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On the existence of stable, causal multipliers for systems with slope-restricted nonlinearities

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Abstract— **The stability of a feedback interconnection of a linear time invariant (LTI) system and a slope-restricted nonlinearity is revisited. Unlike the normal treatment of this problem, in which multipliers are explicitly chosen and then stability conditions checked, this paper derives *existence* conditions for a sub-class of these multipliers, namely those which are $\mathcal{L}_1$ bounded, stable, causal and of order equal to the LTI part of the system. It is proved that for the single-input-single-output (SISO) case, these existence conditions can be expressed as a set of linear matrix inequalities (LMIs) and thus can be solved efficiently with modern optimisation software. Examples illustrate the effectiveness of the results.**

I. Introduction

Many researchers have studied the feedback interconnection of an LTI system and a nonlinearity of a given form; for example see [1], [2], [3], [4], [5], [6], [7] and the references therein. When all that is known is that the nonlinearity is sector bounded, the Circle Criterion gives an efficient method for stability analysis. When the nonlinearity is also time invariant, the Popov Criterion may be used to study stability. When, in addition, the nonlinearity is slope restricted, it is well known that asymptotic stability can be established by proving the existence of an $\mathcal{L}_1$ bounded diagonally dominant multiplier [2], [8], [9], [1]. Such systems are of tremendous importance in control engineering because many control problems of practical interest are fundamentally of this form. One is particularly reminded of constrained control problems, where the saturation element is the nonlinearity under consideration (see for example [10], [11], [12]); and the anti-windup problem, where effectively the deadzone nonlinearity is typically used (see for example [13], [14] and the references therein).

Over recent years, the integral quadratic constraint (IQC) method [1], [9], [15] has become a convenient way of both framing and solving problems involving systems containing LTI parts and slope-restricted nonlinearities. Reference [15] is particularly relevant as this allows the stability analysis problem to be solved by seeking multipliers which are $\mathcal{L}_1$ bounded and diagonally dominant (rather than just diagonal), enabling the conservatism of the previous results to be reduced. Work by Safanov and colleagues ([7], [16]) proved that the whole class of multipliers which one can choose for stability analysis does not even have to be symmetric and thus, by judicious choice of multiplier, one could prove stability of systems which were hitherto not *proven* to be stable despite being suspected of being so.

Although the work of [15] and [7], [16] proves that there exists a very large class of multipliers which can enable a system of the above type to be proved stable, there is currently no *systematic* way of choosing these multipliers. Typically, engineering judgement is used to "guess" the multiplier structure (e.g. order, pole location) and then, for example, the IQC toolbox [9] can be used to check whether indeed a given system can be proved stable - with that particular selection of multiplier. A similar approach is proposed in [17] where again the engineer is required to choose multipliers of a given form and then iterate in order to compute a satisfactory solution. While these approaches seem effective for simple systems and while useful improvements in the stability margins/$\mathcal{L}_2$ gains have been demonstrated, it is likely that, for complex systems, a more systematic way of choosing multipliers is required.

In contrast to the above, Park [6] has studied the same problem by proposing a new type of Lur'e-Postnikov Lyapunov function. The examples included in [6] demonstrate that the method proposed therein is one of the least conservative methods for the Lur'e problem and, moreover, it is convex. The solution given in [6] is derived in a similar manner to the standard Popov criterion, although the manipulations involved are more intricate and care is required in casting the problem as a linear matrix inequality. Furthermore, effectively Park's method imposes a certain choice of multiplier on the system and hence, although it is more general than the Popov Criterion, does not exploit the full freedom in multiplier choice which is present in the work of [2] (or the later results of [15], [16], [17]).

This paper improves on the current results (notably [6]) by translating the choice of multipliers to an *existence* problem using LMIs. Although some conservatism is introduced in this translation (the multipliers are restricted to be causal and of order equal to that of the linear part of the system), the method proposed is systematic and requires few parameters to be chosen when the criterion is applied. We emphasise that we do not actually wish to *compute* these multipliers, we simply wish to prove that they exist - which is all that is required in order to prove stability. The problem is cast in an IQC framework and then a nonlinear change of variables as proposed by [18], along with standard "tricks" popular in convex optimisation, are used to "linearise" the matrix inequalities. We only treat SISO systems in this paper but the results are, in principle, extendable to multivariable systems.

Notation. Notation is standard throughout. The $\mathcal{L}_2$ norm

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of a vector valued function $x(t)$ is defined as $\|x\|_2 := \sqrt{\int_0^\infty \|x(t)\|^2 dt}$ where $\|x\|$ denotes a vector's Euclidean norm; the space of functions where this norm is finite is denoted $\mathcal{L}_2$. Likewise, the $\mathcal{L}_1$ norm is defined as $\|x\|_1 := \int_0^\infty \|x(t)\| dt$; the space where this norm is finite is denoted $\mathcal{L}_1$. The space of real rational transfer function matrices, bounded on the imaginary axis is denoted by $\mathcal{RL}_\infty$; the subspace of $\mathcal{RL}_\infty$ which is analytically continuous in the right half complex plane is denoted $\mathcal{RH}_\infty$. With some abuse of notation we say that a transfer function matrix $H(s) \in \mathcal{L}_1$ if its impulse response, $h(t)$ is in $\mathcal{L}_1$. An operator H is described as bounded if $\|H(u)\| \leq \gamma \|u\|$ for all $u \in \mathcal{L}_2$ and some $\gamma > 0$. A function $\phi(.)$ is said to have a slope restriction $[0, \alpha]$ if

$$0 \leq \frac{\phi(x) - \phi(y)}{x - y} \leq \alpha \quad \forall x, y \in \mathbb{R}, \quad \alpha > 0$$

We use the shorthand notation $\phi \in \partial[0, \alpha]$ to indicate that a function has this property. Note that the slope restriction is stronger than the related sector condition ([19]). Simply multiply all terms of the slope inequality by $(x - y)^2 > 0$ and set $y = 0$. Then if $\phi(0) = 0$ it follows that

$$\phi(x)x \leq \alpha x^2$$

and thus every slope restricted nonlinearity is also sector bounded.

II. IQC FRAMEWORK

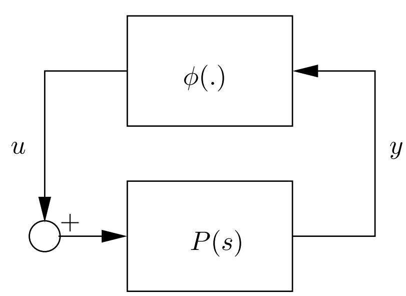

Fig. 1. System under consideration

Consider Figure 1 in which $P(s)$ is the LTI part of the system with state-space realisation

$$P(s) \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array} \right] \tag{1}$$

where $A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times 1}$, $C_p \in \mathbb{R}^{1 \times n}$, $D_p \in \mathbb{R}^{1 \times 1}$ and $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a static nonlinearity satisfying the following assumption.

Assumption 1: $\phi(.) : \mathbb{R} \mapsto \mathbb{R}$ satisfies the following properties:

i) It is bounded, odd and $\phi(0) = 0$
ii) It has slope restriction $\partial\phi \in [0, \alpha]$

A $\phi(.)$ which satisfies the above is said to belong to $\mathcal{N}_S$.

Without loss of generality, the lower gradient of the slope is assumed to be zero; if this is not the case, loop-shifting can be used to pose an equivalent problem where the "loop-shifted" nonlinearity, $\tilde{\phi}$ is such that $\partial\tilde{\phi} \in [0, \tilde{\alpha}]$. It is now reasonably well known ([9], [1], [15]) that $\phi(.) \in \mathcal{N}_S$ satisfies the IQC defined by

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix}^\star \Pi(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{2}$$

where $\hat{u}(j\omega)$ and $\hat{y}(j\omega)$ are the Fourier Transforms of $u(t)$ and $y(t)$ respectively, and $\Pi(j\omega)$ is given by

$$\Pi(j\omega) = \begin{bmatrix} 0 & \alpha M^\star(j\omega) \\ \alpha M(j\omega) & -M^\star(j\omega) - M(j\omega) \end{bmatrix} \tag{3}$$

The transfer function $M(s)$ - the "multiplier" - belongs to the following class, $\mathcal{M}_S$, of functions, normally referred to as the Zames-Falb multipliers ([2]).

Definition 1: A transfer function $M(s) := H_0 - H(s) \in \mathcal{RL}_\infty$ is said to belong to the set $\mathcal{M}_S$ if $H_0 > 0$ and $H(s) \in \mathcal{L}_1$ is such that $\|H(s)\|_1 \leq H_0$.

When $M(s) \in \mathcal{M}_S$, the IQC (2)-(3) captures the largest class of "multipliers" for $\phi(.) \in \mathcal{N}_S$. In the more general case that $\phi(.)$ is vector valued, [15] has provided more general multipliers and more recently [16] has derived the largest class of multipliers. For our work, $M(s) \in \mathcal{M}_S$ will be sufficient. The basic stability result (stated in an IQC context) for the system in Figure 1 can therefore be stated by re-writing the results in [1], [15] as the following Theorem.

Theorem 1: Consider Figure 1 where $P(s) \in \mathcal{RH}_\infty$ and $\phi \in \mathcal{N}_S$ satisfies the IQC defined by (2) and (3) where $M(s) \in \mathcal{M}_S$. Assume that the closed loop system is well-posed. Then the system is asymptotically stable if

$$\begin{bmatrix} P(j\omega) \\ I \end{bmatrix}^\star \Pi(j\omega) \begin{bmatrix} P(j\omega) \\ I \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R} \tag{4}$$

Thus stability of the system essentially reduces to finding suitable $H_0 > 0$ and $H(s) \in \mathcal{L}_1$ such that inequality (4) holds. Our first result, which is derived in a similar manner to [15] shows how (4) can be interpreted as a (nonlinear) matrix inequality.

Proposition 1: The system depicted in Figure 1 is stable if there exists a real symmetric matrix $P = P'$, a scalar $H_0 > 0$ and a transfer function

$$U(s) \sim \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array} \right] \tag{5}$$

where $\|U(s)\|_1 \leq 1$ such that the following matrix inequality is satisfied.

$$\begin{bmatrix} A_I' P + P A_I & P B_I + C_I' H_0 \\ \star & -H_0 D_I - D_I' H_0 \end{bmatrix} < 0 \tag{6}$$

where the matrices A_I, B_I, C_I, D_I are defined in the appendix.

Proof: See appendix.

Remark 1. Inequality (6) closely resembles the Circle Criterion but instead of the original plant matrices (A_p, B_p, C_p, D_p), an "extended" set of matrices (A_I, B_I, C_I, D_I) is involved instead. In fact, when $U(s) \equiv 0$, inequality (6) does indeed reduce to the Circle Criterion.

$$\begin{bmatrix} \mathbf{S_{11}}A_p + A_p'\mathbf{S_{11}} & \mathbf{S_{11}}A_p + A_p'\mathbf{P_{11}} - \alpha C_p'\mathbf{B_u}' + \mathbf{A_u}' & \mathbf{S_{11}}B_p + \alpha C_p' - \alpha C_p'\mathbf{D_u} + \mathbf{C_u'} \\ \star & A_p'\mathbf{P_{11}} + \mathbf{P_{11}}A_p - \mathbf{B_u}\alpha C_p - \alpha C_p'\mathbf{B_u}' & \mathbf{P_{11}}B_p + \mathbf{B_u}(I-\alpha D_p) + \alpha C_p' - \alpha C_p'\mathbf{D_u}' \\ \star & \star & -(I-\mathbf{D_u})(I-\alpha D_p) - (I-\alpha D_p)'(I-\mathbf{D_u})' \end{bmatrix} < 0 \quad (9)$$

$$\begin{bmatrix} -\mathbf{A_u} - \mathbf{A_u}' + \lambda(\mathbf{P_{11}} - \mathbf{S_{11}}) & \mathbf{B_u} \\ \star & -\mu \end{bmatrix} < 0 \quad (10)$$

$$\begin{bmatrix} \lambda(\mathbf{P_{11}} - \mathbf{S_{11}}) & 0 & \mathbf{C_u}' \\ \star & 1-\mu & \mathbf{D_u}' \\ \star & \star & 1 \end{bmatrix} > 0 \quad (11)$$

$$\left[\begin{array}{c|c|c} A_pQ_{11} + Q_{11}A_p' & \begin{array}{c} A_p + Q_{11}A_p'P_{11} \\ -\alpha Q_{11}C_p'B_u'P_{12}' + Q_{12}A_u'P_{12}' \end{array} & \begin{array}{c} B_p + \alpha Q_{11}C_p'H_0 \\ -\alpha Q_{11}C_p'D_u'H_0 + Q_{12}C_u'H_0 \end{array} \\ \hline \star & \begin{array}{c} P_{11}A_p + A_p'P_{11} \\ -\alpha P_{12}B_uC_p - \alpha C_p'B_u'P_{12}' \end{array} & \begin{array}{c} P_{11}B_p + P_{12}B_u(I-\alpha D_p) \\ +\alpha C_p'H_0 - \alpha C_p'D_u'H_0 \end{array} \\ \hline \star & \star & \begin{array}{c} -(I-\alpha D_p)'(I-D_u')H_0 \\ -H_0(I-D_u)(I-\alpha D_p) \end{array} \end{array}\right] < 0 \quad (14)$$

$$\left[\begin{array}{c|c|c} S_{11}A_p + A_p'S_{11} & \begin{array}{c} S_{11}A_p + A_p'P_{11} \\ -\alpha C_p'B_u'P_{12}' + S_{11}Q_{12}A_u'P_{12}' \end{array} & \begin{array}{c} S_{11}B_p + \alpha C_p'H_0 \\ -\alpha C_p'D_u'H_0 + S_{11}Q_{12}C_u'H_0 \end{array} \\ \hline \star & \begin{array}{c} P_{11}A_p + A_p'P_{11} \\ -\alpha P_{12}B_uC_p - \alpha C_p'B_u'P_{12}' \end{array} & \begin{array}{c} P_{11}B_p + P_{12}B_u(I-\alpha D_p) \\ +\alpha C_p'H_0 - \alpha C_p'D_u'H_0 \end{array} \\ \hline \star & \star & \begin{array}{c} -(I-\alpha D_p)'(I-D_u')H_0 \\ -H_0(I-D_u)(I-\alpha D_p) \end{array} \end{array}\right] < 0 \quad (15)$$

III. Main results

The results as they appear in Proposition 1 are not convenient for checking existence of multipliers because they involve, explicitly, the state-space matrices of the multiplier in inequality (6) and then the transfer function $U(s)$ must then be checked to ensure that $\|U(s)\|_1 \leq 1$. In general, $U(s)$ may be of arbitrary order and may be non-causal. In order to manipulate inequality (6) into a more tractable form, we restrict our attention to a limited class of multipliers. Thus throughout the remainder of the paper we make the following assumption.

Assumption 2: The transfer function $U(s)$ is stable with state-space realisation

$$U(s) \sim \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right]$$

where $A_u \in \mathbb{R}^{n\times n}$, $B_u \in \mathbb{R}^{n\times 1}$, $C_u \in \mathbb{R}^{1\times n}$ and $D_u \in \mathbb{R}^{1\times 1}$.

The above assumption ensures that $U(s)$ is stable, causal and is of the same order as $P(s)$.

In [18] it was proved that a transfer function matrix $U(s)$ was such that $\|U(s)\|_1 \leq \xi$ if there exist a matrix $Y = Y' > 0$ and scalars $\lambda > 0$ and $\mu > 0$, such that the following inequalities hold:

$$\begin{bmatrix} A_u'Y + YA_u + \lambda Y & YB_u \\ \star & -\mu I \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} \lambda Y & 0 & C_u' \\ \star & (\xi - \mu)I & D_u' \\ \star & \star & \xi I \end{bmatrix} \geq 0 \quad (8)$$

Although this is conservative in the sense that the above inequalities are only *sufficient* for $\|U(s)\|_1 \leq \xi$, they are relatively straightforward to check and may be combined conveniently with inequality (6). Thus the aim is to combine inequality (6) (the basic "stability" inequality) with the "$\mathcal{L}_1$" inequalities (7) and (8) to arrive at a convenient way of proving the existence of a multiplier which ensures the stability of the system depicted in Figure 1. The following is the main result of the paper.

Proposition 2: Under Assumption 2, Theorem 1 is satisfied if there exist positive definite symmetric matrices $\mathbf{S_{11}} > 0, \mathbf{P_{11}} > 0$, (unstructured) matrices $\mathbf{A_u}, \mathbf{B_u}, \mathbf{C_u}, \mathbf{D_u}$, and scalars $\mu > 0$ and $\lambda > 0$ such that inequalities (9), (10) and (11) are satisfied.

Proof: There exists a transfer function $U(s)$, where $\|U(s)\|_1 \leq 1$, if inequalities (7) and (8) are satisfied with $\xi = 1$. Furthermore, from Proposition 1, we know that if in addition to $\|U(s)\|_1 \leq 1$, inequality (6) holds, then Figure 1 is stable. Thus the proof is essentially one of converting inequalities (6), (7) and (8) into those given in the Proposition. It proceeds in several stages.

Main congruence transformation. Taking inspiration from [18], we consider the matrix $P = P' > 0$ (with $P = P'$ following from Proposition 1) and its inverse $P^{-1} =: Q$. By Assumption 2, $U(s)$ is the same order as $P(s)$ and thus it follows that $Q, P \in \mathbb{R}^{2n\times 2n}$. Partitioning these into $n \times n$ sub-matrices it follows that

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (12)$$

where $P_{11}, P_{22}, P_{12}, Q_{11}, Q_{22}, Q_{12}$ are all full rank. Next consider the full rank matrices

$$\begin{bmatrix} Q_{12}A_u'P_{22}Q_{12}'/H_0 + Q_{12}P_{22}A_uQ_{12}'/H_0 + \lambda Q_{12}P_{22}Q_{12}'/H_0 & Q_{12}P_{22}B_u/H_0 \\ \star & -\mu \end{bmatrix} < 0 \quad (22)$$

$$\begin{bmatrix} -Q_{12}A_u'P_{12}'Q_{11}/H_0 - Q_{11}P_{12}A_uQ_{12}'/H_0 - \lambda(Q_{11} - Q_{11}P_{11}Q_{11})/H_0 & -Q_{11}P_{12}B_u/H_0 \\ \star & -\mu \end{bmatrix} < 0 \quad (23)$$

$$\begin{bmatrix} -S_{11}Q_{12}A_u'P_{12}'/H_0 - P_{12}A_uQ_{12}'S_{11}/H_0 - \lambda(S_{11} - P_{11})/H_0 & P_{12}B_u/H_0 \\ \star & -\mu \end{bmatrix} < 0 \quad (24)$$

$$\Pi_1 := \begin{bmatrix} Q_{11} & I \\ Q_{12}' & 0 \end{bmatrix} \quad \Pi_2 := \begin{bmatrix} I & 0 \\ P_{11} & P_{12} \end{bmatrix} \quad (13)$$

It follows by direct calculation that $\Pi_1'P = \Pi_2$.

Main stability LMI. Applying the congruence transformation $\mathrm{diag}(\Pi_1', I)$ to inequality (6) we obtain, after some manipulations, inequality (14). Applying a further congruence transformation $\mathrm{diag}(Q_{11}^{-1}, I, I) =: \mathrm{diag}(S_{11}, I, I)$ then leads to (15). Dividing inequality (15) by $H_0 > 0$ (as H_0 is scalar), and defining

$$\begin{aligned} \mathbf{S_{11}} &:= S_{11}/H_0 && (16)\\ \mathbf{P_{11}} &:= P_{11}/H_0 && (17)\\ \mathbf{A_u} &:= P_{12}A_uQ_{12}'S_{11}/H_0 && (18)\\ \mathbf{B_u} &:= P_{12}B_u/H_0 && (19)\\ \mathbf{C_u} &:= C_uQ_{12}'S_{11} && (20)\\ \mathbf{D_u} &:= D_u && (21) \end{aligned}$$

then yields the first LMI in the Theorem.

First $\mathcal{L}_1$ LMI. Assuming that $Y = P_{22}/H_0$ in inequality (7) and using the congruence transformation $\mathrm{diag}(Q_{12}, I)$ gives inequality (22). Next, from equation (12) it follows that $Q_{12}P_{22} = -Q_{11}P_{12}$. Using this in (22) and noting further from equation (12) that $Q_{11}P_{12}Q_{12}' = Q_{11}(I - P_{11}Q_{11})$, then yields (23). Then, using the congruence transformation $\mathrm{diag}(Q_{11}^{-1}, -I) = \mathrm{diag}(S_{11}, -I)$ yields (24). Using equations (16)-(19), then yields inequality (10) in the proposition.

Second $\mathcal{L}_1$ LMI. Letting $\xi = 1$ and replacing Y with P_{22}/H_0 in inequality (8) and applying the congruence transformation $\mathrm{diag}(Q_{11}^{-1}Q_{12}, I, I)$ gives, after similar working to the above, the following inequality.

$$\begin{bmatrix} -\lambda(S_{11} - P_{11})/H_0 & 0 & S_{11}Q_{12}C_u' \\ \star & 1-\mu & D_u' \\ \star & \star & 1 \end{bmatrix} \geq 0 \quad (25)$$

Making this inequality strict and recalling equations (16)-(21) then yields inequality (11).

Ensuring $P > 0$. $P > 0$ is equivalent to $\Pi_1'P\Pi_1 > 0$, which can be written as

$$\Pi_1'P\Pi_1 = \begin{bmatrix} Q_{11} & I \\ I & P_{11} \end{bmatrix} > 0 \quad (26)$$

This is equivalent, by the Schur complement, to $P_{11} - S_{11} > 0$. As $H_0 > 0$, this will hold if and only if $\mathbf{P_{11}} - \mathbf{S_{11}} > 0$, which is guaranteed by inequality (11). □□

Remark 1 - conservatism. Proposition 2 is conservative: the system in Figure 1 may be stable even if the inequalities in Proposition 2 are not satisfied. There are two main sources of conservatism.

1) The class of multipliers is restrictive: in general $M(s) \in \mathcal{M}_S$ need only be such that $M(s) \in \mathcal{RL}_\infty$ and $\|H(s)\|_1 \leq H_0$; it need not even be causal. However, as Proposition 2 is proved under Assumption 2, which restricts the multiplier to be stable, causal and of order equal to that of the plant, significant conservatism may be present. Nevertheless, this class of multipliers is still more general than those considered in existing convex results, such as the Circle and Popov Criteria and Park's results. Therefore the stability results offered by Proposition 2 will be *no more conservative* than existing results and possibly much less (see later examples). For fixed $\alpha > 0$ and $\lambda > 0$, the inequalities in Proposition 2 are convex and easy to solve; we trade conservatism for efficiency.
2) To enforce the $\mathcal{L}_1$ bound $\|U(s)\|_1 \leq 1$, we make use of inequalities (7) and (8) which may be very conservative. Furthermore, because P_{22} is part of the main Lyapunov matrix, P, *and* is used in $Y = P_{22}/H_0 > 0$ to enforce the $\mathcal{L}_1$ bound, additional conservatism may be introduced. For this reason, when solving the inequalities in Proposition 2, it is often better to replace inequality (11) with

$$\begin{bmatrix} -\lambda(\mathbf{S_{11}} - \mathbf{P_{11}}) & 0 & \mathbf{C_u}' \\ \star & \gamma_1 - \mu & \mathbf{D_u} \\ \star & \star & \gamma_1 \end{bmatrix} \geq 0 \quad (27)$$

where $\gamma_1 > 1$ is chosen slightly greater than unity to "relax" the LMI's. Normally when the multipliers are constructed and the $\mathcal{L}_1$ gain of $U(s)$ is then calculated it is less than unity. □

Remark 2 - convexity. For fixed $\alpha > 0$ and $\lambda > 0$, the inequalities (9)-(11) in Proposition 2 are LMIs and easily solved by standard convex optimisation software. Thus if the slope of the nonlinearity is known, closed-loop stability can be easily determined. Alternatively, if the objective is to compute the maximum slope, $\alpha > 0$, for which stability holds, the optimisation problem is only quasi-convex. In this case a bisection algorithm (similar to that for computing generalised eigenvalues) can be used in conjection with LMI solvers to compute the largest $\alpha > 0$ yielding stability. It is important to note that a lower bound on α will be zero and an upper bound will be the gain margin of the open-loop system. In the authors' experience, commencing the bisection just below the gain margin gives fast convergence. $\lambda > 0$ must be chosen by the designer, but it usually suffices to choose it reasonably small (e.g. $\lambda = 10^{-5}$) although some tuning maybe required. $\square$

IV. Examples

This section compares the results obtained using the method proposed in this paper to existing methods in the literature. Similar to [6] we consider several systems $P(s)$ and attempt to compute the maximum size slope (or sector) for which we are able to guarantee stability. The methods to which we compare our result are the standard Circle Criterion and the method of Park [6]. The method of [6] contains the Popov Criterion as a special case and has been demonstrated to be less conservative than the methods of Haddad and Kapila [3], Suykens *et al.*[4] and Chen and Wen [20]. Park's method is also convex, making it easy to compute solutions. The transfer functions of the systems we consider are listed in Table I (because we assume a positive feedback convention, the transfer functions have opposite sign to those given in [6]).

The results of the comparison performed are shown in Table II. For the computation of these results, $\lambda = 10^{-5}$, and, as suggested in Remark 1, to add flexibility in the LMI's, inequality (11) was replaced by inequality (27) with $\gamma_1 = 1.1$. The slope size is taken to be equivalent to the sector size when comparing to the Circle Criterion.

The first two examples confirm that our results are no more conservative than Park's results but are a notable improvement on the standard Circle Criteria. The third and fourth examples are more interesting because although Park's algorithm gives vast improvements over the Circle Criterion, the slopes for which stability is guaranteed are still rather small. However, our new results improve upon Park's results by several orders of magnitude and clearly show the benefit of using a wider class of multipliers and the accompanying LMI-based algorithm from Proposition 2. Note that in all cases, despite using the "relaxed" inequality (27) instead of (11), $\|U(s)\|_1 \leq 1$ as required.

Remark 3 - multiplier reconstruction It is emphasized that the results here *do not require the multiplier to be reconstructed*; they simply prove the existence of a multiplier which would then imply stability. However, given solutions to the inequalities in Proposition 2, one can then construct a multiplier on the basis of these. In particular, by selecting an (arbitrary) $H_0 > 0$, from equations (16) and (17) we have

$$P_{11} = H_0 \mathbf{P_{11}}, \quad S_{11} = H_0 \mathbf{S_{11}} \tag{28}$$

Using equation (12) with $P_{22} = I$, it then follows that

$$P_{12}P_{12}' = P_{11}S_{11}^{-1}P_{11} - P_{11} \tag{29}$$

$$Q_{12}'S_{11} = P_{12}^{-1}(S_{11} - P_{11}) \tag{30}$$

Together with equations (18)-(21), these can then be used to determine $U(s) \sim (A_u, B_u, C_u, D_u)$. A list of multipliers returned by the optimisation process is given in Table III. The gain of these multipliers can be scaled by a positive scalar without affecting the stability result (this is equivalent to changing $H_0 > 0$). Finally, it should be mentioned that poor numerical conditioning may arise in the *reconstruction* of multipliers; another reason for simply proving their existence, rather than computing them explicitly. $\square$

Example	P(s)	Source
1	$P_1(s) = -\frac{s^2-0.2s-0.1}{s^3+2s^2+s+1}$	[3]
2	$P_2(s) = -P_1(s)$	[6]
3	$P_3(s) = -\frac{s^2}{s^4+0.0003s^3+10s^2+0.0021s+9}$	[20]
4	$P_4(s) = -P_3(s)$	new

TABLE I

Table of transfer functions $P(s)$

Criteria	Ex 1	Ex 2	Ex 3	Ex 4
Circle	1.2431	0.7640	0.00040	0.00039
Park	4.5894	1.0894	0.00183	0.00183
Proposition 2	4.5894	1.0894	662.5603	1.6599

TABLE II

Sector/Slope bounds obtainable using various stability criteria

Example	Multiplier
1	$M_1(s) = \frac{10^{-4}(0.59s^3+1.64s^2+1.55s+0.36)}{s^3+6.41s^2+0.16s+0.52}$
2	$M_2(s) = \frac{10^{-4}(3.64s^3+2.54s^2+4.53s+2.10)}{s^3+1.02s^2+1.24s+1.21}$
3	$M_3(s) = \frac{10^{-6}(0.10s^4+7.17s^3+1.06s^2+47.2s+1.25)}{s^4+0.82s^3+729.4s^2+6.07s+6.68}$
4	$M_4(s) = \frac{10^{-6}(0.81s^4+5.92s^3+9.26s^2+33.9s+7.62)}{s^4+0.14s^3+7.93s^2+0.19s+8.63}$

TABLE III

Multipliers computed using optimisation algorithm

It is important to point out that results as non-conservative as ours could be obtained using the IQC method of [9] or the multiplier method of [17]. However both those algorithms essentially assume a form of multiplier and then leave the designer to pick parameters (such as order, pole location and so on), making the process somewhat iterative. Proposition 2 is a computationally convenient routine which makes it straightforward to assess stability with our more limited class of multipliers. We also note that the results obtained using this potentially conservative form of multiplier do not, in the examples considered, appear conservative at all.

V. Conclusion

This paper has proposed a new method for testing stability of a feedback interconnection involving an LTI part and a slope-restricted nonlinearity. The approach is based on the multiplier/IQC machinery but, as the optimisation procedure involved is automated and simply involves the solution of a set of linear matrix inequalities, it is believed to be computationally attractive compared to [9] and [17] where a certain amount of iteration and choice is involved. It also appears superior to other Lyapunov based literature, of which [6] seems to be best. This is because, as shown in [6], Park's method is equivalent to choosing IQC's of a particular form whereas our method allows optimisation over a larger class of multipliers.

It would be logical to extend these results to MIMO systems, although (i) it is more difficult to obtain *linear* matrix inequalities in the MIMO version of Proposition 2;

$$\begin{bmatrix} -\alpha P(j\omega)+I \\ I \\ U(j\omega)(-\alpha P(j\omega)+I) \\ -I \end{bmatrix}^{\star} \underbrace{\begin{bmatrix} 0 & -H_0 & 0 & 0 \\ -H_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -H_0 \\ 0 & 0 & -H_0 & 0 \end{bmatrix}}_{W} \begin{bmatrix} -\alpha P(j\omega)+I \\ I \\ U(j\omega)(-\alpha P(j\omega)+I) \\ -I \end{bmatrix} < 0 \tag{31}$$

$$\begin{bmatrix} -\alpha P(j\omega)+I \\ I \\ U(j\omega)(-\alpha P(j\omega)+I) \\ -I \end{bmatrix} \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{cc|c} A_p & 0 & B_p \\ -B_u\alpha C_p & A_u & B_u(I-\alpha D_p) \\ \hline -\alpha C_p & 0 & I-\alpha D_p \\ 0 & 0 & I \\ -D_u\alpha C_p & C_u & D_u(I-\alpha D_p) \\ 0 & 0 & -I \end{array}\right] \tag{32}$$

$$\left[\begin{array}{c|c} A_I & B_I \\ \hline C_I & D_I \end{array}\right] = \left[\begin{array}{cc|c} A_p & 0 & B_p \\ -B_u\alpha C_p & A_u & B_u(I-\alpha D_p) \\ \hline \alpha C_p - D_u\alpha C_p & C_u & (I-D_u)(I-\alpha D_p) \end{array}\right] \tag{34}$$

and (ii) as noted in [16], the class of MIMO multipliers is wider than was previously thought and may be difficult to characterise (non-conservatively) in a similar way.

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APPENDIX

Letting $H(s) = H_0U(s)$, it follows that $M(s) \in \mathcal{M}_S$ if $U(s) \in \mathcal{L}_1$ is such that $\|U(s)\|_1 \leq 1$. Also with $H(s) = H_0U(s)$, inequality (4) can be re-written as inequality (31). Some algebra then shows that, given $U(s) \sim (A_u, B_u, C_u, D_u)$, the state-space realisation in equation (32) can be derived.

Using the KYP Lemma ([21]) it then follows that inequality (31) is satisfied if and only if there exists a symmetric matrix $P = P'$ such that the following matrix inequality is satisfied.

$$\begin{bmatrix} \tilde{A}'P + P\tilde{A} & P\tilde{B} \\ \tilde{B}'P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}' \\ \tilde{D}' \end{bmatrix} W \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0 \tag{33}$$

Using the definitions of $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and W, this then reduces to inequality (6) in Proposition 1 where a realisation for (A_I, B_I, C_I, D_I) is given in equation (34).