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A Method to Construct Viability Kernels for Nonlinear Control Systems

John Turriff and Mireille E. Broucke

Abstract— **This paper proposes a method to construct viability kernels for single output nonlinear control systems affine in the control. The safe set is a manifold with boundary and the control is constrained to take values in a compact polyhedron. The results make use of the Frankowska method and the notion of viable capture basins. Three examples illustrate the methodology: the inverted pendulum, a linear system, and a fisheries management problem.**

I. INTRODUCTION

The purpose of this paper is to develop a methodology to construct viability kernels for nonlinear control systems. The central problem can be roughly described as enforcing a control system to evolve in a "safe set" of the state space starting from any initial condition inside the set, by proper assignment of the control input. When no control exists to satisfy this requirement, then the problem is to find a largest subset inside the safe set, called a viability kernel, and an associated controller, called a viability controller, so that the system remains inside the safe set, starting from any initial condition in the viability kernel, using a viability controller. The theory of viability kernels has been developed over the last two decades by J.-P. Aubin and his co-workers [1]. The viability problem is strongly linked to problems of set invariance [16], [10] and control with state constraints. The recent text by Blanchini and Miani [5] provides a comprehensive treatment of methodologies for solving set invariance problems. The notion of a viability kernel is also closely related conceptually to that of controlled invariance. Recently, viability kernels have been recognized to be of theoretical importance in the development of generalizations of nonlinear regulator theory [9]. Also several interesting results have recently appeared on numerical methods to compute viability kernels [3], [6], [7], [11], [12].

Despite substantial progress on viability theory and set invariance in control theory, there remain many open questions. This paper explores how the geometric structure of nonlinear systems can aid in constructing viability kernels. We propose a framework for the problem, a set of reasonable conditions (with respect to applications), and a constructive methodology to build viability kernels for nonlinear systems. In particular, we study the following situation: we have a multi-input, single output nonlinear system affine in the control. The safe set is the superlevel set of a smooth function and geometrically is a manifold with boundary. We want to find the viability kernel associated with the safe set, and a viability controller, assuming that the control is constrained to take values in a compact, convex set. This scenario was first studied in [13] with the aim to solve a viability problem of collision avoidance. However, that theory was only for the smooth case; an assumption which is generally unrealistic. The problem considered here is a special case of the much larger class of problems studied in [2]. However, our focus is on constructing viability kernels for a class of systems, whereas the aim in [2] was to characterize them. Finally, this paper focuses on the basic results and examples; proofs are omitted for brevity and are included in [8].

Department of Electrical and Computer Engineering, University of Toronto, Toronto ON, M5S 3G4, Canada (e-mail: broucke@control.toronto.edu). Supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

The proposed method to find viability kernels has several advantages over traditional numerical methods to find viability kernels: (1) it is exact, whereas numerical methods give only an approximation of the viability kernel; (2) Numerical methods generally do not allow the designer to specify up front the class of control inputs with respect to which the viability kernel is to be found; (3) Computing viability kernels by hand calculations for low-dimensional (say less than five) benchmark examples has significant pedagogical value; (4) Extensions of our theory will allow the development of viability kernels for bang-bang controls and other control classes such as state feedbacks, of clear relevance to control designers.

Notation. Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The complement of the set is $\neg\mathcal{K} := \mathbb{R}^n \setminus \mathcal{K}$, the closure of the set is denoted $\overline{\mathcal{K}}$, and the interior of the set is denoted $\mathcal{K}^\circ$. The *Bouligand contingent cone* or *tangent cone* of $\mathcal{K}$ at a point $x \in \mathcal{K}$, is denoted by $T_{\mathcal{K}}(x)$ [1]. If $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h : \mathbb{R}^n \to \mathbb{R}$, then $L_f h(x) = \frac{\partial h}{\partial x} f(x)$, $L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$, and we define recursively, $L_f^0 h(x) = h(x)$ and $L_f^k h(x) = \frac{\partial \left(L_f^{k-1} h\right)}{\partial x} f(x)$.

II. PROBLEM FORMULATION

Consider the multi-input, single-output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x)\,, \end{aligned} \tag{1}$$

where $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^{n\times m}$ are smooth and Lipschitz, and $h : \mathbb{R}^n \to \mathbb{R}$ is a smooth submersion, i.e. the gradient ∇h is non-vanishing everywhere in $\mathbb{R}^n$. The input space is a compact, convex polyhedron $U \subset \mathbb{R}^m$. A control $u : [0, \infty) \to U$ is a measurable function in t which takes values in U. Let $\phi(t, x_0)$ denote the unique solution of (1) starting at x_0 and using control u. The set of q vertices of U is denoted as

$$V = \{v^1, \ldots, v^q\}\,.$$

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Let $I := \{1, \ldots, q\}$ be the set of indices. A *bang control* is a control that takes a single constant control value in V. A *bang-bang control* is a control that is piecewise constant and takes values in V.

The domain of the state space that we want to render positively invariant by proper choice of control, called the *safe set*, is

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid h(x) \geq 0\} . \quad (2)$$

Assumption 1: There exists $1 < r \leq n$ such that for all $x \in \mathbb{R}^n$ and for all $k < r - 1$, $L_g L_f^k h(x) = 0$.

Remark 2: The assumption says that each component of the row vector $L_g L_f^k h(x) = \frac{\partial \left(L_f^k h\right)}{\partial x} g(x)$ is zero for $k < r - 1$; that is, no input appears before r differentiations of the output. One interpretation is that the system does not have relative degree less than two at any point. The condition arises from a structural property of the system. It is a reasonable assumption for the given problem in the sense that if $L_g h(x) \neq 0$ on some set, then the viability kernel is trivially computable on that set. It is possible to formulate the present problem even if the system had relative degree one on some points, but it does not significantly contribute to the ideas of the paper.

Assumption 1 implies that the derivative of h along solutions of (1) is $\frac{dh(t)}{dt} = L_f h(\phi(t, x_0))$. Thus, we can define the set of states where h is decreasing as

$$\mathcal{W} := \{ \; x \in \mathbb{R}^n \mid L_f h(x) < 0 \; \} .$$

Definition 3: [1], [2] A subset $\mathcal{K}$ is said to be a *viability domain* if for each $x_0 \in \mathcal{K}$, there exists a control $u(t)$ such that the unique solution $\phi_u(t, x_0)$ of (1) stays in $\mathcal{K}$ for all $t \geq 0$. If $\mathcal{K}$ is not a viability domain, then there exists a largest closed (possibly empty) viability domain $Viab(\mathcal{K})$ contained in $\mathcal{K}$, which is called the *viability kernel* of $\mathcal{K}$. A control u which renders $Viab(\mathcal{K})$ viable is called a *viability controller*.

The notion of a viability kernel with target was introduced in [15]. A related notion is that of viable capture basin of a set.

Definition 4: Let $\mathcal{C} \subset \mathcal{K}$. The subset $Capt(\mathcal{K}, \mathcal{C})$, called the *viable capture basin* is the set of initial states $x_0 \in \mathcal{K}$ such that there exists a control $u(t)$ such that the solution of (1) starting at x_0 with control u stays in $\mathcal{K}$ until reaching $\mathcal{C}$ in finite time.

We are interested in finding the viability kernel of the set $\mathcal{K} := \mathcal{S} \cap \overline{\mathcal{W}}$, where $\mathcal{S} \cap \overline{\mathcal{W}}$ is the closed set of states where the system is safe but in danger of reaching an unsafe state. We also impose the practical requirement that the system reach a target set $\mathcal{C} \subset \mathcal{K}$ from the set $\mathcal{K}$ in finite time. This formulation is meaningful if we can guarantee that the system can remain in $\mathcal{S}$ after arriving at $\mathcal{C}$. To do so, we define the sets

$$\begin{aligned} \mathcal{C}^+ \quad &:= \quad \{x \in \mathbb{R}^n \mid h(x) \geq 0, L_f h(x) \geq 0, \ldots, L_f^{r-1} h(x) \geq 0 \; \} \\ \mathcal{C} \quad &:= \quad \mathcal{C}^+ \cap \mathcal{K} . \end{aligned} \quad (3)$$

Assumption 5: For all $x_0 \in \mathcal{C}$, there exists an open-loop control $u_p : \mathbb{R}^+ \to U$ such that $\frac{d^r}{dt^r} h(\phi_{u_p}(t, x_0)) \geq 0$, for all $t \geq 0$.

Remark 6: When Assumption 5 holds we say that $\mathcal{C}^+$ is the *viability core* of $\mathcal{S}$. Its importance is in providing concrete termination conditions for the viability problem, and it is inspired by applications in ecology, biology and robotics, where a viability core often arises. Without such a termination condition the computation of the viability kernel is significantly more complex. Indeed existence of a viability core can be used as a guideline for classifying the difficulty of a given instance of a viability problem. One way in which Assumption 5 can be achieved is by assuming a well-defined uniform relative degree. However, relative degree is too strong, as can be seen in applications [13]. Finally, we note that u_p need not have any significance as a useful control action.

Our viability problem is formally stated as follows.

Problem 1: Given a control affine system (1), the closed set $\mathcal{K} = \mathcal{S} \cap \overline{\mathcal{W}}$, and a target set $\mathcal{C} = \mathcal{C}^+ \cap \mathcal{K}$, find $u^\star$, a viability controller, and $\mathcal{S}^\star := Capt(\mathcal{K}, \mathcal{C})$, the viable capture basin.

III. Viable Capture Basin

In this section we present a construction of the viable capture basin for the set $\mathcal{K}$ with target $\mathcal{C}$. Our construction is centered on bang controls. This is motivated by the fact that, under reasonable conditions, there always exists a subset of $\mathcal{K}$ that can reach $\mathcal{C}$ in finite time via a bang control (for if $\mathcal{C}$ is not reachable by bang control then it is not reachable by bang-bang control). It is also motivated by applications where it is often known that bang controls are the correct controls for a particular domain, without having explicit knowledge of system trajectories.

Consider $x_0 \in \mathbb{R}^n$ and for each $i \in I$, define $\phi_i(t, x_0)$ to be the unique solution of the autonomous system $\dot{x} = f(x) + g(x) v^i$ with initial condition x_0. For $x_0 \in \mathbb{R}^n$, define the *hitting time* $\overline{t}_i(x_0)$ to be the first time when $\phi_i(t, x_0)$ reaches $\mathcal{C}$ before possibly leaving $\mathcal{K}$. If $\phi_i(t, x_0)$ does not reach $\mathcal{C}$ or it leaves $\mathcal{K}$ before reaching $\mathcal{C}$, set $\overline{t}_i(x_0) = \infty$. For $x_0 \in \mathcal{C}$, set $\overline{t}_i(x_0) = 0$. Define the set $\mathcal{X}_i := \{x_0 \in \mathbb{R}^n \mid \overline{t}_i(x_0) < \infty\}$. It can be shown that for each $i \in I$, $\overline{t}_i$ is lower semicontinuous on $\mathcal{X}_i$ [1].

Next, for $x_0 \in \mathbb{R}^n$, we define $\overline{h}_i(x_0)$ to be the value of h at $\overline{t}_i(x_0)$, i.e., $\overline{h}_i(x_0) := h(\phi_i(\overline{t}_i(x_0), x_0))$. If $\overline{t}_i(x_0) = \infty$, set $\overline{h}_i(x_0) := -\infty$. Notice that by construction $\overline{h}_i$ is constant when evaluated along the trajectory $\phi_i(t, x_0)$ over the interval $[0, \overline{t}_i(x_0)]$.

For $x \in \mathcal{K}$, define the set of indices

$$I^\star(x) = \mathrm{argmax}_{i \in I} \{ \; \overline{h}_i(x) \mid \overline{t}_i(x) < \infty \} . \quad (4)$$

Note the cardinality of this set may vary with x. Define the function $\mu^\star : \mathcal{K} \to V$ by $\mu^\star(x) := v^j$, where $j \in I^\star(x)$ is selected arbitrarily. Finally, for each initial condition $x_0 \in \mathcal{K}$ we define

$$u^\star(t, x_0) := \mu^\star(x_0) , \qquad t \in [0, \overline{t}(x_0)] , \quad (5)$$

where $\overline{t}(x_0) := \overline{t}_j(x_0)$ if $\mu^\star(x_0) = v^j$. Intuitively, this choice of controller maximizes the first local minimum value of h on an interval $[0, \overline{t}]$, by using only a single control value in V. The controller $u^\star$ terminates at the time $\overline{t}$ when, by construction, $\dot{h} = 0$ and the target $\mathcal{C}$ is reached.

Define a function $h^\star : \mathbb{R}^n \to \mathbb{R}$ by

$$h^\star(x) = \max_{i \in I}\{ \ \overline{h}_i(x) \ \} \,.$$

Finally, we define

$$\mathcal{S}^\star := \{x \in \mathbb{R}^n \mid h^\star(x) \geq 0\} \,. \tag{6}$$

Assumption 7: $h^\star$ is continuous on $\mathrm{Dom}(h^\star) := \{x \in \mathbb{R}^n \mid \|h^\star(x)\| < \infty\}$ and $\mathcal{S}^\star$ is closed.

Note that $\mathcal{S}^\star \subset \mathcal{K}$, because if $x_0 \notin \mathcal{K}$ then $\overline{t}_i(x_0) = \infty$, $\forall i \in I$. Our aim is to show that $\mathcal{S}^\star$ is the viable capture basin solving Problem 1, and we do so in three steps depending on the class of controls: bang controls, bang-bang controls, and measurable controls. Our main theoretical tool is the following characterization of viable capture basins, adapted from [2].

Theorem 8: Let $\mathcal{K}$ and $\mathcal{C}$ be closed sets such that $\mathcal{C} \subset \mathcal{K}$. The viable capture basin $Capt(\mathcal{K}, \mathcal{C})$ is the unique closed subset $\mathcal{D}$ satisfying $\mathcal{C} \subset \mathcal{D} \subset \mathcal{K}$ and

(i) For each $x_0 \in \mathcal{D}$, there exists a control $u(t)$ such that the trajectory starting at x_0 and using control u reaches $\mathcal{C}$ in finite time without first exiting $\mathcal{D}$.

(ii) $\mathcal{D}$ is backward invariant relative to $\mathcal{K}$. That is, for every $x_0 \in \mathcal{D}$ and every solution $\phi(\cdot, x_0)$, if there exists $T > 0$ such that $\phi(t, x_0) \in \mathcal{K}$ for $t \in [-T, 0]$, then $\phi(t, x_0) \in \mathcal{D}$ for $t \in [-T, 0]$.

Remark 9: Theorem 8 is a version of Frankowska's method [14] which gives a unique characterization of viability kernels and capture basins. We use Theorem 8 in the following way. First we show in Lemma 10 that by construction $u^\star$ satisfies condition (i). Second, we replace condition (ii) by equivalent tangential conditions (see [2]) given by:

$$-\left(f(x) + g(x)u\right) \in T_{\mathcal{D}}(x) \,, \forall x \in \mathcal{D} \cap \mathcal{K}^\circ, \ \forall u \in U \tag{7}$$

$$-\left(f(x) + g(x)u\right) \in T_{\mathcal{D}}(x) \cup T_{\neg\mathcal{K}}(x) \,, \forall x \in \mathcal{D} \cap \partial\mathcal{K}, \ \forall u \in U \,. \tag{8}$$

These are then adapted to obtain our main condition (9) which guarantees backward invariance of $\mathcal{S}^\star$ relative to $\mathcal{K}$. The difference between (7)-(8) and (9) is that (9) is more precise about identifying those controls important in assuring backward invariance, based on $I^\star$. The most important consequence of this is that computationally, (9) is a finite test, where (7)-(8) generally are not.

Lemma 10: Given a system (1), a safe set (2), and a target set (3), suppose that Assumptions 1, 5, and 7 hold. For each $x_0 \in \mathcal{S}^\star$, the trajectory starting at x_0 and using control $u^\star$ reaches $\mathcal{C}$ in finite time without first exiting $\mathcal{S}^\star$.

A. Bang-Bang Controls

In this section we study the special case when only bang or bang-bang controls are allowed. Due to the properties of bang controls and the special structure of $\mathcal{S}^\star$, we have the following straightforward result.

Proposition 11: Given a system (1), a safe set (2), and a target set $\mathcal{C}$, suppose that Assumptions 1, 5, and 7 hold. Then $\mathcal{S}^\star$ is the viable capture basin of $\mathcal{K}$ with target $\mathcal{C}$ under the restriction of bang controls, and $u^\star$ is a viability controller.

It is interesting to consider the differences between the previous result, which requires no additional assumptions on $\mathcal{S}^\star$, and Frankowska's method. To apply the Frankowska method, we must show that conditions (i)-(ii) of Theorem 8 hold. Condition (i) holds by Lemma 10. The following example shows that, instead, condition (ii) does not generally hold even though $\mathcal{S}^\star$ is the viable capture basin using bang controls.

Example 12: Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -1.5x_1^2 x_2 + (1 + 1.5x_1^2 x_2)u \,. \end{aligned}$$

with $U := [-1, 1]$. Let $h(x) = x_1$, so $\mathcal{S} = \{ \ x \in \mathbb{R}^2 \mid x_1 \geq 0\}$ and $\mathcal{W} = \{ \ x \in \mathbb{R}^2 \mid x_2 < 0 \ \}$. The target set is $\mathcal{C} = \{ \ x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 = 0 \ \}$. Now it can be seen that $\overline{t}_1(x_0) = \infty$ for $x_0 \in \mathcal{S} \cap \mathcal{W}$ because the vector field corresponding to v^1 is $(0, -1)$ along $\mathcal{C}$ so trajectories cannot reach $\mathcal{C}$ from $\mathcal{S} \cap \mathcal{W}$. Also, it is easily verified that for all $x_0 \in \mathcal{S} \cap \overline{\mathcal{W}}$, $\overline{t}_2(x_0) = |x_2(0)|$ and $\overline{h}_2(x_0) := x_1(0) - \frac{1}{2}x_2^2(0)$. Therefore,

$$\mathcal{S}^\star = \{ \ x \in \mathbb{R}^2 \mid x_1 \geq \frac{1}{2}x_2^2, x_2 \leq 0\}$$

and a viability controller is $u^\star = 1$. Let $p(x) := (-1, x_2)$ be the outward normal vector of $\mathcal{S}^\star$ at $x \in \partial\mathcal{S}^\star \cap \mathcal{W}$. Then, $T_{\mathcal{S}^\star}(x) = \{ \ v \in \mathbb{R}^2 \mid \langle v, p(x) \rangle \leq 0\}$. Now consider the point $\tilde{x} := (1, -\sqrt{2}) \in \partial\mathcal{S}^\star \cap \mathcal{W}$. The vector field with control $v^1 = -1$ evaluated at $\tilde{x}$ is $(-\sqrt{2}, 3\sqrt{2} - 1)$. Now we verify (9) at $\tilde{x}$. We have $\begin{bmatrix} \sqrt{2} & -3\sqrt{2} + 1 \end{bmatrix} \begin{bmatrix} -1 \\ -\sqrt{2} \end{bmatrix} = -2\sqrt{2} + 6 > 0$. Therefore we have a situation in which Frankowska's second condition is violated, even though $\mathcal{S}^\star$ is the viable capture basin under the restriction of bang controls.

Let $\partial\mathcal{W} \cap \mathcal{S}$ be partitioned as the disjoint union $\partial\mathcal{W} \cap \mathcal{S} = \partial\mathcal{W}_{1e} \cup \mathcal{W}_{1o} \cup \partial\mathcal{W}_2 \cup \mathcal{C}$ where

$$\begin{aligned} \partial\mathcal{W}_{1e} &= \Big\{x \in \partial\mathcal{W} \cap \mathcal{S} \cap \neg\mathcal{C} \mid (\exists\ 2 \leq k(x) \leq r - 1, k(x) \text{ even}) \\ &\qquad L_f h(x) = \cdots = L_f^{(k(x)-1)} h(x) = 0, L_f^{k(x)} h(x) < 0\Big\} \\ \partial\mathcal{W}_{1o} &= \Big\{x \in \partial\mathcal{W} \cap \mathcal{S} \cap \neg\mathcal{C} \mid (\exists\ 2 < k(x) \leq r - 1, k(x) \text{ odd}) \\ &\qquad L_f h(x) = \cdots = L_f^{(k(x)-1)} h(x) = 0, L_f^{k(x)} h(x) < 0\Big\} \\ \partial\mathcal{W}_2 &= \{x \in \partial\mathcal{W} \cap \mathcal{S} \cap \neg\mathcal{C} \mid (\exists\ 2 \leq k(x) \leq r - 2) \\ &\qquad L_f h(x) = \cdots = L_f^{(k(x)-1)} h(x) = 0, L_f^{k(x)} h(x) > 0\Big\} \,. \end{aligned}$$

Note that for $r = 2$, $\partial\mathcal{W}_{1e} = \partial\mathcal{W}_{1o} = \partial\mathcal{W}_2 = \emptyset$, and for $r = 3$, $\partial\mathcal{W}_{1o} = \partial\mathcal{W}_2 = \emptyset$.

Lemma 13: $\mathcal{S}^\star \cap \partial\mathcal{S} \cap \mathcal{W} = \emptyset$.

Lemma 14: Trajectories arrive at $\mathcal{S} \cap \partial\mathcal{W}_{1e}$ only from $\neg\mathcal{K}$.

Lemma 15: $\mathcal{S}^\star \cap \partial\mathcal{W}_2 = \emptyset$.

Remark 16: Lemma 13 and 15 show that, moreover, for all $x_0 \in (\partial\mathcal{S} \cap \mathcal{W}) \cup \partial\mathcal{W}_2$ and for all trajectories $\phi_u(t, x_0)$, there exists $\delta > 0$ such that $\phi_u(t, x_0) \in \neg\mathcal{K}$, $\forall t \in (0, \delta)$.

Theorem 17: Given a system (1), a safe set (2), and a target set (3), suppose that Assumptions 1, 5, and 7 hold. In addition, suppose that for all $x \in \partial\mathcal{S}^\star \cap \neg\mathcal{C}$ and for all $j \notin I^\star(x)$,

$$-(f(x) + g(x)v^j) \in T_{\mathcal{S}^\star}(x)\,. \tag{9}$$

Then $\mathcal{S}^\star$ is the viable capture basin of $\mathcal{K}$ with target $\mathcal{C}$ under the restriction of bang-bang controls, and $u^\star$ is a viability controller.

This result means that the Frankowska method can be used to distinguish when a viable capture basin can be constructed using only bang controls, even if bang-bang controls are permitted. For instance, suppose we find the viable capture basin $\mathcal{K}^\star$ and therefore condition (9) holds on $\mathcal{K}^\star$, but (our) $u^\star$ is not a viability controller. A candidate viability controller is instead a bang-bang control with possible switching points. The key observation is that $\mathcal{K}^\star \neq \mathcal{S}^\star$, and that condition (9) fails for $\mathcal{S}^\star$. We can summarize by saying that, if $\mathcal{K}^\star$ is the viable capture basin of $\mathcal{K}$ with target $\mathcal{C}$, $v^\star$ is a bang-bang viability controller, and $\mathcal{S}^\star \neq \mathcal{K}^\star$, then (9) fails on at least one point of $\mathcal{S}^\star$ and $v^\star$ is not a bang viability controller.

B. Measurable Controls

In this section we extend the previous results to show that $\mathcal{S}^\star$ is the viable capture basin even when measurable controls are permitted. We would like to retain the finite test in (9). From applications it is observed that the form of $h^\star$ is typically independent of the level value of h which determines the safe set. Similiarly, (9) typically can be verified independently of the level value of $h^\star$ which determines the viability kernel. These observations lead to a suitable notion of robustness of viability kernels (which are inherently fragile): if the level value of h defining the safe set is perturbed by a sufficiently small value, then the new viability kernel is determined by a perturbed level value of $h^\star$.

Let $\epsilon \in \mathbb{R}$ and define the sets $\mathcal{S}_\epsilon := \{x \in \mathbb{R}^n \mid h(x) \geq \epsilon\}$, $\mathcal{K}_\epsilon := \mathcal{S}_\epsilon \cap \overline{\mathcal{W}}$, $\mathcal{C}_\epsilon^+ := \left\{x \in \mathbb{R}^n \mid h(x) \geq \epsilon, L_f h(x) \geq 0, \ldots, L_f^{r-1}h(x) \geq 0\ \right\}$, and $\mathcal{C}_\epsilon := \mathcal{C}^+ \cap \mathcal{K}_\epsilon$. For $x_0 \in \mathbb{R}^n$ and $i \in I$, define $\overline{t}_i^\epsilon(x_0)$ to be the first time when $\phi_i(t, x_0)$ reaches $\mathcal{C}_\epsilon$ before possibly leaving $\mathcal{K}_\epsilon$. If $\phi_i(t, x_0)$ does not reach $\mathcal{C}_\epsilon$ or it leaves $\mathcal{K}_\epsilon$ before reaching $\mathcal{C}_\epsilon$, set $\overline{t}_i^\epsilon(x_0) = \infty$. For $x_0 \in \mathcal{C}_\epsilon$, set $\overline{t}_i^\epsilon(x_0) = 0$. Also, define $\overline{h}_i^\epsilon(x_0) := h(\phi_i(\overline{t}_i^\epsilon(x_0), x_0))$. If $\overline{t}_i^\epsilon(x_0) = \infty$, set $\overline{h}_i^\epsilon(x_0) := -\infty$. Finally, for $x \in \mathcal{K}_\epsilon$, define $h_\epsilon^\star(x) := \max_{i \in I}\{\ \overline{h}_i^\epsilon(x)\ \}$.

Remark 18: It is easy to show that for all $\epsilon \leq 0$ and for all $x_0 \in \mathcal{S}^\star \cup \mathcal{S}_0^\star$, $h^\star(x_0) = h_\epsilon^\star(x_0)$.
Given $\epsilon < 0$, define $\mathcal{N}_\epsilon := \{x \in \mathbb{R}^n \mid \epsilon < h_\epsilon^\star(x) < 0\}$. Also for each $\delta \in [\epsilon, 0]$, define $\mathcal{S}_\delta^\star := \{x \in \mathbb{R}^n \mid h_\epsilon^\star(x) \geq \delta\}$.

Assumption 19: There exists $\epsilon < 0$ such that $h_\epsilon^\star$ is continuous on $\mathrm{Dom}(h_\epsilon^\star) := \{x \in \mathbb{R}^n \mid \|h_\epsilon^\star(x)\| < \infty\}$. For each $\delta \in [\epsilon, 0]$, $\mathcal{S}_\delta^\star$ is closed and $\partial\mathcal{S}_\delta^\star \cap \mathcal{W} = \{x \in \mathcal{W} \mid h_\epsilon^\star(x) = \delta\}$.

Theorem 20: Given a system (1), a safe set (2), and a target set (3), suppose that Assumptions 1 and 5 hold. In addition, suppose there exists $\epsilon < 0$ such that Assumption 19 holds; for all $x \in \mathcal{N}_\epsilon \cap \neg\mathcal{C}_\epsilon$ and for all $j \notin I_\epsilon^\star(x)$, $-(f(x) + g(x)v^j) \in T_{\mathcal{S}_\epsilon^\star}(x)$; and for all $x \in \partial\mathcal{W}_{1o} \cap \mathcal{C}$ and $u \in U$, $-(f(x) + g(x)u) \in T_{\mathcal{S}^\star}(x) \cup T_{\neg\mathcal{K}}(x)$. Then $\mathcal{S}^\star$ is the viable capture basin of $\mathcal{K}$ with target $\mathcal{C}$, and $u^\star$ is a viability controller.

IV. Examples

Example 21: We illustrate the steps of the design for a second-order model of the pendulum on a cart assuming the cart mass is neglible with respect to the pendulum mass and all parameters are set to 1. If x_1 is the position of the pendulum from the upright vertical and x_2 is its angular velocity, then the model is:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - u\cos x_1\,,\end{aligned}$$

where $x \in \mathbb{R}^2$ and $U := [-1, 1] \subset \mathbb{R}$. We assume that the pendulum angle x_1 is unwrapped, meaning that we distinguish between angles differing by multiples of 2π. Let $v^1 = -1$ and $v^2 = 1$. The viability problem is to keep the pendulum in a region about the upright (unstable) equilibrium such that $x_1 \in [-c, c]$ where $c > 0$. To simplify the computations, we assume $c < \frac{\pi}{4}$. We choose $h(x) = c^2 - x_1^2$ so that $\mathcal{S} = \{x \in \mathbb{R}^2 \mid c^2 - x_1^2 \geq 0\}$. (Note this choice of h is consistent with the convention that angles are unwrapped.) Assumption 1 holds with $r = 2$ so $\mathcal{W} = \{\ x \in \mathbb{R}^2 \mid x_1 x_2 > 0\ \}$ and $\mathcal{C} = \{x \mid c^2 - x_1^2 \geq 0, x_1 x_2 = 0\}$. It is easily verified that Assumption 5 holds.

It can be determined that for $u = \pm 1$, the set of initial conditions in $\mathcal{S} \cap \overline{\mathcal{W}}$ that can reach $\mathcal{C}$ in finite time are:

$$\mathcal{X}_1 = \{x \in \mathcal{S} \cap \overline{\mathcal{W}} : |x_2| < \sqrt{2\sqrt{2} + 2\sin x_1 - 2\cos x_1}\},$$
$$\mathcal{X}_2 = \{x \in \mathcal{S} \cap \overline{\mathcal{W}} : |x_2| < \sqrt{2\sqrt{2} - 2\sin x_1 - 2\cos x_1}\}.$$

To obtain formulas for $\overline{h}_i$, we note that for constant values of u the system admits a first integral $\frac{1}{2}x_2^2 + \cos x_1 + u\sin x_1 = a$, where $a \in \mathbb{R}$ is determined by the initial condition $(x_1(0), x_2(0))$. We set $a = \frac{1}{2}x_2^2(0) + \cos x_1(0) + u\sin x_1(0)$. Second, solve for $x_1(\overline{t})$ at the first time the trajectory reaches $\partial\mathcal{W}$. (Because $\arccos(\cdot)$ and $\arcsin(\cdot)$ appear in this step, care must be taken so that the range of x_1 allows to use the principle values $\mathrm{Arccos}(\cdot)$ and $\mathrm{Arcsin}(\cdot)$). Finally, the expressions for x_1 are substituted in h to yield $\overline{h}_i$. For $x \in \mathcal{X}_1 \cup \mathcal{X}_2$,

$$h^\star(x) = c^2 - \left[\frac{\pi}{4} + \mathrm{Arcsin}\left(-\frac{1}{\sqrt{2}}x_2^2(0) - \frac{1}{\sqrt{2}}\cos x_1(0) - \frac{1}{\sqrt{2}}\left|\sin x_1(0)\right|\right)\right]^2.$$

The final step of the design is to verify condition (9). Since the computations are symmetric we only consider the region $x_1 \in [-c, 0]$, where the boundary of the viable capture basin is given by $\overline{h}_1(x) = 0$. Since $\overline{h}_1$ is differentiable, condition (9) reduces to verifying that for all $x \in \partial\mathcal{S}^\star \cap \neg\mathcal{C}$ and $x_1 \in [-c, 0]$, $\nabla\overline{h}_1(x) \cdot (f(x) + g(x)v^2) \leq 0$. We obtain $\nabla\overline{h}_1(x) \cdot (f(x) + g(x)v^2) =$

$$\frac{2cx_2}{\sqrt{1 - \left(\sin\left(x_1 - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}x_2^2\right)^2}}\left[\cos\left(x_1 - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\left(\sin x_1 - \cos x_1\right)\right].$$

Now for $x_1 \in [-c, 0]$ with $c \in \left(0, \frac{\pi}{4}\right)$, we have that $\cos\left(x_1 - \frac{\pi}{4}\right) \geq 0$ and $\sin x_1 - \cos x_1 \leq 0$; therefore the last term is positive. However, since $x_2 \leq 0$ for this region of the boundary of $\partial\mathcal{S}^\star$, we obtain the desired result.

Example 22: We consider a fourth-order single output linear system given by:

$$\dot{x} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} u , \quad (10)$$

$$y = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} x . \quad (11)$$

Define $U := [-1, 1] \subset \mathbb{R}$ and let $v^1 = -1$ and $v^2 = 1$. The viability problem is to maintain $h(x) := y - c \geq 0$, where $c \in \mathbb{R}$ is a given constant. Then $\mathcal{S} = \left\{x \in \mathbb{R}^3 \mid y - c \geq 0\right\}$ and since Assumption 1 is satisfied we compute $\mathcal{W} = \left\{ x \in \mathbb{R}^3 \mid x_2 < 0 \right\}$. The first step is to compute $\overline{t}_i$ and for this we first compute the system trajectories for constant u. We obtain $x_1(t) = \frac{1}{24}ut^4 + \frac{1}{6}(x_{20} + x_{40} + u)t^3 + \frac{1}{2}(2x_{20} + x_{40})t^2 + (x_{10} - x_{30} - x_{40})t$, $x_2(t) = \frac{1}{2}ut^2 + (x_{40} + x_{20})t + x_{20}$, $x_3(t) = \frac{1}{24}ut^4 + \frac{1}{6}(x_{40} + x_{20})t^3 + \frac{1}{2}x_{20}t^2 + (x_{10} - x_{20} - x_{30} - x_{40} - u)t + x_{30}$, $x_4(t) = -\frac{1}{2}ut^2 + (u - x_{40} - x_{20})t + x_{40}$. To solve for $\overline{t}_i$, we set $x_2(t) = 0$ and solve for t. For $x_0 \in \mathcal{S} \cap \overline{\mathcal{W}}$ this yields

$$\overline{t}_1(x_0) = \begin{cases} x_{20} + x_{40} - \sqrt{(x_{40} + x_{20})^2 + 2x_{20}} \,, \\ \qquad \text{if } x_{40} + x_{20} \geq 0,\ (x_{40} + x_{20})^2 + 2x_{20} \geq 0 \\ -\infty \,, \\ \qquad \text{otherwise} \,. \end{cases} \quad (12)$$

$$\overline{t}_2 = -x_{20} - x_{40} + \sqrt{(x_{40} + x_{20})^2 - 2x_{20}} \,. \quad (13)$$

The analysis shows that for $u = \pm 1$, the set of initial conditions in $\mathcal{S} \cap \overline{\mathcal{W}}$ that can reach $\partial\overline{\mathcal{W}}$ in finite time are:

$$\mathcal{X}_1 = \left\{ x \in \mathcal{S} \cap \overline{\mathcal{W}} \;\middle|\; x_4 + x_2 \geq 0, (x_4 + x_2)^2 + 2x_2 \geq 0 \right\} .$$
$$\mathcal{X}_2 = \mathcal{S} \cap \overline{\mathcal{W}} .$$

We observe that Assumptions 5 and 7 are satisfied on $\mathcal{K}$. Next we want to compute $\overline{h}_i(x) = x_1(\overline{t}_i) - x_2(\overline{t}_i) - x_3(\overline{t}_i) - x_4(\overline{t}_i) - c$. We obtain

$$\overline{h}_1(x) = x_1 - x_2 - x_3 - x_4 + x_2(x_4 + x_2) + \frac{1}{3}(x_2 + x_4)^3 - \frac{1}{3}(x_2^2 + 2x_2x_4 + x_4^2 + 2x_2)^{\frac{3}{2}} - c \,.$$

$$\overline{h}_2(x) = x_1 - x_2 - x_3 - x_4 - x_2(x_4 + x_2) + \frac{1}{3}(x_2 + x_4)^3 - \frac{1}{3}(x_2^2 + 2x_2x_4 + x_4^2 - 2x_2)^{\frac{3}{2}} - c \,.$$

The next step is to compute $h^\star(x) = \max\{\overline{h}_1(x), \overline{h}_2(x)\}$ for all $x \in \mathcal{S} \cap \overline{\mathcal{W}}$. On $\mathcal{X}_2 \setminus \mathcal{X}_1$, $h^\star(x) = \overline{h}_2(x)$. On $\mathcal{X}_1 \cap \mathcal{X}_2$ we must calculate which is larger. Let $c_k := \binom{\frac{3}{2}}{k}$. Skipping some algebraic steps, we obtain

$$\overline{h}_2(x) - \overline{h}_1(x) = -2x_2^2 - 2x_2x_4 - \frac{1}{3}(x_2^2 + 2x_2x_4 + x_4^2 - 2x_2)^{\frac{3}{2}} + \frac{1}{3}(x_2^2 + 2x_2x_4 + x_4^2 + 2x_2)^{\frac{3}{2}} = \frac{16}{3}c_3(x_2 + x_3)^{-3}x_2^3 + \frac{32}{3}c_5(x_2 + x_3)^{-7}x_2^5 + \frac{64}{3}c_7(x_2 + x_3)^{-9}x_2^7 + \cdots .$$

Now we know that on $\mathcal{X}_1 \cap \mathcal{X}_2$, $x_2 \leq 0$ and also $c_k < 0$ for $k = 3, 5, 7, \ldots$. Therefore every term in the sum above is positive. Thus we obtain that $h^\star(x) = \overline{h}_2(x)$ and $u^\star = +1$ for all $x \in \mathcal{S} \cap \overline{\mathcal{W}}$.

The final step of the design is to verify condition (9). For all $x \in \partial\mathcal{S}^\star \cap \mathcal{W}$, we have that $I^\star(x) = \{2\}$. Therefore, for all $x \in \mathcal{S} \cap \overline{\mathcal{W}}$, the boundary of the viable capture basin is given by $\overline{h}_2(x) = 0$ and since $\overline{h}_2$ is differentiable, condition (9) reduces to verifying that for all $x \in \partial\mathcal{S}^\star \cap \mathcal{W}$, $\nabla\overline{h}_2(x) \cdot (f(x) + g(x)v^1) \leq 0$. We obtain

$$\nabla\overline{h}_2(x) \cdot \big(f(x) + g(x)v^1\big) = 2\left[x_2 - (x_2 + x_4)^2 + (x_2 + x_4)((x_2 + x_4)^2 - 2x_2)^{\frac{1}{2}}\right] .$$

Now we observe that

$$\begin{aligned} 0 &\leq (x_2 + x_4)[(x_2 + x_4)^2 - 2x_2]^{\frac{1}{2}} \\ &= \left[(x_2 + x_4)^4 - 2x_2(x_2 + x_4)^2\right]^{\frac{1}{2}} \\ &\leq \left[(x_2 + x_4)^4 - 2x_2(x_2 + x_4)^2 + x_2^2\right]^{\frac{1}{2}} \\ &= -(x_2 - (x_2 + x_4)^2) . \end{aligned}$$

Therefore condition (9) is satisfied.

Example 23: We consider an example of fisheries management adapted from [3], which models the effect of fishing activity on a prey-predator system. Let x_1 denote the population level of a prey species, let x_2 denote the population level of a predator species and let x_3 denote the effort expended by humans in fishing the predator species. We assume that in the absence of any predation, the prey population follows an exponential growth model with intrinsic growth rate $r_1 > 0$. Similarly, in the absence of any fishing activity, the predator population follows an exponential growth model with intrinsic growth rate $r_2 > 0$. We do not assume any carrying capacity limitations on either the prey or predator populations. The system model is given by

$$\begin{aligned} \dot{x}_1 &= (r_1 - x_2)x_1 \\ \dot{x}_2 &= (r_2 - x_3)x_2 \\ \dot{x}_3 &= u \end{aligned}$$

where $x \in \mathbb{R}^3$ and $U := [-1, 1] \subset \mathbb{R}$.

Let $v^1 = -1$ and $v^2 = 1$. The viability problem is to keep the stock level of the prey above some positive level $c > 0$. We define $h(x) = x_1 - c$, so $L_f h(x) = (r_1 - x_2)x_1$ and $\mathcal{S} = \left\{x \in \mathbb{R}^3 \mid x_1 - c \geq 0\right\}$. Assumption 1 holds with $r = 3$, so $\mathcal{W} = \left\{ x \in \mathbb{R}^3 \mid (r_1 - x_2)x_1 < 0 \right\}$. If $x_0 \in \mathcal{S} \cap \mathcal{W}$, then $x_1(0) \geq c > 0$ and $x_2(0) > r_1 > 0$. Thus, we compute $\mathcal{C}^+ = \{x \;:\; x_1 \geq c, x_2 \leq r_1, (r_1 - x_2)^2 x_1 - (r_2 - x_3)x_1x_2 \geq 0\}$ and $\mathcal{C} = \{x \;:\; x_1 \geq c, x_2 = r_1, x_3 \geq r_2\}$. Using the expression for $\mathcal{C}$ it can be easily verified that Assumption 5 holds with $u_p = 1$.

Define the functions $m_1(t) := \int_0^t e^{(r_2 - x_3(0))\tau + \frac{1}{2}\tau^2}\, d\tau$ and $m_2(t) := \int_0^t e^{(r_2 - x_3(0))\tau - \frac{1}{2}\tau^2}\, d\tau$. Note that these are expressible in terms of the error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. For constant values of u we have that

$$x_1(t) = \begin{cases} x_1(0) e^{r_1 t - x_2(0) m_1(t)}, & \text{if } u = v^1 \\ x_1(0) e^{r_1 t - x_2(0) m_2(t)}, & \text{if } u = v^2 . \end{cases} \tag{14}$$

$$x_2(t) = x_2(0) e^{(r_2 - x_3(0))t - \frac{1}{2} u t^2} \tag{15}$$

$$x_3(t) = ut + x_3(0) . \tag{16}$$

To compute $\bar{t}_i$, we remark that for $u = \pm 1$ the set $\{x \in \mathbb{R}^3 \mid x_1 = 0\}$ is an asymptote of the system and hence the $x_1 = 0$ component of $\partial \overline{\mathcal{W}}$ cannot be reached in finite time. Therefore, we must consider (15) to determine if there exists a time $\bar{t}_i$ such that $x_2(\bar{t}_i) = r_1$. Substituting $x_2(\bar{t}_i) = r_1$ in (15) and solving for $\bar{t}_i$ we get

$$\bar{t}_1 = -(r_2 - x_3(0)) - \sqrt{(r_2 - x_3(0))^2 + 2 \ln \frac{r_1}{x_2(0)}} . \tag{17}$$

$$\bar{t}_2 = (r_2 - x_3(0)) + \sqrt{(r_2 - x_3(0))^2 - 2 \ln \frac{r_1}{x_2(0)}} . \tag{18}$$

The analysis shows that for $u = \pm 1$, the set of initial conditions in $\mathcal{S} \cap \overline{\mathcal{W}}$ that can reach $\mathcal{C}$ in finite time are:

$$\mathcal{X}_1 = \left\{ x \in \mathcal{S} \cap \overline{\mathcal{W}} \;\middle|\; x_3 \geq r_2 + \sqrt{-2 \ln \frac{r_1}{x_2}} \right\} .$$
$$\mathcal{X}_2 = \mathcal{S} \cap \overline{\mathcal{W}} .$$

Finally, substituting (17) and (18) into the expression for h we get

$$\overline{h}_1(x_0) = x_1(0) e^{r_1 \bar{t}_1 - x_2(0) m_1(\bar{t}_1)} - c ,$$
$$\overline{h}_2(x_0) = x_1(0) e^{r_1 \bar{t}_2 - x_2(0) m_2(\bar{t}_2)} - c .$$

It can be shown that $h^\star(x) = \overline{h}_2(x)$ for all $x \in \mathcal{S} \cap \overline{\mathcal{W}}$; therefore, $u^\star = 1$. The final step of the design is to verify condition (9). For all $x \in \partial \mathcal{S}^\star \cap \mathcal{W}$, we have that $I^\star(x) = \{2\}$. Therefore, for all $x \in \mathcal{S} \cap \overline{\mathcal{W}}$, the boundary of the viable capture basin is given by $\overline{h}_2(x) = 0$ and since $\overline{h}_2$ is differentiable, condition (9) reduces to verifying that for all $x \in \partial \mathcal{S}^\star \cap \neg \mathcal{C}$, $\nabla \overline{h}_2(x) \cdot (f(x) + g(x) v^1) \leq 0$. We obtain

$$\nabla \overline{h}_2(x) \cdot (f(x) + g(x) v^1) = \left((r_1 - x_2) - (r_2 - x_3) x_2 m_2(\bar{t}_2) \right) 2c .$$

For $x \in \mathcal{S} \cap \overline{\mathcal{W}}$, we have that $x_2 \geq r_1 > 0$. Moreover, since $erf(\cdot)$ is an increasing function, the value of $m_2(\bar{t}_2)$ is always nonnegative (this is also obvious from the integral definition of $m_2(t)$). Therefore, if $(r_2 - x_3) \geq 0$ the result follows immediately. Now, if $(r_2 - x_3) < 0$, then

$$\begin{aligned} &(r_1 - x_2) - (r_2 - x_3) x_2 m_2(\bar{t}_2) \\ &\leq (r_1 - x_2) - (r_2 - x_3) x_2 \int_0^{\bar{t}_2} e^{(r_2 - x_3)\tau}\, d\tau \\ &= (r_1 - x_2) - (r_2 - x_3) x_2 \frac{1}{(r_2 - x_3)} \left(e^{(r_2 - x_3)\bar{t}_2} - 1 \right) \\ &\leq r_1 - x_2 e^{(r_2 - x_3)\bar{t}_2 - \frac{1}{2}\bar{t}_2^2} \leq 0 . \end{aligned}$$

Therefore condition (9) is satisfied.

V. Conclusion

The paper proposes and solves a viability problem for control affine systems. The problem formulation is based on the notion of viable capture basins, it is shaped by the practical concern to be able to conclude execution of the viability controller in a finite time, and it is relevant in almost all nonlinear control applications of current interest. An explicit formula for the viability kernel and a viability controller are derived, and these formulas are shown to be valid using the Frankowska method, which provides the essential backward invariance condition to obtain the result. A natural next step would be to extend the results to multi-output systems.

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