2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009



# Stabilizing Model Predictive Control for LPV Systems Subject to Constraints with Parameter-Dependent Control Law

Shuyou Yu, Christoph Böhm, Hong Chen, Frank Allgöwer

*Abstract*— **This paper presents an infinite horizon model predictive control (MPC) scheme for constrained linear parameter-varying systems. We assume that the time-varying parameter can be measured online and exploited for feedback. The proposed method is based on a parameter-dependent control law which is obtained via the repeated solution of a convex optimization problem involving linear matrix inequalities (LMIs). Closed-loop stability is guaranteed by the feasibility of the LMIs at initial time. Compared to existing algorithms with static linear control law and more restrictive LMI conditions, the proposed scheme reduces conservatism and improves performance, which is confirmed by a simulation example.**

## I. Introduction

Linear parameter-varying (LPV) systems play an important role in both control theory and application. LPV systems represent a class of nonlinear systems which can be controlled using linear-like control techniques. This explains that in numerous practical control problems LPV systems are used for controller design as e.g. in automotive [10, 11] and aerospace [8, 18] applications. In the field of control theory many research activities have focussed on the development of control methods for LPV systems in the past, see for example the results presented [1, 2, 14, 17, 23–25] for an overview.

Since model-predictive control (MPC) has well-known advantageous properties such as optimal solutions with respect to the considered cost function and guaranteed satisfaction of state and input constraints, see e.g. [6] and [7], clearly also several MPC schemes that are able to deal with LPV systems have been published in the literature [4, 5, 13, 15, 16, 19–22]. In most of those methods the control law is calculated by repeatedly solving a convex optimization problem based on linear matrix inequalities (LMIs) such that an upper bound of a worst-case cost function is minimized. The approaches [5] and [13] have not explicitly been developed for LPV systems and therefore suffer from rather conservative LMI conditions that have to be satisfied. However, they are a suitable choice as MPC controllers for LPV systems since they robustly stabilize an LPV system for all possible parameter variations. The controllers suggested in [19] and [21] are restricted to LPV systems with bounded rates of parameter variation. Those approaches are not applicable to the case considered in this paper where we assume that the parameters may vary arbitrarily within a given set. The approach presented in [20] assumes the parameter to be measurable in real-time. This knowledge on the parameter allows to obtain in the first step an exact prediction of the future system behavior and therefore reduced conservatism. However, as discussed in [3] feasibility of the optimization problem cannot be guaranteed. In the MPC controllers proposed in [15] and [16], the control law is independent of the system parameter. As [5] and [13] those approaches robustly stabilize the considered LPV system. Thus, if the parameter is measurable, this knowledge cannot be exploited. We will show in this paper that the incorporation of the parameter measurement in the control law may reduce conservatism and improve the controller performance. A solution involving the parameter measurement in the controller design is suggested in [4]. However, this approach relies on conservative LMI conditions. As will be shown those conditions can be relaxed using results presented in [9],[12],[26] and [27].

The goal of this work is to derive a computationally attractive MPC controller with guaranteed closed-loop stability for discrete-time LPV systems subject to state and input constraints. The control law is calculated efficiently via solving a convex optimization problem at each sampling instant such that an upper bound of an infinite horizon worst-case cost function is minimized. The obtained LMI conditions are less restrictive than those of comparable approaches, as for example [4, 5, 13]. Furthermore, the solution to the optimization problem delivers a control law which depends on the time-varying system parameter, which is assumed to be measurable in real-time. The exploitation of this knowledge on the parameter in the controller design in combination with the relaxed LMI conditions reduces the conservatism and improves controller performance when compared to many MPC approaches for LPV systems, as for example [4, 5, 13].

The paper is organized as follows: After a short overview on the notation used in the paper the following section will introduce the considered system class, namely discrete-time LPV systems, and present the MPC problem setup. Section III derives the main result of this paper which is a novel, stabilizing MPC controller for LPV systems. The parameter-dependent control law is calculated via the solution of a convex optimization problem based on LMI conditions which are less conservative compared to existing MPC approaches for LPV systems [4, 5, 13]. In Section IV we apply the proposed controller to a simulation example and compare the obtained performance with existing MPC schemes. It is shown that controller performance can be improved significantly by our approach. Section V concludes the paper with a brief summary.

S. Yu, C. Böhm, and F. Allgöwer are with the Institute of Systems Theory and Automatic Control, University of Stuttgart, Germany. {shuyou,cboehm,allgower}@ist.uni-stuttgart.de.
S. Yu and H. Chen are with the Jilin University, China. chenh@jlu.edu.cn.

978-1-4244-4524-0/09/$25.00 ©2009 AACC 

## A. Notation

We denote $\psi_{i,k}$ as the $i$-th element of the vector $\psi_k$. The expression $x_{k+v|k}$ ($u_{k+v|k}$) denotes the predicted state $x$ (input $u$) at the time instant $k+v$, where the prediction has been calculated at the sampling instant $k$. With $I$ and $0$ we denote an identity matrix and a zero matrix, respectively, of suitable dimension. The vectors $e_m$, $m = 1, \ldots, m_{max}$, represent the column vectors of an identity matrix of dimension $m_{max} \times m_{max}$. With the expression $Co\{F_1, \ldots, F_N\}$ we denote the convex hull of the $N$ matrices $F_1, \ldots, F_N$.

# II. Problem Setup

Consider discrete-time linear parameter-varying (LPV) systems of the form

$$\begin{aligned} x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k, && \text{(1a)} \\ z_k &= C(\theta_k)x_k + D(\theta_k)u_k, && \text{(1b)} \end{aligned}$$

subject to

$$-z_{m,max} \leq z_{m,k} \leq z_{m,max}, \quad m = 1, 2, \ldots, n_z, \tag{2}$$

where $x_k \in R^{n_x}$ denotes the system states, $u_k \in R^{n_u}$ is the control input, and $z_k \in R^{n_z}$ denotes the constraints output vector, which is not necessarily measurable. The constant vector $z_{max}$ defines the state and input constraints for system (1). The system matrices $A(\theta_k) \in \mathbb{R}^{n_x \times n_x}$, $B(\theta_k) \in \mathbb{R}^{n_x \times n_u}$, $C(\theta_k) \in \mathbb{R}^{n_z \times n_x}$ and $D(\theta_k) \in \mathbb{R}^{n_z \times n_u}$ are assumed to depend on the parameter vector $\theta_k := [\theta_{1,k}, \theta_{2,k}, \cdots, \theta_{N,k}]^T \in \mathbb{R}^N$, which belongs to a convex polytope $\mathcal{P}$ defined by

$$\sum_{j=1}^{N} \theta_{j,k} = 1, \qquad 0 \leq \theta_{j,k} \leq 1. \tag{3}$$

We assume that the parameter $\theta_k$ can be measured online. Clearly, as $\theta_k$ varies inside the polytope $\mathcal{P}$, the matrices of system (1) vary inside a corresponding polytope $\Omega$

$$\begin{bmatrix} A(\theta_k) & B(\theta_k) \\ C(\theta_k) & D(\theta_k) \end{bmatrix} \in \Omega, \tag{4}$$

which is defined by the convex hull of $N$ local extremal matrices $[A_i, B_i, C_i, D_i]$, $i = 1, 2, \cdots, N$,

$$\Omega := Co\left\{ \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}, \ldots, \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix} \right\}. \tag{5}$$

Therefore, we can write the matrices of system (1) as

$$\begin{aligned} A(\theta_k) &= \sum_{j=1}^{N} \theta_{j,k} A_j, \qquad B(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} B_j, \\ C(\theta_k) &= \sum_{j=1}^{N} \theta_{j,k} C_j, \qquad D(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} D_j. \end{aligned}$$

The control task is to stabilize the origin of system (1) with a model predictive controller such that the constraints (2) are satisfied. The MPC controller will be derived such that an upper bound on the infinite horizon cost function

$$J_{\infty|k} = \max_{\theta \in \mathcal{P}} \sum_{v=0}^{\infty} \left\{ x_{k+v|k}^T Q x_{k+v|k} + u_{k+v|k}^T R u_{k+v|k} \right\} \tag{6}$$

is minimized at each sampling instant $k$ based on a prediction of the system behavior into the future. In the considered cost function $Q > 0$ and $R > 0$ are weighting matrices of suitable dimension. Throughout this paper we assume that the full state $x_k$ is measurable in real-time. Since we also measure the parameter $\theta_k$, at every sampling instant $k$ the current system matrices are known exactly. However, all future systems matrices are uncertain and vary inside the polytope $\Omega$ since we cannot predict the future behavior of the system parameter $\theta_{k+v|k}$, $v = 1, \ldots, \infty$. Therefore, in the cost function (6) the worst case over all possible future parameters has to be considered.

In the following section we derive an MPC controller based on the parameter-dependent control law

$$u_k = K(\theta_k)x_k, \tag{7}$$

which is updated at each sampling instant via the minimization of an upper bound on cost function (6). The parameter dependency allows more degree of freedom in the controller design and leads to less restrictive LMI conditions in the optimization problem.

# III. MPC using linear parameter-dependent feedback law

In this section, we propose a new model predictive controller for system (1) subject to the constraints (2) by using a parameter-dependent state feedback control law, which is obtained via the solution of a convex optimization problem. The conservatism of the LMI conditions inherent to this optimization problem is reduced following the ideas presented in [26] and [27]. In combination with the parameter dependency of the feedback law the obtained LMI conditions provide more degree of freedom in the controller design such that the obtained controller reduces the conservatism of the methods proposed in [4, 5, 13].

Suppose that $K_j \in \mathbb{R}^{m \times n}$ is a time-invariant feedback gain of the $j$-th vertex system. A suitable, parameter-dependent feedback law for the whole LPV system is obtained via the weighted average of the control laws designed for each vertex

$$K(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} K_j. \tag{8}$$

Using control law (7), for system (1) we obtain the closed-loop representation

$$\begin{aligned} x_{k+1} &= A_{cl}(\theta_k)x_k && \text{(9a)} \\ z_k &= C_{cl}(\theta_k)x_k. && \text{(9b)} \end{aligned}$$

where the system matrices $A_{cl}(\theta_k)$ and $C_{cl}(\theta_k)$ are given by

$$\begin{aligned} A_{cl}(\theta_k) &= \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k}\theta_{j,k}(A_i + B_i K_j), && \text{(10a)} \\ C_{cl}(\theta_k) &= \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k}\theta_{j,k}(C_i + D_i K_j). && \text{(10b)} \end{aligned}$$

The following theorem derives conditions to obtain an upper bound on cost function (6) using the system description (10).

**Theorem 1:** Suppose that there exist a symmetric, positive definite matrix $X_k \in \mathbb{R}^{n_x \times n_x}$, matrices $Y_{1,k}, Y_{2,k}, \dots, Y_{N,k} \in \mathbb{R}^{n_u \times n_x}$, and a constant $\gamma_k \in \mathbb{R}^+$ such that the optimization problem at time instant $k$

$$\min_{\gamma_k, X_k, Y_{1,k}, Y_{2,k}, \dots, Y_{N,k}} \gamma_k \tag{11a}$$

subject to

$$\begin{bmatrix} 1 & x_k^T \\ x_k & X_k \end{bmatrix} \geq 0, \tag{11b}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k+v|k} \theta_{j,k+v|k} L_{ij} > 0, \tag{11c}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k+v|k} \theta_{j,k+v|k} F_{ij,m} \geq 0, \; m = 1, 2, \dots, n_z, \tag{11d}$$

with the matrices

$$L_{ij} = \begin{bmatrix} X_k & * & * & * \\ A_i X_k + B_i Y_{j,k} & X_k & * & * \\ Q^{\frac{1}{2}} X_k & 0 & \gamma_k I & * \\ R^{\frac{1}{2}} Y_{j,k} & 0 & 0 & \gamma_k I \end{bmatrix}, \tag{11e}$$

$$F_{ij,m} = \begin{bmatrix} z_{m,max}^2 & e_m^T (C_i X_k + D_i Y_{j,k}) \\ * & X_k \end{bmatrix}, \tag{11f}$$

has a feasible solution which holds for all $\theta_{k+v|k} \in \mathcal{P}$, $v = 1, \dots, \infty$, where $x_k$ is the measured system state at the sampling instant $k$. Then, with $P_k = \gamma_k X_k^{-1}$, $K_{j,k} = Y_{j,k} X_k^{-1}$, $j = 1, \dots, N$, and with the parameter-dependent control law

$$u_{k+v|k} = K(\theta_{k+v|k}) x_{k+v|k}, \tag{12}$$

where $K(\theta_{k+v|k}) = \sum_{j=1}^{N} \theta_{j,k+v|k} K_{j,k}$, the following holds:

(a) The predicted states $x_{k+v|k}$ with $x_{k|k} = x_k$ converge to the origin as $v \to \infty$.
(b) The expression $V_k = x_k^T P_k x_k$ is minimized and represents an upper bound on cost function (6) at the sampling instant $k$.
(c) The predicted states $x_{k+v|k}$ and inputs $u_{k+v|k}$ satisfy the constraints (2).

*Proof: The proof is divided into three parts in order to show separately that the properties (a)-(c) hold.*

*Part (a): Multiplying (11c) from the left and from the right with diag$\{X_k^{-1}, I, I, I\}$ and substituting $P_k = \gamma_k X_k^{-1}$, $K_{j,k} = Y_{j,k} X_k^{-1}$, we obtain that*

$$\begin{bmatrix} \gamma_k^{-1} P_k & * & * & * \\ A_{cl}(\theta_{k+v|k}) & \gamma_k P_k^{-1} & * & * \\ Q^{\frac{1}{2}} & 0 & \gamma_k I & * \\ R^{\frac{1}{2}} K(\theta_{k+v|k}) & 0 & 0 & \gamma_k I \end{bmatrix} \geq 0, \tag{13}$$

*holds for all $\theta_{k+v|k} \in \mathcal{P}, v = 0, \dots, \infty$. By the Schur complement this is equivalent to*

$$\begin{aligned} A_{cl}^T(\theta_{k+v|k}) P_k A_{cl}(\theta_{k+v|k}) - P_k & \\ + Q + K(\theta_{k+v|k})^T R K(\theta_{k+v|k}) & \leq 0. \end{aligned} \tag{14}$$

*Multiplying from both sides with $x_{k+v|k}^T$ and $x_{k+v|k}$, respectively, plugging in the system dynamics (1) and using (12), it follows that the inequality*

$$\begin{aligned} x_{k+v+1|k}^T P_k x_{k+v+1|k} - x_{k+v|k}^T P_k x_{k+v|k} & \\ + x_{k+v|k}^T Q x_{k+v|k} + u_{k+v|k}^T R u_{k+v|k} & \leq 0 \end{aligned} \tag{15}$$

*is satisfied. Since $Q > 0$ and $R > 0$, clearly $V_{k+v|k} = x_{k+v|k}^T P_k x_{k+v|k}$ is a Lyapunov function and therefore the predicted states $x_{k+v|k}$ converge to zero as $v \to \infty$.*

*Part (b): Using $x_{k+v|k} \to 0$ for $v \to \infty$, by summing up (15) from $v = 0$ to $v = \infty$ we obtain*

$$x_{k|k}^T P_k x_{k|k} \geq \sum_{v=0}^{\infty} x_{k+v|k}^T Q x_{k+v|k} + u_{k+v|k}^T R u_{k+v|k}. \tag{16}$$

*Since this inequality is satisfied for all $\theta_{k+v|k} \in \mathcal{P}$, $v = 1, \dots, \infty$, with $x_{k|k} = x_k$ it follows that*

$$V_k = x_k^T P_k x_k \geq J_{\infty|k}. \tag{17}$$

*Thus, $V_k$ is an upper bound on cost function (6) at the sampling instant k. Applying the Schur complement on (11b) and substituting $P_k = \gamma_k X_k^{-1}$ we conclude that*

$$x_k^T P_k x_k = V_k \leq \gamma_k \tag{18}$$

*holds. Thus, minimizing $\gamma_k$ implies the minimization of $V_k$, see [13] for details.*

*Part (c): The predicted states and inputs clearly satisfy the constraints (2) if*

$$x_{k+v|k}^T C_{cl}^T(\theta_{k+v|k}) e_m e_m^T C_{cl}(\theta_{k+v|k}) x_{k+v|k} \leq z_{m,max}^2, \tag{19}$$

*$m = 1, 2, \dots, n_z$, holds for all $\theta_{k+v|k} \in \mathcal{P}$ and all $v \geq 0$. It follows from (15) and (18) that*

$$x_{k+v|k}^T P_k x_{k+v|k} \leq \gamma_k, \; \forall v \geq 0. \tag{20}$$

*Thus, inequality (19) is satisfied if*

$$\frac{x_{k+v|k}^T C_{cl}^T(\theta_{k+v|k}) e_m e_m^T C_{cl}(\theta_{k+v|k}) x_{k+v|k}}{z_{m,max}^2} - \frac{x_{k+v|k}^T P_k x_{k+v|k}}{\gamma_k} \leq 0 \tag{21}$$

*holds, which is clearly the case if*

$$\frac{P_k}{\gamma_k} - \frac{C_{cl}^T(\theta_{k+v|k}) e_m e_k^T C_{cl}(\theta_{k+v|k})}{z_{m,max}^2} \geq 0, \tag{22}$$

*$m = 1, 2, \dots, n_z$, holds for all $\theta_{k+v|k} \in \mathcal{P}$ and all $v \geq 0$. Using the definition of $C_{cl}(\theta_{k+v|k})$ in (10b), with standard modifications we obtain (11d). Thus, satisfaction of the matrix inequalities (11d) implies that (19) holds, and therefore, the predicted states and inputs satisfy the constraints (2).* □

*Remark 3.1:* Note that in Theorem 1 for simplicity of notation we have skipped the index $k$ in the matrices $L_{ij}$ and $F_{ij,m}$. It is clear from the definition of those matrices in (11e) and (11f) that they change with $k$ since they depend on $X_k$ and $Y_{j,k}$.

Theorem 1 gives conditions for the minimization of an upper bound on the infinite horizon cost function (6). However, the matrix inequalities (11c) and (11d) depend on the unknown future parameter $\theta_{k+v|k}$. This makes it impossible to find a solution to the optimization problem (11) in Theorem 1. The following lemma gives conditions to reformulate the conditions of Theorem 1 in terms of LMIs, which allow the calculation of the solution to the optimization problem.

*Lemma 1:* [9, 12] If there exist matrices $\Lambda_{ij} = \Lambda_{ji}^T$, $i = 1, \ldots, N$, $j = 1, \ldots, N$, such that the LMIs

$$\Gamma_{ii} \geq \Lambda_{ii}, \quad i = 1, \ldots, N, \tag{23a}$$

$$\Gamma_{ij} + \Gamma_{ji} \geq \Lambda_{ij} + \Lambda_{ij}^T, \quad i = 1, \ldots, N, \ j < i, \tag{23b}$$

$$[\Lambda_{ij}]_{N \times N} \geq 0, \tag{23c}$$

are satisfied, where

$$[\Lambda_{ij}]_{N \times N} = \begin{bmatrix} \Lambda_{11} & \cdots & \Lambda_{1N} \\ \vdots & \ddots & \vdots \\ \Lambda_{N1} & \cdots & \Lambda_{NN} \end{bmatrix}, \tag{24}$$

then with $\alpha_{i,k} \geq 0$, $\sum_{i=1}^{N} \alpha_{i,k} = 1 \ \forall k$, the parameter-dependent matrix inequalities

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{i,k} \alpha_{j,k} \Gamma_{ij} \geq 0, \tag{25}$$

are satisfied for all $k$.

Lemma 1 allows us to formulate LMI conditions as in (23) such that a parameter-dependent matrix inequality of the form (25) is satisfied. This can be used to reformulate the optimization problem (11) in Theorem 1 in terms of LMIs. In the following theorem, which derives the main result of this paper, namely a novel, computationally attractive MPC controller for LPV systems subject to constraints with guaranteed stability and reduced conservatism, the parameter-dependent matrix inequalities (11c) and (11d) are reformulated as LMIs by applying Lemma 1.

**Theorem 2:** Consider the LPV system (1) subject to the constraints (2) and the cost function (6). The MPC controller with the optimization problem

$$\min_{\gamma_k, X_k, Y_{1,k}, Y_{2,k}, \ldots, Y_{N,k}, T_{ij}, S_{ij}} \gamma_k, \tag{26a}$$

subject to

$$\begin{bmatrix} 1 & x_k^T \\ x_k & X_k \end{bmatrix} \geq 0, \tag{26b}$$

$$L_{ii} \geq T_{ii}, \quad i = 1, 2, \cdots, N, \tag{26c}$$

$$L_{ij} + L_{ji} \geq T_{ij} + T_{ij}^T, \ i = 1, \ldots, N, \ j < i, \tag{26d}$$

$$[T_{ij}]_{N \times N} \geq 0, \tag{26e}$$

$$F_{ii,m} \geq S_{ii,m}, \quad i = 1, 2, \cdots, N, \tag{26f}$$

$$F_{ij,m} + F_{ji,m} \geq S_{ij,m} + S_{ij,m}^T, \ i = 1, \ldots, N, \ j < i, \tag{26g}$$

$$[S_{ij,m}]_{N \times N} \geq 0, \tag{26h}$$

that is solved repeatedly at each sampling instant $k$ based on the state measurement $x_k$, and where $L_{ij}$ and $F_{ij,m}$ are as defined in Theorem 1, has the following properties with $P_k = \gamma_k X_k^{-1}$ and $K_{j,k} = Y_{j,k} X_k^{-1}$, $j = 1, \ldots, N$:

(a) The optimization problem (26) is convex. Furthermore, it is feasible at the sampling instant $k+1$ if it is feasible at the sampling instant $k$.

(b) The solution to the optimization problem (26) minimizes the upper bound $V_k = x_k^T P_k x_k$ on cost function (6) at each sampling instant $k$.

(c) If the optimization problem (26) is initially feasible, the control law

$$u_k = K(\theta_k) x_k = K(\theta_{k|k}) x_k, \tag{27}$$

asymptotically stabilizes the origin of system (1), where $K(\theta_{k|k})$ is the first part of the optimal feedback sequence $K(\theta_{k+v|k}) = \sum_{j=1}^{N} \theta_{j,k+v|k} K_{j,k}$, $v = 0, \ldots, \infty$, calculated at the sampling instant $k$.

(d) The MPC control law (27) is such that the input and state constraints (2) are satisfied for all $k$.

*Proof: The proof is divided into four parts in order to show separately that the properties (a)-(d) hold.*

*Part (a): It is trivial to show that the optimization problem is convex since the conditions (26b)-(26h) are LMI conditions. By applying Lemma 1 to the LMIs (26c)-(26h) it can be shown that the solution to the optimization problem (26) at the sampling instant $k$ has the same properties as the solution to the optimization problem (11) in Theorem 1. Thus, it follows from (15) that*

$$x_{k+1|k}^T P_k x_{k+1|k} < x_{k|k}^T P_k x_{k|k} \tag{28}$$

*is satisfied for all $k$. The first part of the input sequence $u_{k+v|k} = K(\theta_{k+v|k}) x_{k+v|k}$ predicted at the sampling instant $k$ is applied to the real system, i.e. $u_k = K(\theta_k) x_k = K(\theta_{k|k}) x_{k|k} = u_{k|k}$. Furthermore, no model plant mismatch is present, i.e. $x_{k+1|k} = x_{k+1}$. Thus, it follows from (28) that*

$$x_{k+1}^T P_k x_{k+1} < x_k^T P_k x_k \tag{29}$$

*holds for all $k$. This implies that the solution to the optimization problem (26) at the sampling instant $k$ satisfies the LMIs (26b)-(26h) at the sampling instant $k+1$ and therefore is a feasible solution to the optimization problem (26) at sampling instant $k+1$. It follows by induction that initial feasibility implies feasibility at all future sampling instants.*

*Part (b): This property follows directly from the proof of Theorem 1.*

*Part (c): It follows from part (a) that the feedback law $K(\theta_k)$ and the matrix $P_k$ can be calculated at each sampling instant $k$ if the optimization problem is feasible at the first sampling instant. Under this assumption the expression $V_{k+1} = x_{k+1}^T P_{k+1} x_{k+1}$ is minimized at the sampling instant $k+1$. Since $P_k$ is a feasible, however suboptimal solution to the optimization problem (26) at $k+1$, with (29) it follows that*

$$x_{k+1}^T P_{k+1} x_{k+1} \leq x_{k+1}^T P_k x_{k+1} < x_k^T P_k x_k \tag{30}$$

*holds for all $k$. Clearly, $V_k = x_k^T P_k x_k$ is a Lyapunov function and thus, system (1) is asymptotically stabilized by the control law (27).*

*Part (d): It follows from the proof of Theorem 1 that at each sampling instant $k$ the predicted state and input trajectories $x_{k+v|k}$ and $u_{k+v|k}$ satisfy the constraints (2) for all $v \geq 0$. Since $u_k = u_{k|k}$ and $x_{k+1|k} = x_{k+1}$, this clearly implies satisfaction of the constraints (2) for all $k$.* $\square$

The proposed MPC controller is less conservative than those suggested in [4, 5, 13]. For example, the solution to the optimization problem in [4] and [13] would have to satisfy the condition $L_{ij} > 0$, $\forall i, j = 1, \ldots, N$. Here, this condition is relaxed by the parameter-dependent matrix inequality (11c) which is satisfied according to Lemma 1 if the (less restrictive) LMIs (26c)-(26e) hold. Furthermore, the exploitation of the measurable parameter $\theta_k$ in the feedback law (27) reduces conservatism of the schemes presented in [5] and [13].

In the following section we apply the proposed MPC controller to a simulation example which demonstrates the improvements obtained compared to the controllers suggested in [5] and [13].

## IV. SIMULATION EXAMPLE

To illustrate the effectiveness of the proposed approach we consider, as in [5] and [13], the two-mass-spring model (obtained from the continuous time model using a first-order Euler approximation with sampling time $\delta = 0.1s$)

$$x_{k+1} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.1\frac{\mu}{m_1} & 0.1\frac{\mu}{m_1} & 1 & 0 \\ 0.1\frac{\mu}{m_2} & -0.1\frac{\mu}{m_2} & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0 \\ \frac{0.1}{m_1} \\ 0 \end{bmatrix} u_k \quad (31)$$

where $m_1$ and $m_2$ are the two masses and $\mu$ is the spring constant. The positions of the masses are represented by the states $x_{1,k}$ and $x_{2,k}$, whereas $x_{3,k}$ and $x_{4,k}$ describe their velocities. For the simulation the constant masses $m_1 = 1$ and $m_2 = 1$ have been chosen. The spring constant has been assumed to be a time-varying function of the sampling instant $k$

$$\mu_k = 5.25 - 4.75 \sin(0.5k). \quad (32)$$

Note that as in [5] and [13] $\mu_k \in [0.5, 10]$. Introducing the parameters $\theta_{1,k} = 1 - \frac{\mu_k - 0.5}{9.5}$ and $\theta_{2,k} = 1 - \theta_{1,k}$ system (31) can be written in the form as considered in this paper, i.e. the parameters $\theta_{i,k}$, $i = 1, 2$, satisfy condition (3) and the matrices $A_i$ and $B_i = B$, $i = 1, 2$, are as follows:

$$A_1 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.05 & 0.05 & 1 & 0 \\ 0.05 & -0.05 & 0 & 1 \end{bmatrix}, \quad (33)$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}. \quad (34)$$

The control objective is to steer the example system (31) from the initial condition $x_0 = [1\ 1\ 0\ 0]^T$ to the origin while satisfying the input constraint $|u_k| \leq 0.05$ for all $k$. Since only one input constraint is considered, we obtain $n_z = 1$ and the matrices $C = [0\ 0]$ and $D = 1$ which are independent of the parameter $\theta_k$. We have applied both approaches suggested in [5] and [13] to illustrate the reduced conservatism and the improved performance obtained by the MPC controller proposed in this paper. In the simulation the matrices of the infinite horizon cost function (6) have been chosen as $Q = I \in \mathbb{R}^{4 \times 4}$ and $R = 1$.

Figure 1 shows the obtained simulation results. Compared to the MPC approaches [5] and [13] the proposed MPC controller steers the example system significantly faster to the origin. The behavior of the input $u_k$ shows that the novel controller is able to react more efficiently on the varying parameter $\theta_k$. This results from the parameter-dependency of the feedback law and from the less conservative LMI conditions in the optimization problem. The reduced conservatism of the proposed controller is also illustrated well by the behavior of $\gamma_k$ which represents the minimized upper bound on the worst-case cost function. Figure 1 clearly shows that with the novel approach a significantly smaller upper bound can be calculated at each sampling instant $k$. Thus, the obtained feedback law is closer to the optimal solution that would be obtained if the optimization problem could be solved analytically.

To summarize, the application of the proposed MPC controller to the example system shows, that the parameter-dependency of the feedback law and the reduced conservatism of the LMI conditions can lead to significant performance improvements when compared to the MPC approaches [5] and [13].

## V. CONCLUSIONS

In this paper a novel, computationally attractive MPC approach for linear parameter-varying systems has been derived. The control law, which depends on the measurable system parameter, is the solution to a convex optimization problem based on linear matrix inequalities that is solved repeatedly at each sampling instant. If the optimization problem is initially feasible, the approach guarantees closed-loop stability and satisfaction of state and input constraints. The obtained optimal solution minimizes an upper bound on the considered worst-case infinite horizon cost function. It has been shown that, due to the parameter-dependency of the control law and due to relaxed LMI conditions, the novel approach reduces the conservatism of the well-known MPC methods presented in [5] and [13]. The obtained results have been confirmed by a simulation example.

## REFERENCES


[1] F. Blanchini and S. Miani. Stabilization of LPV systems: state feedback, state estimation and duality. In *Proceedings of the 42nd IEEE Conference on Decision and Control CDC*, Maui, Hawaii, 2003.

[2] P.A. Bliman. Stabilization of LPV systems. In D. Henrion and A. Garulli, editors, *Positive Polynomials in Control*, volume 312 of *Lecture Notes in Control and Information Sciences*, pages 103–117. Springer, 2005.

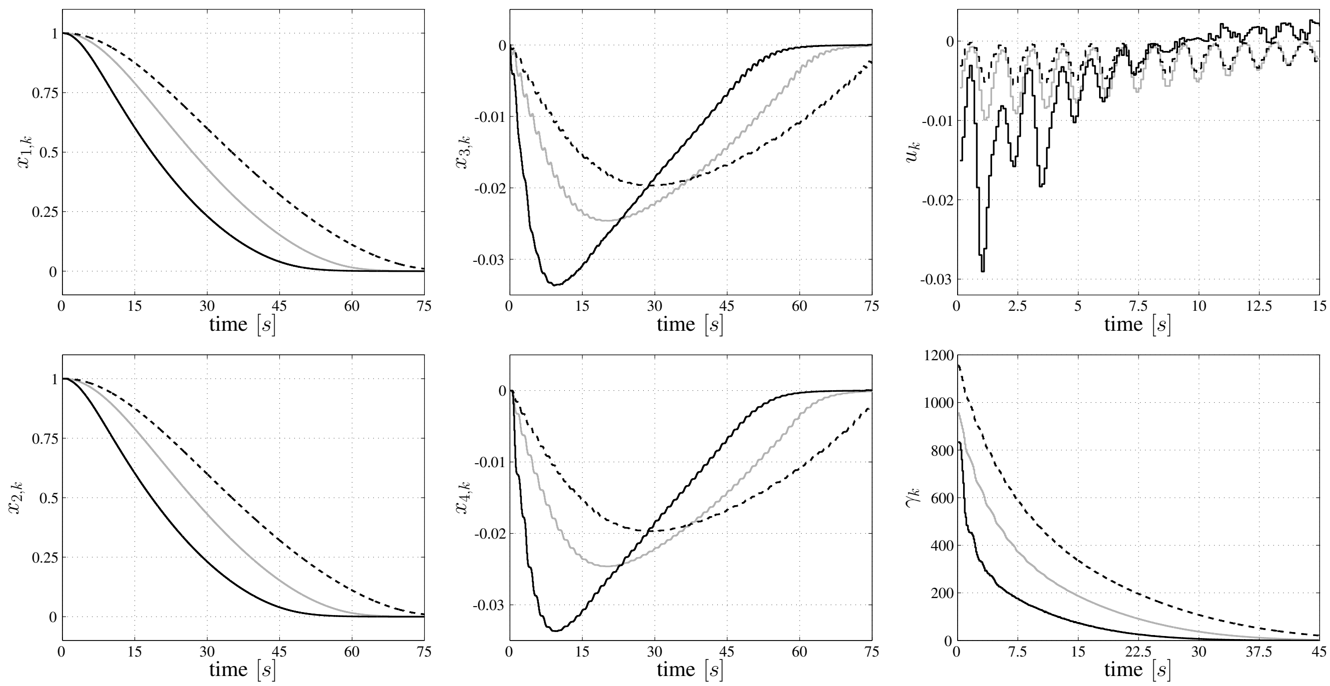


Fig. 1. Comparison of the proposed MPC approach (solid black line) with the controllers [5] (gray line) and [13] (dashed black line). Four plots on the left part: States $x_k$ of the two-mass-spring system. Upper right plot: Input $u_k$. Lower right plot: Upper bound on the considered cost function $\gamma_k$.


[3] Y.Y. Cao and A. Xue. Comments on 'Quasi–min–max MPC algorithm for LPV systems' by Y. Lu and Y. Arkun, Automatica 36 (2000) 527–540. *Automatica*, 40(7):1281–1282, 2004.

[4] A. Casavola, D. Famularo, and G. Franze. A feedback min-max MPC algorithm for LPV systems subject to bounded rates of change of parameters. *IEEE Transactions on Automatic Control*, 47(7):1147–1153, 2002.

[5] F. Cuzzola, J.C. Geromel, and M. Morari. An improved approach for constrained robust model predictive control. *Automatica*, 38(7):1183–1189, 2002.

[6] R. Findeisen, L. Imsland, F. Allgöwer, and B. Foss. State and output feedback nonlinear model predictive control: An overview. *European Journal of Control*, 9:190–206, 2003.

[7] F.A. Fontes. A general framework to design stabilizing nonlinear model predictive controllers. *System and Control Letters*, 42(2):127–142, 2000.

[8] S. Ganguli, A. Marcos, and G. Balas. Reconfigurable LPV control design for Boeing 747-100/200 longitudinal axis. In *Proceedings of the American Control Conference ACC*, Anchorage, Alaska, 2002.

[9] X. Gao. *Control for T-S fuzzy systems based on LMI optimization.* PhD thesis, Jilin University, 2006, in Chinese.

[10] P. Gaspar, I. Szaszi, and J. Bokor. Active suspension design using linear parameter varying control. *International Journal of Vehicle Autonomous Systems*, 1(2):206–221, 2003.

[11] B. He and M. Yang. Robust LPV control of diesel auxiliary power unit for series hybrid electric vehicles. *IEEE Transactions on Power Electronics*, 21(3):791–798, 2006.

[12] E. Kim and H. Lee. New approaches to relaxed quadratic stability condition of fuzzy control systems. *IEEE Transactions on Automatic Control*, 8(5):523–534, 2000.

[13] M.V. Kothare, V. Balakrishnan, and M. Morari. Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379, 1996.

[14] J.-W. Lee. On uniform stabilization of discrete-time linear parameter-varying control systems . *IEEE Transactions on Automatic Control*, 51(10):1714–1721, 2006.

[15] S. Lee and S. Won. Model predictive control for linear parameter varying systems using a new parameter dependent terminal weighting matrix. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, 89(8):2166–2172, 2006.

[16] S.M. Lee, J.H. Park, D.H. Ji, and S.C. Won. Robust model predictive control for LPV systems using relaxation matrices. *IET Control Theory and Applications*, 1(6):1567–1573, 2007.

[17] S. Lim. *Analysis and Control of Linear Parameter-varying Systems.* Stanford University, 1999.

[18] B. Lu. *Linear parameter-varying control of an F-16 aircraft at high angle of attack.* PhD thesis, Department of Mechanical and Aerospace Engineering, North Carolina State University, 2004.

[19] Y. Lu and Y. Arkun. A quasi-min-max MPC algorithm for linear parameter varying systems with bounded rate of change of parameters. In *Proceedings of the American Control Conference ACC*, pages 3234–3238, Chicago, Illinois, 2000.

[20] Y. Lu and Y. Arkun. Quasi-min-max MPC algorithms for LPV systems. *Automatica*, 36(4):527–540, 2000.

[21] P.G. Park and S.C. Jeong. Constrained RHC for LPV systems with bounded rates of parameter variations. *Automatica*, 40(5):865–872, 2004.

[22] B. Pluymers, J.A. Rossiter, J.A.K. Suykens, and B. De Moor. Interpolation based MPC for LPV systems using polyhedral invariant sets. In *Proceedings of the American Control Conference ACC*, pages 810–815, Portland, Oregon, 2005.

[23] C.W. Scherer. LPV control and full block multipliers. *Automatica*, 37(3):361–375, 2001.

[24] F. Wang and V. Balakrishnan. Robustness analysis and gain-scheduled controller synthesis for rational parameter-dependent systems using parameter-dependent Lyapunov functions. In *Proceedings of the 38th IEEE Conference on Decision and Control CDC*, pages 281–286, Phoenix, Arizona, 1999.

[25] F. Wu. A generalized LPV system analysis and control synthesis framework. *International Journal of Control*, 74(7):745–759, 2001.

[26] S. Yu, H. Chen, C. Böhm, and F. Allgöwer. Enlarging the terminal region of NMPC with parameter-dependent terminal control law. In *Proceedings of the International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, Pavia, Italy, 2008.

[27] S. Yu, H. Chen, C. Böhm, and F. Allgöwer. Moving horizon nonlinear $\mathcal{H}_\infty$ control based on T-S-models. In *Proceedings of the International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, Pavia, Italy, 2008.