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An Individual-Based Evolutionary Dynamics Model for Networked Social Behaviors

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Abstract— **In this paper, an evolutionary dynamics model over a graph of connected individuals choosing between multiple behaviors is developed. This model emphasizes the individuality of the nodes, which arrive at individual behavioral choices primarily based on subjective individual preferences as well as individual mutation characteristics. We use the replicator-mutator dynamical equations to model the process of building individual behavioral inclinations. A dynamic graph, whose vertices are the individual members of society, is then constructed and the weighted adjacency matrix and individual fitness parameters are used to effect a social interaction model that is itself modeled based on the replicator-mutator dynamical equations. A notion of social diversity is defined for this individual-based social choice model. The individual-based social evolutionary model presented here relates to and generalizes three previous models appearing in the literature: the replicator-mutator social choice model, consensus algorithms, and an evolutionary dynamic model on graphs. The basic properties and conditions for the emergence of an absolutely dominant behavior over the social network are derived, and how the proposed model generalizes and relates to other work is also discussed.**

I. Introduction

Evolutionary dynamics have been used not only to model biological [1], [2] and molecular [3] evolution, but have also been used to model language evolution [4], [5], learning [6], cooperation in social systems [7], and in the study of emergence of behavioral norms in social networks [8]–[10]. The latter of these problems, the emergence of behavioral norms in social networks, is the focus of this paper. In general, an approach may adopt either a collective or an individual-based viewpoint. In the former approach, distinctions between the different individual nodes are ignored and the entire society is treated as a single evolutionary organism. The second approach, which is the viewpoint adopted in this paper, emphasizes the individuality of the nodes, which arrive at individual behavioral choices primarily based on subjective individual preferences, individual mutation characteristics, and through interactions with other individual nodes in the network.

Unlike other individual-based approaches [11], [12], in this paper we employ the replicator-mutator dynamics [6] to model how individuals build up their own personal behavioral inclinations based on subjective valuation of the various behaviors as well as the individual's mutation rates. However, similar to [11], a dynamic evolutionary graph theoretic approach is adopted here to model the effect of social interaction on the individual behavioral inclinations. While in [11] a probabilistic model is proposed to study the propagation (or suppression) of a single mutant behavior across a social network with a single dominant behavior, the model developed here allows for an arbitrary number of behaviors over an arbitrary number of individuals. Moreover, our dynamic graph is constructed based on the replicator-mutator dynamics [6], [13], [14] and, hence, both the individual choice and social interaction model are based on the replicator-mutator evolutionary equations originally developed by Eigen and Schuster [15].

The goal of this paper is, thus, *to develop and analyze an individual-based evolutionary dynamics model, based on the replicator-mutator equations, for networked social behaviors. We rely on the replicator-mutator dynamics to model both the individual choice as well as the networked social interaction model*. After introducing the replicator-mutator (collective) social choice model of [10] in Section II, we first adapt that model to an individual choice model in Section III-A. We then introduce a graph theoretic approach to include the effect of social interactions on individual decision making in Section III-B. In Section III-C, we define a notion of social diversity analogous to that introduced in [10]. We give a basic result that guarantees the emergence of dominant social behaviors in Section III-D. Finally, in Section IV, we show how the proposed model relates to and generalizes three previous models appearing in the literature: the replicator-mutator-based social choice model of [10], consensus algorithms [16]–[18], and an evolutionary dynamic model on graphs [11]. We conclude the paper with a summary and future research directions.

II. Evolutionary Dynamics and Collective Behavior Networks

A. Replicator-Mutator Social Choice Model

In this section, we first review the replicator-mutator social choice model of [10]. Consider the problem where we have N possible behaviors (different brands of a product, political candidates in a race for public office, etc) with a *vector of frequencies* $\mathbf{x} = (x_1, x_2, \ldots, x_N)^{\mathrm{T}} \in \mathbb{R}^N$ associated with the *vector of behaviors* $\mathbf{b} = (b_1, b_2, \ldots, b_N)$

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satisfying the normalized condition $\mathbf{x}^{\mathrm{T}}\mathbf{c} = 1$, where $\mathbf{c} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^N$ is the vector of 1's. A component x_i in the vector of frequencies $\mathbf{x}$ describes the proportion of the population that subscribes to behavior b_i. Let $a_{ij} \geq 0$ be the reward to convert from behavior b_i to behavior b_j, with $a_{ii} = 1$. Hence, $\mathbf{A} = [a_{ij}]$ is the *matrix of rewards*. The *fitness of behavior* b_i is defined as $\mathbf{f} = \mathbf{f}_0 + \mathbf{A}\mathbf{x} \in \mathbb{R}^N$, where $\mathbf{f}_0$ is the vector of *base fitness*. How fit a behavior is depends on the matrix of rewards $\mathbf{A}$ (or how rewarding it is to adopt a certain behavior instead of some other behavior) and the vector of frequencies. Expanding this expression, one gets $f_i = f_{0i} + \sum_{j=1}^{N} a_{ij}x_j$. As in [10], we will assume that $\mathbf{f}_0 = \mathbf{0}$. The replicator-mutator dynamics are then given by [5], [6], [10]

$$\dot{x}_i = \sum_{j=1}^{N} x_j f_j q_{ji} - \phi x_i, \ i = 1, \ldots, N, \tag{1}$$

where $\phi = \mathbf{f}^{\mathrm{T}}\mathbf{x}$ is the *average fitness* and $\mathbf{Q} = [q_{ij}]$ is the *mutation matrix*. The mutation matrix $\mathbf{Q}$ is a row stochastic matrix (i.e., one satisfying $\sum_{j=1}^{N} q_{ij} = 1$). The component q_{ij} describes the rate of conversion from behavior b_i to behavior b_j ($j \neq i$). The pair $\mathbf{A}$ and $\mathbf{Q}$ define the *social choice model*.

The mutation matrix $\mathbf{Q}$ can be chosen several ways. In [10], the author chooses $\mathbf{Q}$ to be the Perron matrix (see [18]) given by $\mathbf{Q} = \mathbf{I} - \mu\mathbf{L}$, where μ is the *mutation parameter* and $\mathbf{L}$ is the graph Laplacian associated with the graph $G = (V, E)$ whose vertices correspond to the N behaviors $\mathbf{b}$ and whose interaction matrix is $\mathbf{A}$ [16], [18]. According to this social choice model, mutation rates are proportional to the weighted matrix of rewards $\mathbf{W} = \mathbf{D}^{-1}\mathbf{A}$, where $\mathbf{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ with $d_i = \sum_{j=1}^{N} a_{ij}$, and the mutation rates are controlled by the choice of the mutation parameter μ. This choice for the mutation matrix $\mathbf{Q}$ provides us with a model consistent with the general properties of the mutation matrix (appropriate dependence on the weighted awards matrix and the mutation parameters). It is also a natural choice to make given the network graphical model we adopt in this paper. Hence, we have $\mathbf{Q} = \mathbf{I} - \mu\mathbf{L} = \mathbf{I} - \mu(\mathbf{I} - \mathbf{W}) = (1 - \mu)\mathbf{I} + \mu\mathbf{W}$. In [10], the author calls this a first order social choice model since it represents a first order approximation of a higher order (exponential) model briefly discussed in [10]. Note also that if $\mu = 0$, we get $\mathbf{Q} = \mathbf{I}$ and the dynamic model (1), as one would expect for a valid choice of the mutation matrix, reduce to the replicator dynamics [13], [19]. Note that $q_{ij} = \mu w_{ij}$ for $i \neq j$ and that $q_{ii} = 1 - \mu(1 - w_{ii})$.

One can check that the dynamic model (1) can be written in matrix form as[1]

$$\dot{\mathbf{x}} = \mathbf{Q}^{\mathrm{T}}\mathbf{F}\mathbf{x} - \phi\mathbf{x}, \tag{2}$$

where $\mathbf{F} = \mathrm{diag}(\mathbf{f})$ (a matrix whose diagonal elements are the components of $\mathbf{f}$) that satisfies $\mathbf{F}\mathbf{c} = \mathbf{f}$. Note that $\sum_{i=1}^{N} \dot{x}_i = \mathbf{c}^{\mathrm{T}}\dot{\mathbf{x}} = \mathbf{c}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{F}\mathbf{x} - \phi\mathbf{c}^{\mathrm{T}}\mathbf{x} = \mathbf{c}^{\mathrm{T}}\mathbf{F}\mathbf{x} - \phi = \mathbf{f}^{\mathrm{T}}\mathbf{x} - \phi = \phi - \phi = 0$, and, hence, solutions of (2) satisfy $\mathbf{c}^{\mathrm{T}}\mathbf{x} = \sum_{i=1}^{N} x_i = 1$ if $\mathbf{c}^{\mathrm{T}}\mathbf{x}(0) = 1$.

There are two main types of emergent social behaviors depending on how diverse the steady state is. The notion of *diversity* is used to characterize the emergent steady state $\mathbf{x}^*$ [10]. Diversity is a number N_e, that satisfies $1 \leq N_e \leq N$, and is defined as

$$N_e = 1/\left\|\mathbf{x}\right\|^2. \tag{3}$$

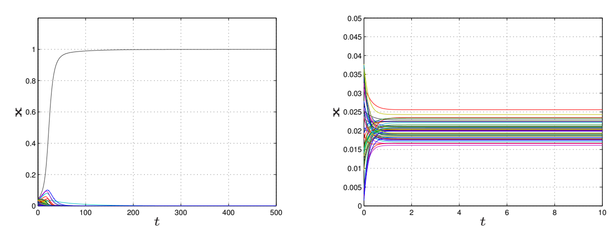
One can check that if $\mathbf{x}^* = \mathbf{e}_i$ (a vector that is zero everywhere except for the i^{th} component), then $N_e = 1$ indicating a *dominant behavior* (also called *behavioral flocking*). On the other hand, if each behavior receives an equal share of the state $\mathbf{x}^*$ at steady state, then $\mathbf{x}^* = \frac{1}{N}\mathbf{c}$ and the diversity is $N_e = N$. This indicates a state of *total* or *complete collapse*. If $N \gg N_e > 1$, we have *cohesion*, where a few dominant behaviors emerge. If $N > N_e \gg 1$, we have *collapse*, where many dominant behaviors emerge.

For a given reward matrix $\mathbf{A}$, the mutation parameter μ determines the value of N_e and the number of behaviors that emerge. As shown in [10], a low value of μ (i.e., low mutation rates) results in a dominant behavior. This behavior is shown in Figure 1(a) with $N = 50$, $\mu = 5 \times 10^{-5}$, and a randomly chosen reward matrix. The diversity in this case is $N_e = 1.0008$, indicating a clear dominant behavior. At the other end of the spectrum, for $N = 50$ and $\mu = 10$ (i.e., high mutation rates) results in collapse (see Figure 1(b)) with a diversity of $N_e = 49.45$.

Theorem II.1 (Olfati-Saber [10])**.** Let $\mathbf{x}^*$ be an equilibrium of the replicator-mutator dynamics with $\mathbf{Q} = \mathbf{I} - \mu\mathbf{L}$. Then the following statements hold:

1) An absolutely dominant behavior results from $\mu = 0$.
2) For large behavior networks $N \gg 1$, a single relatively dominant behavior (with $x_i = 1 - O(\epsilon)$ and $x_j = O(\epsilon/n)$ for all $j \neq i$) can only emerge from evolution with a relatively small μ.

Hence, we see that the mutation parameter μ under the above model determines whether or not dominant behaviors emerge in a society.



(a) $\mu = 5 \times 10^{-5}$ (b) $\mu = 10$

Fig. 1. (a) dominant behavior, (b) total collapse.

There are two main concerns with the above model if it is used to model a society and how social hubs or behaviors emerge to become the norm. Firstly, the above model assumes that society is homogeneous. This assumption is

[1]The author was inspired to write the replicator-mutator dynamics based on the exposition in [14] for a slightly different replicator-mutator dynamical model.

implicitly made once one chooses a single rewards matrix $\mathbf{A}$. By utilizing a single rewards matrix, one assumes that the reward of switching from one behavior to another is the same across the population. This assumption may hold true for biological systems, where there is a uniform "agreement" among the members of "society" as to the rewards of adopting one behavior over another. In other words, there is an *objective* way of assessing the rewards matrix regardless of the *subjective* assessment of the individual members of the population. However, if one is discussing a society of individuals choosing between different brand names or political candidates in a political campaign, the value of the different brand names, political candidates or, in general, behaviors is completely subjectively assessed. In other words, each member of the population derives its own rewards matrix based on a subjective assessment of the potential costs and benefits that each behavior offers to the individual.

The second concern has to do with the mutation parameter μ. Again, the underlying assumption in [10] is that a society composed of individuals is as homogeneous in its mutation rate as a collection of identical cells in a biological setting. Different members of society do have different mutation rates. Each individual, based on a myriad of factors (such as psychological, physical, financial, societal, etc) can change its mind independently, and this change can be reflected using an individual-based mutation parameter μ. Hence, in a single society, as for the rewards matrix, one will have as many and as varied mutation parameters as there are individuals. While it is true that segments of society could share similar mutation rates or rewards matrices, any model that attempts to account for social change and behavioral evolution has to first recognize the fact that both the rewards and mutation rates are subjective and can change from one member of society to another. Capturing this diversity in an individual-based evolutionary dynamic model is the goal of this paper.

III. An Individual-Based Evolutionary Dynamics Model for Networked Social Behaviors

In this section the replicator-mutator dynamic model described above will be adopted to model (1) how an individual may arrive at a choice among a number of possibilities, and (2) how social interaction in a social network affect that choice and the ultimate emergence of one or more dominant behaviors. The model developed here relies on the assumption that the two main model parameters, the reward matrix $\mathbf{A}$ and the mutation parameter μ, are inherent properties of individuals and vary from one individual to another. The mutation parameter μ reflects a trait of an individual (this trait being how frequently the individual in question is willing to change its mind), and the choice of the reward matrix is fundamentally subjective in nature. Given any two brands of some product (e.g., laptops), what one individual may consider a more important property of one brand over others, another individual may not consider as important. Hence, each individual in society selects, based on very subjective criteria, the elements of its reward matrix. While the individual reward matrices may vary greatly in society, those individuals with similar rewards matrices would be expected to aggregate. Whether they reinforce their beliefs or degenerate depends on the properties of the constituents' mutation parameters. In this section, we will describe the replicator-mutator dynamics for individual choice, and in following sections we describe how to transition from individual choice to the emergence of social behaviors in a given social network.

A. Individual Choice Model

Assume that there are n individuals in a social network. We begin with the individual and the choices offered to it. As before, we assume we have N choices or behaviors $\mathbf{b}$. Instead of having x_k denote the frequency of behavior b_j in the sense of a "market share", we will define x_j^k, with $\sum_{j=1}^N x_j^k = 1$, as the inclination of individual k towards behavior b_j. Hence, $\mathbf{x}^k \in \mathbb{R}^N$ denotes the vector of *individual inclinations* towards behaviors $\mathbf{b}$. Let a_{ij}^k be the *subjective* reward assigned by individual k to switch from behavior b_i to behavior b_j, with $a_{ii}^k = 1$. Let $\mathbf{A}^k = [a_{ij}^k] \neq \mathbf{I}$ be the *subjective matrix of individual rewards*. $\mathbf{A}^k$ can not be the identity matrix because if $a_{ij}^k = 0$ then the individual absolutely prefers b_i to b_j, which implies that $a_{ji}^k \neq 0$, for otherwise this would imply that the individual absolutely prefers b_j to b_i, which is a contradiction if $a_{ij}^k = 0$. The vector $\mathbf{f}^k = \mathbf{f}_0^k + \mathbf{A}^k\mathbf{x}^k$ would then describe how individual k perceives the fitness of the different behaviors. We will call it the *perceived fitness vector* of the behaviors $\mathbf{b}$ associated with individual k. Let $\mathbf{f}_0^k = 0$ as before. Finally, let $\phi^k = \sum_{j=1}^N f_j^k x_j^k$ be the *average perceived fitness of the behaviors* $\mathbf{b}$ by individual k.

Before we state how the vector of individual inclinations $\mathbf{x}^k$ evolves, we first have to introduce the vector space of behaviors. This vector space is N-dimensional with the basis vectors denoted by $\mathbf{e}_i$, $i = 1, \ldots, N$, and the vector of individual inclinations will be expressed as the linear sum

$$\mathbf{x}^k = \sum_{j=1}^{N} x_j^k \mathbf{e}_j, \ k = 1, \ldots, n \tag{4}$$

where $\mathbf{e}_j$ is a unit vector whose elements are zero except for the j^{th} element which is 1. We will constrain $\mathbf{x}^k$ to have unity magnitude and have its components satisfy the replicator-mutator dynamics:

$$\dot{x}_i^k = \sum_{j=1}^{N} x_j^k f_j^k q_{ji}^k - \phi^k x_i^k. \tag{5}$$

Note that if the initial condition has a unity magnitude, then the magnitude of $\mathbf{x}^k$ retains its unity value while the dynamics of an individual's inclinations evolve according to the replicator-mutator model. The matrix $\mathbf{Q}^k = [q_{ij}^k]$ is the mutation matrix associated with individual k. The component q_{ij}^k represents the likelihood that individual k changes its inclinations from behavior i to behavior j. In matrix form, the individual replicator-mutator equation can be written as

$$\dot{\mathbf{x}}^k = \left(\mathbf{Q}^k\right)^{\mathrm{T}} \mathbf{F}^k \mathbf{x}^k - \phi^k \mathbf{x}^k, \tag{6}$$

where $\mathbf{F}^k = \mathrm{diag}(\mathbf{f}^k)$. The equation (6) models how an individual, if *isolated from society* (for we have not added a social interaction component to the model yet) would tend to change its inclinations to the various behaviors available to it.

The mutation matrix could be any one from a wide range of possibilities, though here we will use the same model as that used in [10] and that is described above in Section II. That is, let $\mathbf{Q}^k = \mathbf{I} - \mu^k \mathbf{L}^k = \mathbf{I} - \mu^k(\mathbf{I} - \mathbf{W}^k) = (1-\mu^k)\mathbf{I} + \mu^k \mathbf{W}^k$, where μ^k will now denote the *mutation parameter associated with individual* k and $\mathbf{W}^k = \left(\mathbf{D}^k\right)^{-1} \mathbf{A}^k$, with $\mathbf{D}^k = \mathrm{diag}(d_1^k, \ldots, d_N^k)$ and where $d_i^k = \sum_{j=1}^N a_{ij}^k$. Note that $q_{ij}^k = \mu^k w_{ij}^k$ for $i \neq j$ and that $q_{ii}^k = 1 - \mu^k(1 - w_{ii}^k)$. The mutation parameter μ^k is a parameter that describes how easily the individual in question tends to change its inclinations between the various behaviors.

B. The Effect of Social Interactions

We now introduce a social interaction model. The social network is modeled as a graph $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$. Individual i is represented by a vertex $v_i \in \mathcal{V}$. An edge $e_{ij} \in \mathcal{E}$ connects v_i with v_j. Let $\mathbf{A}^s = [a_{ij}^s]$, with $a_{ii}^s = 0$, denote the interaction matrix of $\mathcal{S}$. Let $\mathbf{W}^s = [w_{ij}^s]$ be the weighted interaction matrix given by $w_{ij}^s = \frac{a_{ij}^s}{\sum_{j=1}^n a_{ij}^s}$. Hence, $\mathbf{W}^s$ is a row stochastic matrix. Each individual v_k has a fitness $f_k^s > 0$. Note that we use a superscript s to differentiate social parameters from their individual counterparts. Hence, f_j^s is the social fitness parameter associated with individual v_j and f_k^j is the fitness of behavior k as seen by individual v_k.

Consider the following modified individual replicator-mutator dynamic model that includes the effects of social interaction

$$\dot{\mathbf{x}}^k = \left(\mathbf{Q}^k\right)^{\mathrm{T}} \mathbf{F}^k \mathbf{x}^k - \phi^k \mathbf{x}^k + \sum_{j=1}^n f_j^s w_{kj}^s \mathbf{x}^j - \phi_k^s \mathbf{x}^k. \tag{7}$$

According to this model, an individual v_k's inclination vector updates according to its own valuation of the different behaviors b_j, $j = 1, \ldots, N$, given by the first two terms as before, as well as the influence of all other individuals' inclinations $\mathbf{x}^j$ in the social network. The individual social fitness f_k^s may be a function of the individual's own inclination vector $\mathbf{x}^k$ as well as other social parameters. Here we keep the choice of f_k^s generic, including the possibility that it is a constant parameter as done in [11], for example. The parameter ϕ_k^s is introduced to guarantee that $\mathbf{c}^{\mathrm{T}}\mathbf{x}^k$ is unity. It is given by

$$\phi_k^s = \sum_{j=1}^n f_j^s w_{kj}^s. \tag{8}$$

Hence, ϕ_k^s is the *weighted average social fitness of all individuals* in the social network *with respect to individual* v_k, where the weighting is given by the components of the weighted interaction matrix w_{ij}^s. To show that under this model trajectories that satisfy $\mathbf{c}^{\mathrm{T}}\mathbf{x}^k(t) = 1$ for all $t \geq 0$ and all k, we take the inner product of $\dot{\mathbf{x}}^k$ with $\mathbf{c}$ and check that trajectories of the form $\mathbf{c}^{\mathrm{T}}\mathbf{x}^k = 1$ is a solution to the equation. Doing that, we get

$$\begin{aligned} \mathbf{c}^{\mathrm{T}}\mathbf{x}^k &= \mathbf{c}^{\mathrm{T}} \left(\mathbf{Q}^k\right)^{\mathrm{T}} \mathbf{F}^k \mathbf{x}^k - \phi^k \mathbf{c}^{\mathrm{T}} \mathbf{x}^k \\ &\quad + \sum_{j=1}^n f_j^s w_{kj}^s \mathbf{c}^{\mathrm{T}} \mathbf{x}^j - \phi_k^s \mathbf{c}^{\mathrm{T}} \mathbf{x}^k \\ &= 0 + \sum_{j=1}^n f_j^s w_{kj}^s - \phi_k^s, \end{aligned}$$

which is zero since the first term is zero as discussed in previous sections, and since the second term is zero by definition of ϕ_k^s. This gives the following lemma.

Lemma III.1. The individual-based replicator-mutator dynamic model for social described by equation (7) guarantees that $\mathbf{c}^{\mathrm{T}}\mathbf{x}^k = 1$ for all $k = 1, \ldots, n$ if $\mathbf{x}^k$ satisfies the initial condition $\mathbf{c}^{\mathrm{T}}\mathbf{x}^k(0) = 1$.

To summarize, the above individual-based evolutionary dynamics model for networked societies (including behavioral systems, in general) can be graphically represented as seen in Figure 2. Each node represents an individual. Associated with each node is an individual behavioral choice model. All individuals interact through the social network $\mathcal{S}$.

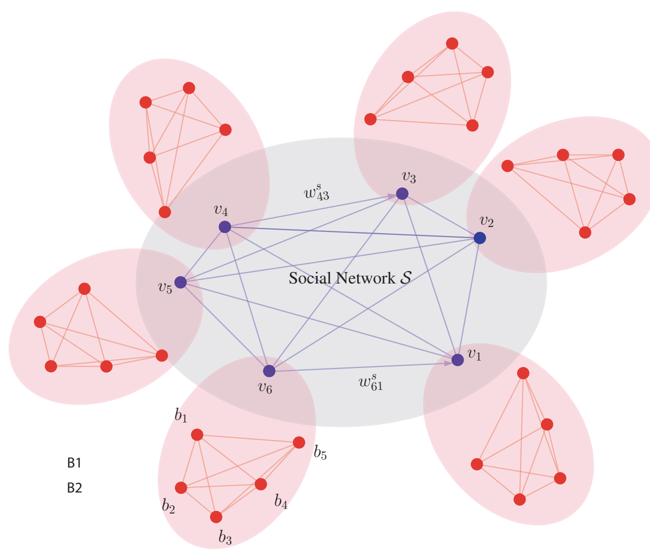


Fig. 2. Individual-based model for social behavioral networks with $n = 6$ and $N = 5$.

Remarks.

1. Individual Social Fitness. The fitness f_k^s determines how fit individual k is in propagating its inclinations in the network. The higher the value of f_k^s, the more fit v_k is in the network. This is an *individual* property. In [11], the fitness is used as the probability for choosing an individual for reproduction. Here, we take a slightly different interpretation, which is that f_k^s determines how effectively the inclination of individual v_k will be transmitted in the network.

2. Inclination- and Time-Dependent Social Connectivity. The role of the weights w_{ij}^s is to reflect the connectedness of individual v_j to individual v_i. For no matter how fit v_j is, if it is not connected or is weakly connected to v_i, very little influence will v_j have on v_i. The components of $\mathbf{W}^s$ derive from $\mathbf{A}^s$ and if the connection between v_i and v_j is stronger, the higher the influence of one's individual inclinations on the other will be. The components of the interaction matrix $\mathbf{A}^s$ may depend on several factors, especially geographic location and/or communications. These components may also be time dependent and may reflect time-dependent migratory behaviors of individuals or groups of individuals in society. Or, if one considers non-migratory dynamics, for example, one can define a measure of distance between individuals and use the inverse of that quantity to determine a_{ij}^s between two individuals v_i and v_j. With the spread in the use of the Internet and other advanced communications methods (text messaging, e-mail, online message boards, etc) as media of exchange of ideas, geographic proximity is becoming less important in social interaction problems. In such cases, one can use the magnitude of the difference between $\mathbf{x}^i$ and $\mathbf{x}^j$ as a measure of distance between v_i and v_j:

$$d(v_i, v_j) = \|\mathbf{x}_i - \mathbf{x}_j\|. \tag{9}$$

In general, one can define the distance between a pair of nodes in a manner consistent with the application at hand.

Note also that the subjective matrix of individual rewards $\mathbf{A}^j$ may also be explicitly time-dependent. This is important in studying, for example, sudden changes in the perceived rewards of the behaviors. In this case, $\mathbf{A}^j$ for a group of individuals could undergo a sudden change from one value to another in reaction to unexpected revelations about a product (that, for example, it has a faulty component) or political candidate (for example, unexpected revelations of corruption). Such studies of time-dependent events may be important for forecasting future volatility of a product or political candidate in response to unexpected (good or bad) events.

C. Social Diversity

The next task is to define diversity N_e under the model given in equation (7). First let

$$\mathbf{y} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}^k \tag{10}$$

be the *average social inclination vector*. One can then define diversity as

$$N_e = 1/\|\mathbf{y}\|^2. \tag{11}$$

Similar to the definition of diversity in [10], N_e ranges from 1 to N. If $N_e = 1$, we have a single dominant behavior and if $N_e = N$, we have complete collapse. We have the following necessary and sufficient condition for an absolutely dominant behavior.

Lemma III.2. An absolutely dominant behavior is algebraically equivalent to $\mathbf{x}^k = \mathbf{e}_i$ for all $k = 1, \ldots, n$ and some unique $i \in \{1, \ldots, N\}$.

Proof. An absolutely dominant behavior, by definition, corresponds to $N_e = 1$. This implies that $\|\mathbf{y}\|^2 = 1$ or

$$y_1^2 + y_2^2 + \cdots + y_N^2 = 1. \tag{12}$$

Note that $\mathbf{c}^{\mathrm{T}}\mathbf{y} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{c}^{\mathrm{T}}\mathbf{x}^k = 1$. Hence, we have

$$y_1 + y_2 + \cdots + y_N = 1. \tag{13}$$

Taking the square of both sides of this equation and applying the multinomial expansion we obtain

$$\begin{aligned}\left(y_1^2 + \cdots + y_N^2\right) + 2\big(&y_1y_2 + \cdots + y_1y_N \\ &+ y_2y_3 + \cdots + y_2y_N \\ &+ \cdots \\ &+ y_{N-1}y_N\big) = 1.\end{aligned}$$

Combining with equation (12) we are left with

$$\begin{aligned} y_1y_2 + \cdots + y_1y_N + y_2y_3 + \cdots + y_2y_N \\ + \cdots + y_{N-1}y_N = 0. \end{aligned} \tag{14}$$

Since $y_i \geq 0$, every term on the left hand side has to be zero. However, there can only be one nonzero y_i for otherwise one of the terms on the left hand side of the equation will be nonzero. Since $\mathbf{c}^{\mathrm{T}}\mathbf{y} = 1$, that nonzero $\mathbf{y}$ component has to be 1. Hence, we must have $\mathbf{y} = \mathbf{e}_i$. Thus we have $\mathbf{y} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}^k = \mathbf{e}_i$. Since all components of $\mathbf{x}^k$ are nonnegative, it must be that $x_j^k = 0$ for all $j \neq i$. Since $\mathbf{c}^{\mathrm{T}}\mathbf{x}^k = 1$ for all k, we must then have $x_i^k = 1$ for all k. This proves that an absolutely dominant behavior implies that $\mathbf{x}^k = \mathbf{e}_i$ for all $k = 1, \ldots, n$ and some unique $i \in \{1, \ldots, N\}$.

The converse is easily proven as follows. If $\mathbf{x}^k = \mathbf{e}_i$ for all $k = 1, \ldots, n$ and some unique $i \in \{1, \ldots, N\}$, then $\mathbf{y} = \mathbf{e}_i$ (by definition of $\mathbf{y}$), which results in $N_e = 1$ (by definition of N_e). This completes the proof. ■

It can easily be checked that if every member of the network has an inclination vector that is given by $\mathbf{x}^k = \frac{1}{N}\mathbf{c}$ for all k then $\mathbf{y} = \frac{1}{N}\mathbf{c}$ and we have total collapse with $N_e = N$. This is a sufficient condition for complete collapse. In this paper, we focus our attention on the emergence of dominant behaviors and further studies of total collapse will be studied in future work.

D. Conditions for the Emergence of Dominant Behaviors

In this section, we generalize the two main results in [10] to the modified individual-based replicator-mutator dynamics given in equation (7). We begin by giving the *balance condition* [10].

Lemma III.3 (Balance Condition)**.** Consider the modified individual replicator-mutator dynamics (7). Define the functions

$$\eta_i^k(\mathbf{x}) = \frac{x_i^k(f_i^k - \phi^k - \phi_k^s) + \sum_{j=1,\neq k}^{n} f_j^s w_{kj}^s x_i^j}{(1 - w_{ii}^k)x_i^k f_i^k - \sum_{j=1,\neq i}^{N} x_j^k f_j^k w_{ji}^k}, \tag{15}$$

where $\mathbf{x} = \left(\left(\mathbf{x}^1\right)^{\mathrm{T}}, \ldots, \left(\mathbf{x}^n\right)^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{nN}$. Let $\mathbf{x}_*$be an equilibrium of the system. The mutation rates μ^k and $\mathbf{x}_*$

must satisfy the *balance conditions*:

$$\mu^k = \eta_1^k(\mathbf{x}_*) = \eta_2^k(\mathbf{x}_*) = \cdots = \eta_N^k(\mathbf{x}_*). \tag{16}$$

Proof. First express equation (7) in coordinates as:

$$\dot{x}_i^k = \sum_{j=1}^{N} x_j^k f_j^k q_{ji}^k - \phi^k x_i^k + \sum_{j=1}^{n} f_j^s w_{kj}^s x_i^j - \phi_k^s x_i^k, \tag{17}$$

which is equivalent to:

$$\begin{aligned}\dot{x}_i^k &= \mu^k \left[\sum_{j=1,\neq i}^{N} x_j^k f_j^k w_{ji}^k - (1 - w_{ii}^k) x_i^k f_i^k \right] \\ &\quad + \sum_{j=1,\neq k}^{n} f_j^s w_{kj}^s x_i^j + x_i^k \left(f_i^k + f_k^s w_{kk}^s - \phi^k - \phi_k^s \right),\end{aligned} \tag{18}$$

where we have used the fact that $q_{ij}^k = \mu^k w_{ij}^k$ for $i \neq j$ and that $q_{ii}^k = 1 - \mu^k(1 - w_{ii}^k)$. If $\mathbf{x}_*$ is an equilibrium, then $\dot{\mathbf{x}}_*^k = 0$ and $\dot{x}_{i*}^k = 0$, $\forall i = 1, \ldots, n$. Applying this equilibrium condition to the above equation, rearranging and noting that $w_{kk}^s = 0$, one obtains the balance conditions expressed in equation (16). ■

We now derive the analogue of the main result in [10] (which was stated above in Theorem II.1), but for the modified individual-based replicator-mutator social dynamic model (7). In the following theorem, we give a necessary and sufficient condition for the emergence of an absolutely dominant behavior in the evolutionary network $\mathcal{S}$.

Theorem III.1. Let $\mathbf{x}_*$ be an equilibrium of the modified individual-based replicator-mutator social dynamic model (7), then an absolutely dominant equilibrium behavior results if and only if $\mathbf{x}_*^k = \mathbf{e}_i$ for some i (i.e., $x_{i*}^k = 1,\ x_{j*}^k = 0$ for all $j \neq i$) and for all $k = 1, \ldots, n$ and $\mu^k = 0$ for all $k = 1, \ldots, n$.

Proof. We first prove necessity. From Lemma III.2 an absolutely dominant behavior is equivalent to requiring that $\mathbf{y} = \mathbf{e}_i$ and that $\mathbf{x}^k = \mathbf{e}_i$ for some i (i.e., $x_i^k = 1,\ x_j^k = 0$ for all $j \neq i$) and for all $k = 1, \ldots, n$. We need to show that if the dominant behavior is an equilibrium, then $\mu^k = 0$ for all $k = 1, \ldots, n$. First note that an absolutely dominant behavior implies that $f_i^k = a_{ii}^k x_i^k + \sum_{j=1,\neq i}^{N} a_{ij}^k x_j^k = 1$ since $x_i^k = 1$, $x_j^k = 0$, $a_{ii}^k = 1$ and, by assumption, $\mathbf{f}_0^k = 0$. Also note that $\phi^k = \sum_{j=1}^{N} x_j^k f_j^k = 1$. Hence, and using the definition of ϕ_k^s and the facts that $x_i^k = x_i^j = 1$ and $w_{kk}^s = 0$, we have

$$\eta_i^k = \frac{-x_i^k \phi_k^s + \sum_{j=1,\neq k}^{n} f_j^s w_{kj}^s x_i^j}{(1 - w_{ii}^k)} = 0.$$

Note that $w_{ii}^k \neq 1$ due to the earlier stated assumption that $\mathbf{A}^k \neq \mathbf{I}$. Thus, we see that for $\mathbf{x}^k = \mathbf{e}_i$, for all k and some i, to be an equilibrium we must have $\mu^k = \eta_1^k = \cdots = \eta_N^k$ for all $k = 1, \ldots, n$ by virtue of the balance conditions. This proves that an absolutely dominant behavior implies that $\mu^k = 0$ in addition to the fact that $\mathbf{x}^k = \mathbf{e}_i$ for some i (i.e., $x_i^k = 1,\ x_j^k = 0$ for all $j \neq i$) and for all $k = 1, \ldots, n$ (from Lemma III.2).

The converse is easily proven as follows. With $\mu^k = 0$, equation (18) is simplified to

$$\dot{x}_i^k = \sum_{j=1,\neq k}^{n} f_j^s w_{kj}^s x_i^j + x_i^k \left(f_i^k + f_k^s w_{kk}^s - \phi^k - \phi_k^s \right).$$

Since $\mathbf{x}^k = \mathbf{e}_i$ for some i (i.e., $x_i^k = 1,\ x_j^k = 0$ for all $j \neq i$) and for all $k = 1, \ldots, n$ then (as shown above) we have $f_i^k = \phi^k = 1$, $x_i^k = x_i^j = 1$ and, thus,

$$\begin{aligned}\dot{x}_i^k &= \sum_{j=1,\neq k}^{n} f_j^s w_{kj}^s - \sum_{j=1}^{k} f_j^s w_{kj}^s = \sum_{j=1}^{n} f_j^s w_{kj}^s - \sum_{j=1}^{k} f_j^s w_{kj}^s \\ &= 0,\end{aligned}$$

where the first equality results from the fact that $w_{kk}^s = 0$. This completes the proof of the theorem. ■

For the individual-based approach, with multiple participants, the counterpart of the second statement in Theorem II.1 can be derived. However, there are multiple different scenarios under which a dominant behavior can arise. For example, for all individuals to have inclinations such that $x_i^k \approx 1 - \epsilon$ and $x_j^k \approx \epsilon$ for $j \neq i$, one can show (along the same lines of the proof of the second statement in Theorem II.1, which can be found in [10]) that $\mu^k \approx O(\epsilon)$ for a small ϵ. However, this condition is not necessary for a dominant (non-absolute) behavior to emerge. For example, (non-absolute) dominance will also emerge if a single individual has a relatively high μ^k and has its inclination degenerate to $\mathbf{x}^k \approx \frac{1}{N}\mathbf{c}$ while all other mutation parameters are small and the corresponding individuals converge to a single dominant behavior. A full analysis of the various conditions that lead to behavioral dominance and collapse will be addressed in an archival version.

IV. Relation to Previous Work

In this section, we demonstrate how the above model relate to and generalizes previous results on behavioral evolution in social networks, consensus over networks, and evolutionary dynamics on graphs.

If one sets $n = 1$, and since $w_{11}^s = 0$, then clearly the modified individual based replicator-mutator dynamics (7) reduce to the social choice model used in [10]. This single individual then models the entire population using a single mutation parameter and rewards matrix. The components of $\mathbf{x} = \mathbf{x}^k$ are now viewed as the fraction of the population adopting each behavior. This, as discussed above, is where the collective population model fails to take into account the subjective valuation of the rewards matrices as well as the diversity in the individual mutation parameters μ^k of the individuals in society.

If, on the other hand, we set $\mathbf{f}^k = 0$ and $f_j^s = 1$, the dynamic equation for each $\dot{\mathbf{x}}^k$ is now given by

$$\dot{\mathbf{x}}^k = \mathbf{W}_k^s \mathbf{x} - \mathbf{x}^k,$$

where we note that $\phi_k^s = \sum_{j=1}^{n} w_{kj}^s = 1$ since $f_j^s = 1$, and where

$$\mathbf{W}_k^s = \left[\begin{array}{cccc} w_{k1}^s \mathbf{I}_N & w_{k2}^s \mathbf{I}_N & \cdots & w_{kn}^s \mathbf{I}_N \end{array} \right].$$

Concatenating the above set of n differential equations, we obtain

$$\dot{\mathbf{x}} = \mathbf{W}^s\mathbf{x} - \mathbf{x} = -\left(\mathbf{I} - \mathbf{W}^s\right)\mathbf{x} = -\mathbf{L}^s\mathbf{x}, \tag{19}$$

where

$$\mathbf{W}^s = \begin{bmatrix} \mathbf{W}_1^s \\ \mathbf{W}_2^s \\ \vdots \\ \mathbf{W}_n^s \end{bmatrix}$$

and where $\mathbf{L}^s$ is the Laplacian matrix of the social network $\mathcal{S}$ as before. One recognizes equation (19) as the consensus algorithm [16]–[18]. Hence, in a social network where individuals have equal social fitness parameters f_k^s, the effect of the social network on the individual decision making is that of a consensus or averaging effect. With uneven fitness parameters and different edge weights, individuals within close proximity to one another, especially in the sense of the distance measure given in equation (9), would consolidate their inclinations, whereas further away members of society introduce a slight averaging effect.

The above discussion shows that the modified individual based replicator-mutator dynamics is essentially a weighted average of evolutionary decision-making and evolutionary consensus-reaching ("evolutionary consensus" is used here since the graph is dynamic and evolves according to an evolutionary replicator-mutator process).

Finally, we establish a connection with the work of Lieberman, Hauert and Nowak [11]. In their work, a graph $\mathcal{S}$ is constructed from a set of individuals v_i who are represented as the vertices of the network as above (we adapt their notation to the one we use here). Each individual has a fitness f_k^s and the graph has a weighted interaction matrix $\mathbf{W}^s$. In [11], the authors seek to find "the probability that a newly introduced mutant generates a lineage that takes over the whole population". In other words, they assume that there are only two behaviors ($N = 2$), one being the prevalent behavior, say b_2, that is assumed by all individuals except one, and a mutant behavior b_1 possessed by a single individual. In their model, Lieberman et al. view f_k^s as the probability that individual v_k is chosen for reproduction (i.e., for transmitting v_k's behavior to other individuals) and w_{ki}^s as the probability that the behavior of individual v_k be transmitted to individual v_i and replace v_i's behavior with that of v_k's. Replicator-mutator dynamics are not considered in their work, and no individual decision-making (or inclination building) model is employed. That is, each individual is endowed with a fixed behavior that can only be replaced by another behavior from a neighboring individual. The authors study the behavior of a single mutant behavior b_1, possessed only by v_1, in different kinds of graph structures, including the star structure, the super-star structure, the funnel, the meta-funnel, and other extensions (see [11] for definitions). It is argued in [11] that the star structure shown in Figure 3 acts as an evolutionary amplifier that favors advantageous mutants and that inhibits disadvantageous mutants. We will verify that this property holds under the modified individual-based evolutionary modeled proposed in this paper. Further rigorous analytical verification of these results, which are beyond the scope of this paper, will be included in a future archival version of this paper.

In the star structure the individual with the mutant behavior is located at the center of the graph and is assigned the node v_1. In the star structure we have $a_{1j}^s = a_{j1}^s = 1$ and $a_{ij}^s = 0$ if neither $i = 1$ nor $j = 1$. Recall that we have only two behaviors, with b_1 representing the mutant behavior associated with v_1. Hence, we set a higher reward for the fist behavior for individual v_1 with

$$\mathbf{A}^1 = \begin{bmatrix} 1 & 0.1 \\ 10 & 1 \end{bmatrix}.$$

Letting $n = 9$, the reward matrices for the remaining individuals with the dominant behavior b_2 is assumed to be the same for $v_j,\ j = 2, \ldots, 9$:

$$\mathbf{A}^j = \begin{bmatrix} 1 & 10 \\ 0.1 & 1 \end{bmatrix},\ j = 2, \ldots, 9.$$

It will be demonstrated that both the mutation parameters and the social fitness of the individuals affect the behavioral outcome over the network. In the first simulation we set $\mu^k = 0.01$ and $f_k^s = 1$ for all $k = 1, \ldots, 9$. The average social inclination vector $\mathbf{y}$ is shown in Figure 4(a). We see here that the slightly dominant behavior is b_2 with $N_e = 1.7275$ at steady state. In other words, the mutant behavior introduced by v_1 is not amplified even though v_1 interacts with all other individuals in the network and the latter is not able to consolidate its dominant behavior.

Next, we show the impact of increasing the mutation rate of the dominant individuals (ones having a higher reward for the dominant behavior b_2). In this case we set $\mu^k = 2$ for all $k = 2, \ldots, 9$. The result is shown in Figure 4(b). By simply increasing the mutation rate of the individuals with the initially dominant behavior gives a social advantage for the mutant b_1, where the diversity is now $N_e = 1.0370$, heavily leaning towards the mutant b_1. In the third case, we use the same parameters as in the first case (case (a)), except that we only change the fitness parameter of v_1 to be 10 instead of 1. Hence, v_1 is socially more fit than its neighbors, though all have equal mutation parameters of 0.01. The result is shown in Figure 4(c), where we see that there is a strong advantage of the mutant behavior b_1 over the previously dominant behavior b_2. Finally, Figure 4(d) shows the impact of having both a lower mutation parameter ($\mu^1 = 0.01,\ \mu^j = 2, j = 2, \ldots, 9$) and a higher social fitness ($f_1^s = 10,\ f_j^s = 1, j = 2, \ldots, 9$) on the emergence of a previously non-dominant behavior b_1 which is now strongly dominant with $N_e = 1.0318$. *These numerical examples show that both the fitness of the individual carrying a mutant and the mutation rate of the individuals in society can result in the evolutionary amplification of advantageous mutants and suppression of disadvantageous mutants.*

V. Conclusion

Recognizing the fact that, in a social network, individuals make subjective valuation of the different available behaviors and arrive at individual inclinations towards them, in this

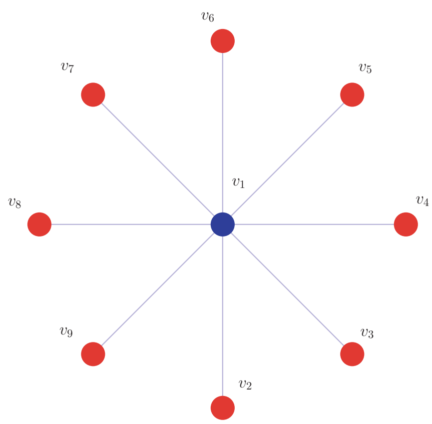


Fig. 3. The star structure with $n = 9$ and $N = 2$.

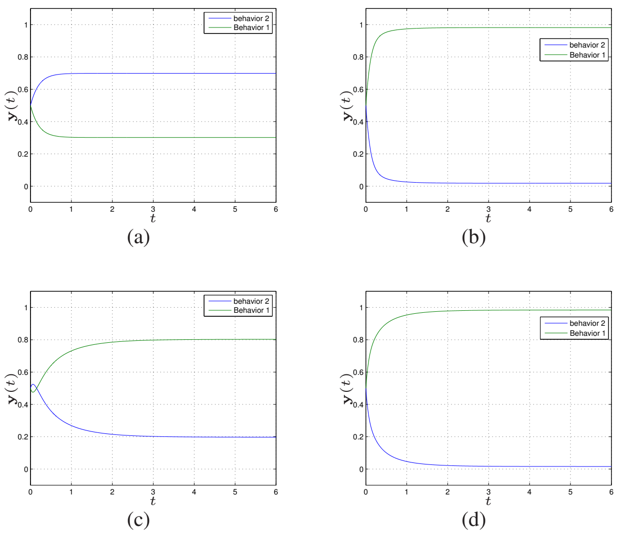


Fig. 4. Response of the star structure.

paper we develop an individual-based, replicator-mutator dynamics that model how an individual makes choices. Based on this individual choice model, we introduce the effect of a social network on an individual's choice through an interaction network whose vertices represent the individual decision-makers and whose edges measure the proximity of the individuals. Some basic properties and implications of the model were shown, and how the individual-based replicator-mutator dynamics with social interaction relate to previous results on behavioral evolution in social networks, consensus over networks, and evolutionary dynamics on graphs was also discussed. Current work focuses on further analysis of the individual-based replicator-mutator model and using it in resource allocation and control of informational and multi-agent robotic systems.

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