 1 to determine if existence and uniqueness of the IHE is satisfied based on the agents models. To satisfy Condition 1) Out_k is defined as the union of of all the transition guards on agent k, i.e. $Out_k \triangleq \{\vec{h} \in H_I; \vec{h}$ satisfies $G_{q_k}(s), \forall s \in S_{q_k}$ where q_k is a component of $\vec{h}\}$. Then as soon as a transition guard is satisfied, the continuous evolution is blocked and the transition occurs.

Condition 2) is satisfied by the definition of the transition guard G_{q_k}: Assume there is a $\vec{h} \in Reach_{\mathbf{H}^*}$ that satisfies the guards $G_{q_k}(s)$, and $G_{q_k}(s')$ of two different transitions s and s' in S_{q_k}. We denote as c the current node of agent k, and with some abuse of notation we denote as s the node that agent k reaches if it takes transition s and s' if it takes transition s'. Then according to (1b) $\mathrm{mig}(x_s, s) < \mathrm{mig}(x_j, j)$ for all $j \in V(q_k)$ including s', and also $\mathrm{mig}(x_{s'}, s') < \mathrm{mig}(x_j, j)$ for all $j \in V(q_k)$ including s, which implies that $\mathrm{mig}(x_s, s) < \mathrm{mig}(x_{s'}, s')$ and $\mathrm{mig}(x_{s'}, s') < \mathrm{mig}(x_s, s)$. Contradiction.

Finally, condition 3) is satisfied by the definition Z_{q_k} for all $k \in I_a$. A similar analysis can be performed on the hybrid model of the nodes to reach the conclusion that these satisfy all conditions in Corollary 1.

[1]The dynamics of the nodes are designed in [10]. The state of the nodes represent the price the agents pay to access a particular resource. Therefore the state of the nodes is inversely related to the availability of resources.

VI. Conclusion

We present an interconnected hybrid systems framework: a set of hybrid systems with interweaved continuous and discrete dynamics that form a multi-agent system with hybrid interacting dynamics. We extend the work in [7] defining reachable sets and executions for interconnected hybrid systems. We prove necessary and sufficient conditions for the existence and uniqueness of interconnected hybrid executions that are written in terms of the local model of each hybrid agent, and apply these conditions to the problem of designing future communication networks.

References

[1] M. Branicky, V. Borkar, and S. Mitter. A unified framework for hybrid control: Model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, Jan. 1998.

[2] J. Cortez and F. Bullo. Coordination and geometric optimization via distributed dynamical systems. *SIAM Journal on Control and Optimization*, 44(5):1543–1574, 2005.

[3] J. Finke, K. Passino, and A. Sparks. Stable task load balancing strategies for cooperative control of networked autonomoues vehicles. *IEEE Transactions on Control Systems Technology*, 14(5):789–803, September 2006.

[4] V. Gazi and M. Passino. Stability analysis of swarms. *IEEE Transactions on Automatic Control*, 48(4):692–697, Apr. 2003.

[5] A. Jadbabaie, J. Lin, and S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, June 2003.

[6] H. Jerez, C. Abdallah, and J. Khoury. A mobile transient network architecture. Pre-print available at `http://hdl.handle.net/2118/hj_tran_06`, 2006.

[7] J. Lygeros, K. Johansson, S. Simić, J. Zhang, and S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48(1):2–16, Jan. 2003.

[8] L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, Feb. 2005.

[9] R. Olfati-Saber, J. Fax, and R. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, January 2007.

[10] J. Piovesan, C. Abdallah, and H. Tanner. A hybrid framework for resource allocation among multiple agents moving on discrete environments. *Asian Journal of Control, Special Issue on Collective Behavior and Control of Multi-Agent Systems*, 10(2):171–186, March 2008.

[11] J. Piovesan, C. Abdallah, and H. Tanner. Preliminary results on interconnected hybrid systems. In *16th Mediterranean Conference on Control and Automation*, pages 101–106, Ajaccio, France, June 2008.

[12] W. Ren, R. Beard, and E. Atkins. Information consensus in multi-vehicle cooperative control. *IEEE Control Systems Magazine*, 27(2):71–82, April 2007.

[13] M. Shields. *An introduction to automata theory*. Blackwell Scientific Publications, 1987.

[14] R. Srikant. *The Mathematics of Internet Congestion Control*. Birkhauser, 2004.