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ℓ_∞-Gain Model Reduction for Discrete-Time Systems via LMIs

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***Abstract*— This paper deals with model order reduction for linear stable discrete-time systems while minimizing the ℓ_∞-gain of the error system. Our approach considers explicitly time domain performance and not frequency domain performance like most other approaches do. A convex programming problem expressed in terms of linear matrix inequalities combined with a line search algorithm is formulated to provide suboptimal solutions to the model reduction problem. The proposed method is compared to the well-known balanced truncation method via simulation. It is shown that the proposed method results in superior performance when time domain performance is of importance.**

I. Introduction

Model order reduction is an important and interesting problem for several reasons. In many applications or when doing simulations it is difficult to deal with a system of high order. Therefore a lower order approximation of the original system that preserves properties of the original system is desirable. The properties that are preserved depend on the model order reduction method that is used. Typically system norms as e.g Hankel singular values, $\mathcal{H}_\infty$- or $\mathcal{H}_2$-gain are considered. In our approach the ℓ_∞-gain of a system is considered. A reduced model with lower pre-specified order is searched for, which approximates the original model according to a given minimum norm criterion. Such an approximation problem is in general hard to solve due to its non-convexity which comes from constraints of involved variable (see [1], [2]).

Many authors have made efforts to transfer this non-convex optimization problem into a convex one by introducing a small degree of suboptimality based on lower and upper bounds on the norm of the reduction error, see e.g. the seminal paper on Hankel norm approximation [3], the survey paper on model order reduction techniques [4] and references therein. Another method was presented by [1], [2] and [5] where the authors propose an alternating projection approach to handle rank constraints on optimization problems, the so called cone complementary linearization (CCL, [6]). Approaches which formulate the model order reduction problem in terms of linear matrix inequalities (LMIs, [7]) to get classical bounds on $\mathcal{H}_\infty$- and $\mathcal{H}_2$- norm of the reduction error are presented in [8], [9] and [10].

All these approaches have in common that their main goal is to preserve frequency domain properties of the original system, as e.g. energy or bandwidth. However, in some situations it is desired to preserve time domain properties as e.g. maximum amplitude of a signal. To cover this case the ℓ_∞-norm of the error between the original system and the reduced order system is considered in our paper. The ℓ_∞-gain characterizes the worst case time domain amplitude of a system output $y = Hu$ normalized by the maximum amplitude of the input u under the assumption of zero initial conditions. Using this as a measure of error between the original and the reduced order system implies to explicitly consider time domain properties as minimization criterion. The ℓ_∞-gain of the reduction error system is then a value of practical meaning as it corresponds to the worst case amplitude of the error system. On the other hand this approach is also of course very interesting in combination with ℓ_1-optimal controller design [11], [12]. To the best of the authors knowledge our approach is the first that considers time domain performance as a criterion for model reduction. Therefore this new method supplements the existing model order reduction techniques, which cannot deal explicitly with time domain properties like amplitude or slope of a signal.

In our paper the ℓ_∞-gain model reduction problem for linear multivariable discrete-time systems is formulated in terms of LMIs combined with a line search. LMIs are computationally attractive because they can be solved efficiently with existing solvers (see [13], [14]). Transforming the non-convex problem into a convex one is possible by an a-priori choice of a certain matrix variable, which is kept fixed during the optimization. By an appropriate choice of this matrix variable the degree of suboptimality is kept small and the proposed method is a suitable alternative to the classical balanced truncation method [15], when time domain performance is of explicit interest.

The article is organized as follows: Next, the problem statement is formulated. Then the model reduction algorithm is developed in three steps. First, we will introduce the star-norm which is an upper bound to the ℓ_∞-gain of a system and can be efficiently computed via LMIs combined with a line search. Second, we will parameterize the original system to obtain systems for which the error to the original system is below a certain value. In the third step the parameterized model is truncated and the reduced order system is introduced. The paper concludes with an example which compares reduced models derived by this method to models derived by the well known balanced truncation approach.

The notation $X < 0$ (≤ 0) stands for X being negative (semi-) definite; likewise for $>$ $(\geq)$ and positive (semi-) definiteness. For ease of notation of partitioned symmetric matrices, the symbol $(\star)$ denotes generically its symmetric blocks; I and 0 represent identity and zero matrices, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. A

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state-space realization of a transfer matrix $H(z)$ is written as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(zI - A)^{-1}B + D = H(z).$$

II. Problem Statement

Let a linear discrete-time system be given by its input/output transfer function representation

$$H(z) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{1}$$

where matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$ and $D \in \mathbb{R}^{r\times m}$ are known. It is assumed that this system is stable as well as controllable and observable, i.e. the system is given in minimal realization. Our goal is to determine a reduced order model associated to (1), with transfer function

$$H_r(z) := \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right], \tag{2}$$

with matrices $A_r \in \mathbb{R}^{q\times q}$, $B_r \in \mathbb{R}^{q\times m}$, $C_r \in \mathbb{R}^{r\times q}$ and $D_r \in \mathbb{R}^{r\times m}$ with $0 \leq q < n$. It is desired that the reduced order model (2) approximates the original system (1) such that

$$\inf_{H_r(z)\in\mathcal{H}} \|H(z) - H_r(z)\|_{\ell_\infty - ind} \tag{3}$$

holds, where $\mathcal{H}$ denotes the set of all stable $H_r(z)$ with minimal realization (2) and order $q < n$, the *-ind* stands for induced norm. Problem (3) is a classical approximation problem which has been considered by many authors (e.g. [4], [3]). Although the objective function is convex, the feasible set $\mathcal{H}$ is generally not convex due to the fact that the search for the transfer functions $H_r(z)$ is restricted to functions of order q strictly smaller than n. In the next section an approach is derived that overcomes this problem. It will be shown that suboptimal solutions of (3) can be obtained by solving a convex optimization problem based on computationally attractive LMI techniques combined with a line search. A further advantage of this approach is that the reduced order model is explicitly calculated from the solution of the optimization problem.

III. Main Results

In this section the main result is introduced in three steps. First, the star-norm is introduced as an upper bound to the ℓ_∞-gain of a system. Afterwards, based on the star-norm a parameterization of the system is shown that yields all systems H_r with order $n = q$ and $\|H(z) - H_r(z)\|_{\ell_\infty - ind} < \gamma$. Third, this parameterized model is then truncated to get the reduced order model.

A. Star-Norm Performance

The space ℓ_∞^n is the Banach space of right-sided bounded real vector sequences of dimension n, with the ℓ_∞-norm

$$\|x\|_{\ell_\infty} := \sup_k \max_{1\leq i\leq n} |x_i(k)|.$$

Thus the ℓ_∞-norm measures the maximum absolute value of a vector sequence. An alternative norm on ℓ_∞^n is the peak-norm

$$\|x\|_{peak} := \sup_k \sqrt{x(k)^T x(k)}.$$

For $x \in \ell_\infty^n$ it holds $\|x\|_{\ell_\infty} \leq \|x\|_{peak} \leq \sqrt{n}\|x\|_{\ell_\infty}$ [16]. The ℓ_∞-induced norm (or ℓ_∞-gain) of a map $H : \ell_\infty^n \to \ell_\infty^m$ is

$$\|H\|_{\ell_\infty - ind} := \sup_{0<\|w\|_{\ell_\infty}<\infty} \frac{\|Hw\|_{\ell_\infty}}{\|w\|_{\ell_\infty}}$$

and hence measures the worst-case amplification of persistent inputs in terms of the maximally attained amplitude. It can be shown that for linear discrete-time invariant systems the ℓ_∞-gain is equal to the ℓ_1-norm of the system's impulse response [17]. However, in this paper we follow a different road. To compute the ℓ_∞-gain of system (1) we use the so called star-norm performance which is an upper bound on the peak-to-peak gain and the ℓ_∞-induced norm. Moreover it holds that $\frac{1}{\sqrt{m}}\|H\|_{peak-ind} \leq \|H\|_{\ell_\infty - ind} \leq \sqrt{r}\|H\|_{peak-ind}$ for $\dim(H) = m \times r$ (see [18], Appendix A).

Theorem 1 ([19]): Consider system (1) with initial conditions $x(0) = 0$. The following statements are equivalent:

- There exist $\gamma > 0$, $\mu > 0$, $0 < \lambda < 1$ and $X = X'$ satisfying

$$\begin{bmatrix} \lambda X & 0 & A'X \\ \star & \mu I & B'X \\ \star & \star & X \end{bmatrix} > 0 \tag{4a}$$

$$\begin{bmatrix} (1-\lambda)X & 0 & C' \\ \star & \left(\frac{\gamma^2}{m} - \mu\right) I & D' \\ \star & \star & I \end{bmatrix} > 0. \tag{4b}$$

- $\|H(z)\|_{\ell_\infty - ind} < \gamma$, and
- $\|y\|_{peak} < \frac{\gamma}{\sqrt{m}}$ for $\|u\|_{peak} \leq 1$, and moreover $\|H(z)\|_{peak-ind} < \frac{\gamma}{\sqrt{m}}$,
- A has all its eigenvalues in the open unit disk.

Remark 1: γ is just an upper bound on the system's ℓ_∞-induced norm. The smallest achievable γ is called star-norm $\|H\|_\star$ of H.

Since the ℓ_∞-gain is a measure for maximum amplitude of a signal, this allows to consider time domain performance criteria explicitly. In the following the inequalities (4) will be used to compute a reduced order system $H_r(z)$ that is the solution to the minimization problem

$$\inf_{H_r(z)} \gamma \text{ subject to } \|H(z) - H_r(z)\|_{\ell_\infty - ind} < \gamma. \tag{5}$$

It is also possible to consider a pre-specified γ and search for a reduced order system, i.e. find a $H_r(z)$ such that

$$\|H(z) - H_r(z)\|_{\ell_\infty - ind} < \gamma$$

holds. In this case $H_r(z)$ is a suboptimal solution to (5).

B. Parameterization of the Model

With the star-norm performance above introduced, as a first result we can formulate the following lemma. It gives the set of all asymptotically stable transfer functions of order $n = q$ such that the error is bounded from above by an ℓ_∞-norm level $\gamma > 0$.

Lemma 1: For $\gamma > 0$, $\mu > 0$ and $0 < \lambda < 1$, all matrices F, M, L, and K of compatible dimensions satisfying the linear matrix inequalities

$$X > Z > 0, \tag{6a}$$

$$\begin{bmatrix} \lambda Z & \lambda Z & 0 & A'Z & A'X - M' \\ \star & \lambda X & 0 & A'Z & A'X \\ \star & \star & \mu I & B'Z & B'X - L' \\ \star & \star & \star & Z & Z \\ \star & \star & \star & \star & X \end{bmatrix} > 0, \tag{6b}$$

$$\begin{bmatrix} (1-\lambda)Z & (1-\lambda)Z & 0 & C' - F' \\ \star & (1-\lambda)X & 0 & C' \\ \star & \star & \left(\frac{\gamma^2}{m} - \mu\right) I & D' - K' \\ \star & \star & \star & I \end{bmatrix} > 0. \tag{6c}$$

produce asymptotically stable transfer functions with order $n = q$ of the form

$$H_r(z) := \left[\begin{array}{c|c} (X-Z)^{-1}M & (X-Z)^{-1}L \\ \hline F & K \end{array}\right], \tag{7}$$

that satisfy $\|H(z) - H_r(z)\|_{\ell_\infty - ind} < \gamma$.

Proof: The proof is similar to the proofs in [10] and [9]. The transfer function of the approximation error is given by

$$\begin{aligned} E(z) :=& H(z) - H_r(z) \\ =& \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & A_r & B_r \\ \hline C & -C_r & D - D_r \end{array}\right]. \end{aligned}$$

$\|E(z)\|_{\ell_\infty - ind} < \gamma$ is satisfied following Theorem 1 if there exists a symmetric and positive matrix $\mathcal{X}$, $\gamma > 0$, $\mu > 0$ and $0 < \lambda < 1$ such that

$$\begin{bmatrix} \lambda\mathcal{X} & 0 & \mathcal{A}'\mathcal{X} \\ \star & \mu I & \mathcal{B}'\mathcal{X} \\ \star & \star & \mathcal{X} \end{bmatrix} > 0, \tag{8a}$$

$$\begin{bmatrix} (1-\lambda)\mathcal{X} & 0 & \mathcal{C}' \\ \star & \left(\frac{\gamma^2}{m} - \mu\right) I & \mathcal{D}' \\ \star & \star & I \end{bmatrix} > 0. \tag{8b}$$

This is a bilinear matrix inequality due to the multiplication between the system matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ (including the unknowns A_r, B_r, C_r and D_r) and the decision variable $\mathcal{X}$. In the following steps, a linearizing change of variables is introduced to transform the BMI into an LMI combined with a line search. Using the partitions

$$\mathcal{X} = \begin{bmatrix} X & U \\ U' & X_2 \end{bmatrix} \quad \mathcal{X}^{-1} = \begin{bmatrix} Y & V \\ V' & Y_2 \end{bmatrix} \quad \mathcal{T} = \begin{bmatrix} Y & I \\ V' & 0 \end{bmatrix},$$

and multiplying (8a) to the right by $\operatorname{diag}(\mathcal{T}, I, \mathcal{T})$ and to the left by its transpose, multiplying the result from both sides by the symmetric matrix $\operatorname{diag}(Y^{-1}, I, I, Y^{-1}, I)$, we obtain the inequality (6b), where $Z = Y^{-1}$, $M = -UA_rV'Z$ and $L = -UB_r$.

Multiplying (8b) to the right by $\operatorname{diag}(\mathcal{T}, I, I)$ and to the left by its transpose, multiplying the result from both sides by the symmetric matrix $\operatorname{diag}(Y^{-1}, I, I, I)$, we obtain the inequality (6c), where $F = C_rV'Z$ and $K = D_r$. Additionally, by multiplying matrix $\mathcal{T}'\mathcal{X}\mathcal{T}$ from both sides by $\operatorname{diag}(Y^{-1}, I)$ and applying the Schur complement it can be concluded that $\mathcal{T} > 0$ holds if $X > Z > 0$. Since the square matrices $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{n \times n}$ can always be assumed nonsingular, we obtain the state-space realization

$$H_r(z) := \left[\begin{array}{c|c} -U^{-1}M(V'Z)^{-1} & -U^{-1}L \\ \hline F(VZ')^{-1} & K \end{array}\right],$$

which reduces to (7) from the choice of V such that $V'Z = I$ and the determination of U from $X + UV'Z = Z$. The matrices U and V can be fixed with no loss of generality. They only change the state-space realization of the system but not its transfer function $H_r(z)$. This concludes the proof of the proposed lemma. ■

To simplify the results of Lemma 1 significantly we will eliminate variables L, M, F and K. The idea is to determine these matrix variables in terms of X and Z without introducing conservatism concerning the feasibility of $H_r(z)$.

Theorem 2: For each $\gamma > 0$, $\mu > 0$, $0 < \lambda < 1$, all symmetric matrices satisfying the matrix inequalities,

$$X > Z > 0, \tag{9a}$$

$$\begin{bmatrix} \mu I - B'ZB & B'ZA \\ \star & \lambda Z - A'ZA \end{bmatrix} > 0, \tag{9b}$$

$$A'XA - \lambda X < 0, \tag{9c}$$

$$\frac{\gamma^2}{m} > \mu, \quad C'C - (1-\lambda)X < 0, \tag{9d}$$

produce asymptotically stable transfer functions with order $n = q$ of the form

$$H_r(z) = \left[\begin{array}{c|c} \lambda A\Theta(X-Z) & B + A\Theta A'ZB \\ \hline CX^{-1}(X-Z) & D \end{array}\right] \tag{10}$$

with $\Theta = (\lambda X - A'ZA)^{-1}$ that satisfy $\|H(z) - H_r(z)\|_{\ell_\infty - ind} < \gamma$.

Proof: The proof follows along the lines of [10]. First, we will consider inequality (6b). The Schur complement is performed twice. First with respect to matrix Z in the fourth row and column and second with respect to the resulting matrix $\lambda X - A'ZA$ placed in the third row and column. The result is a new three by three block matrix whose out-diagonal elements give the unknowns

$$\begin{aligned} M &= \lambda(X-Z)A(\lambda X - A'ZA)^{-1}(X-Z), \\ L &= (X-Z)B + (X-Z)A(\lambda X - A'ZA)^{-1}A'ZB. \end{aligned}$$

Second inequality (6c) is considered. The Schur complement is performed on matrix $(1-\lambda)X$ on the second row and column. The result is a three by three block matrix whose out-diagonals give the unknowns

$$\begin{aligned} F &= CX^{-1}(X - Z), \\ K &= D. \end{aligned}$$

Inserting L, M, F, and K as obtained above in equation (7) leads to equation (10) which only depends on the unknowns X and Z. This concludes the proof of Theorem 2. ■

If the matrix inequalities (9) are feasible then also inequalities (6b). Inequalities (9) are no LMIs due to the products between λ and X and λ and Z. Still the global minimum of γ^2 is found by combining the minimization of $\gamma^2(\lambda)$ for a fixed λ with a line-search over $0 < \lambda < 1$. By Theorem 2 a parameterization for all $H_r(z)$ that satisfy $\|H(z)-H_r(z)\|_{peak-ind} < \gamma$ is found. From this parameterization the reduced order model is derived in the following section.

C. Model Reduction

Solving the inequalities of Theorem 2 always has $H_r(z) = H(z)$ as a solution which is not helpful. To avoid this, we restrict the solution space. This is motivated by the following observation: For $Z \to 0$, $H_r(z)$ approaches $H(z)$ yielding γ arbitrarily close to zero as expected. For $Z \to X$, A_r and C_r go to zero and consequently the zero order approximation $H_r = D$ is obtained. The associated minimum cost γ becomes $\|C(zI - A)B\|_{\ell_\infty}$. In conclusion equation (10) generates not only nth order transfer functions but also $0 \le q \le n$ order transfer functions in the limit case as Z approaches X.

To use this observation for model reduction, we fix the relation between X and Z in the following similar to [10]. Consider that the difference $X - Z$ can be written as

$$X = Z + \begin{bmatrix} W & J \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & O(\epsilon) \end{bmatrix} \begin{bmatrix} W' \\ J' \end{bmatrix}, \quad (11)$$

where $S := \begin{bmatrix} W & J \end{bmatrix} \in \mathbb{R}^{n\times n}$ is a non-singular matrix partitioned accordingly, $\Sigma \in \mathbb{R}^{q\times q}$ is a positive definite matrix and $O(\epsilon)$ is an arbitrarily small (of order $\epsilon > 0$) positive definite matrix. Since the term $(X - Z)$ is included in A_r and C_r, making $\epsilon \to 0$, $n - q$ poles of the transfer function are unobservable poles placed at the origin. Deleting these poles leads to the reduced order transfer function

$$H_r(z) = \left[\begin{array}{c|c} \lambda W' A\Theta W\Sigma & W'(B + A\Theta A' Z B) \\ \hline CX^{-1}W\Sigma & D \end{array}\right], \quad (12)$$

with $\Theta = (\lambda X - A'ZA)^{-1}$ as defined before.

The equality constraint (11) is nonlinear with respect to the involved matrices W and Σ. It becomes linear for a fixed W, which we want to determine a-priori. Then we can consider a convex optimization problem which can be solved in terms of LMIs combined with a line search over λ. In principle, W could be chosen arbitrarily. Different choices of W lead to reduced models of different quality with respect to the considered minimization criterion. Since we want to place $n-q$ poles at the origin, we introduce P as the observability gramian and Q as the controllability gramian. Assume S as the decomposition of P, then we obtain

$$\begin{aligned} P &= S'S \\ \Lambda &:= SQS' = \begin{bmatrix} \Lambda_W & 0 \\ 0 & \Lambda_J \end{bmatrix} = \mathrm{diag}(\sigma_i^2) \end{aligned}$$

where Λ consists of the squared Hankel singular values σ_i^2 in a decreasing ordering. We now choose W as the q columns of S corresponding to the q largest diagonal element. This is motivated by the idea of moving poorly observable poles of the original model to the origin, while keeping the well observable ones. Using Theorem 2 and the considerations above, we can formulate the following algorithm to derive a reduced order model of order q.

Algorithm 1: For a given system $H(z)$ with order n a reduced order system $H_r(z)$ with order $0 \le q \le n$ so that the ℓ_∞-norm of the error system $\|H(z) - H_r(z)\|_{\ell_\infty-ind}$ is minimized can be found by

$$\inf_{\substack{X,Z>0,\Sigma>0,\\ \mu>0,0<\lambda<1}} \gamma \text{ subject to}$$

$$X = Z + W\Sigma W', \quad (13a)$$

$$X > Z > 0, \quad (13b)$$

$$\begin{bmatrix} \mu I - B'ZB & B'ZA \\ \star & \lambda Z - A'ZA \end{bmatrix} > 0, \quad (13c)$$

$$A'XA - \lambda X < 0, \quad (13d)$$

$$\frac{\gamma^2}{m} > \mu, \quad C'C - (1 - \lambda)X < 0. \quad (13e)$$

The reduced order system $H_r(z)$ is given by equation (12).

Remark 2: This algorithm is computationally very attractive since it can be solved via LMIs combined with a line search.

IV. Examples and Comparison

In this section we show simulations of the method introduced in this paper. Transfer functions with dimensions $m = 1$, $r = 1$ and $2 \le n \le 20$ were generated randomly. Matrix A has been multiplied by a scalar in order to obtain a new matrix with all eigenvalues placed in a circle with radius $5/6$ around the origin to avoid numerical problems. For each model of dimension n a reduced order model H_{ℓ_∞} has been calculated for all $q = 1, 2, \ldots, (n - 1)$.

We also generated reduced order models by the very well-known balanced truncation method. For each transfer function of order q the original system $H(z)$ has been transformed into a balanced realization and truncated to obtain the reduced order model $H_{BT}(z)$. Then the following normalized indexes have been calculated

$$\begin{aligned} e_{BT} &:= \frac{\|H(z) - H_{BT}(z)\|_{\ell_\infty-ind}}{\|H(z)\|_{\ell_\infty-ind}} \\ e_{\ell_\infty} &:= \frac{\|H(z) - H_{\ell_\infty}(z)\|_{\ell_\infty-ind}}{\|H(z)\|_{\ell_\infty-ind}}. \end{aligned}$$

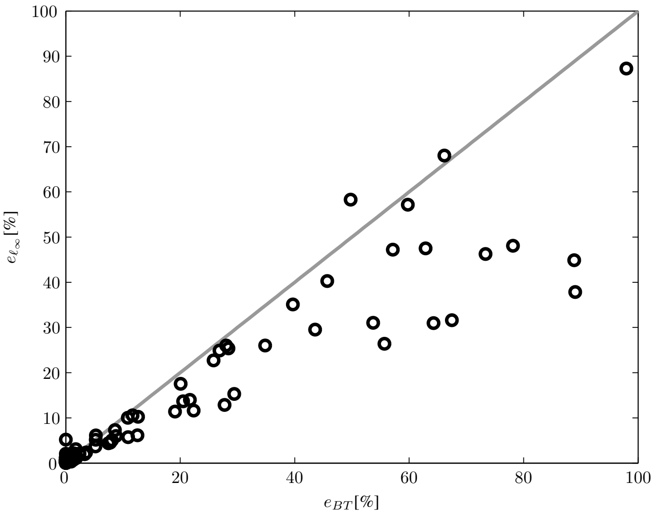

Fig. 1. Comparison of ℓ_∞-gain model reduction with balanced truncation.

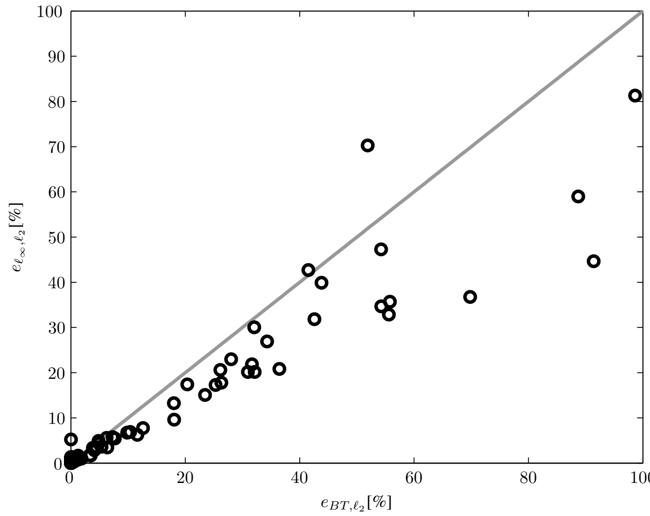

Fig. 2. Comparison of ℓ_2 performance of ℓ_∞-gain model reduction with balanced truncation.

In Figure 1 e_{ℓ_∞} over e_{BT} is plotted. It can be seen that for the same number of reduced states a reduced order model computed by the proposed method has a smaller ℓ_∞-gain than a model reduced by balanced truncation in almost every case. This shows that the proposed method is a promising alternative to classical balanced truncation when time domain properties are of interest for the reduced order model, although balanced truncation is computationally less demanding than the proposed method.

Another interesting question is the ℓ_2-performance of the proposed method. In Figure 2 the ℓ_2 performance of the proposed method compared with balanced truncation is shown for the above randomly generated systems. Again the following normalized indexes have been calculated

$$e_{BT,\ell_2} := \frac{\|H(z) - H_{BT}(z)\|_{\ell_2-ind}}{\|H(z)\|_{\ell_2-ind}}$$
$$e_{\ell_\infty,\ell_2} := \frac{\|H(z) - H_{\ell_\infty}(z)\|_{\ell_2-ind}}{\|H(z)\|_{\ell_2-ind}}.$$

It can be seen, that the reduced order models have good ℓ_2 performance in addition to the minimized ℓ_∞ performance. This is a useful property which makes the method even more attractive.

In both plots are a lot of dots close to zero. This comes from the cases where only one or two states are truncated. Then both methods perform equally well with only a very small error between the reduced order system and the original system.

For completeness we have also considered the forth-order system used in [20] and [21]

$$H(z) = \left[\begin{array}{cccc|c} -1.1 & 1 & 0 & 0 & 1 \\ 0.01 & 0 & 1 & 0 & 0 \\ 0.275 & 0 & 0 & 1 & 0 \\ 0.06 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right], \qquad (14)$$

TABLE I

ℓ_∞-GAIN OF REDUCED ORDER SYSTEMS FOR SYSTEM (14).

q	BT	Algorithm 1	γ
1	2.8700	1.9701	2.3411
2	0.8279	0.2613	0.4300
3	0.0281	0.0162	0.3174

with ℓ_∞-gain $\|H(z)\|_{\ell_\infty-ind} = 9.5238$.

Table I shows the ℓ_∞-gain of the reduced order system computed with balanced truncation and Theorem 1 as well as the upper bound for the star-norm performance γ as it results from optimization problem (13). As the number of reduced states grows (i.e. q small) the proposed algorithm achieves much better results than the balanced truncation. For a growing number of reduced states not only the ℓ_∞-gain of the error systems is much smaller but already γ the upper bound on the star-norm, as can be seen in Table I. This shows that if a larger number of states will be reduced the proposed method is very suitable.

V. Conclusion

In this paper a computationally attractive method for model order reduction for stable linear multivariable discrete-time systems in an ℓ_∞-gain framework was presented. This novel approach minimizes the ℓ_∞-gain of the error system. Thus it considers time domain performance as minimization criteria for the norm of the error system. Since existing approaches mainly consider frequency domain properties like energy of bandwidth of a system, this new approach supplements existing model order reduction techniques that cannot deal with time domain performance.

An upper bound on the ℓ_∞-gain namely the star-norm performance was used so that the problem could be formulated in terms of LMIs combined with a line search. Via simulation it has been shown that the proposed method performs equally or better than the classical balanced truncation method in the ℓ_∞ case. A useful additional property is the good ℓ_2-

performance of the reduced order systems. Thus, in the case when time domain properties are of interest, the proposed approach is a promising alternative to the balanced truncation method. Further research is necessary to reduce suboptimality of the presented approach by choosing the degrees of freedom in the problem formulation in an optimal way.

VI. ACKNOWLEDGEMENT

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REFERENCES

[1] K. M. Grigoriadis, "Optimal $\mathcal{H}_\infty$ model reduction via linear matrix inequalities: continuous and discrete-time cases," *Syst. Contr. Lett.*, vol. 26, no. 5, pp. 321–333, 1995.

[2] K. M. Grigoriadis, "$\mathcal{L}_2$ and $\mathcal{L}_2$-$\mathcal{L}_\infty$ model reduction via linear matrix inequalities," *Int. J. Control*, vol. 68, no. 3, pp. 485–498, 1997.

[3] K. Glover, "All optimal hankel-norm approximations of linear multivariable systems and their $\mathcal{L}^\infty$-error bounds," *Int. J. Control*, vol. 39, no. 6, pp. 1115–1193, 1984.

[4] B. D. O. Anderson and Y. Liu, "Controller reduction: concepts and approaches," *IEEE Trans. Automat. Control*, vol. 34, pp. 802–812, 1989.

[5] R. E. Skelton, T. Iwasaki, and K. Grigoriadis, *A Unified Algebraic Approach to Control Design*, Taylor and Francis, London, 1997.

[6] L. El Ghaoui, F. Oustry, and M. Ait Rami, "A cone complementarity linearization algorithm for static output-feedback and related problems," *IEEE Trans. Automat. Control*, vol. 42, no. 8, pp. 1171–1176, 1997.

[7] S. P. Boyd, l. El Ghaoul, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Siam Philadelphia, 1994.

[8] Y. Ebihara and T. Hagiwara, "On $\mathcal{H}_\infty$-model reduction using LMIs," *IEEE Trans. Automat. Control*, vol. 49, no. 7, pp. 1187–1191, 2004.

[9] J. C. Geromel, R. G Egas, and F. R. R. Kawaoka, "$\mathcal{H}_\infty$ model reduction with application to flexible systems," *IEEE Trans. Automat. Control*, vol. 50, no. 2, pp. 402–406, 2005.

[10] J. C. Geromel, F. R. R. Kawaoka, and R. G. Egas, "Model reduction of discrete time systems trough linear matrix inequalities," *Int. J. Control*, vol. 77, no. 10, pp. 987–984, 2004.

[11] M. A. Dahleh and M. H. Khammash, "Controller design for plants with uncertainties," *Automatica*, vol. 29, no. 1, pp. 37–56, 1993.

[12] M. A. Khammash, "A new approach to the solution of the ℓ_1 control problem: The scaled-*q* method," *IEEE Trans. Automat. Control*, vol. 45, no. 2, pp. 180–187, 2000.

[13] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in matlab," in *Proc. CACSD Conf.*, Taipei, Taiwan, 2004, pp. 284–289.

[14] J. F. Sturm, "Using SeDuMi," *Optimization Methods and Software*, vol. 11–12, no. 1–4, pp. 625–653, 1999.

[15] U. M. Al-Saggaf and G. F. Franklin, "An error bound for a discrete reduced order model of a linear multivariable system," *IEEE Trans. Automat. Control*, vol. 32, no. 9, pp. 815–819, 1987.

[16] J. Bu and M. Sznaier, "A linear matrix inequality approach to synthesizing low order suboptimal $\ell_1/\mathcal{H}_p$ controllers," *Automatica*, vol. 36, no. 7, pp. 957–963, 2000.

[17] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*, Upper Saddle River, NJ: Prentice Hall, 1996.

[18] H. Khalil, *Nonlinear Systems*, Prentice Hall. Englewood Cliffs, NJ, third edition, 2002.

[19] J. M. Rieber, C. W. Scherer, and F. Allgöwer, "Robust ℓ_1 performance analysis in face of parametric uncertainties," in *Proc. 45th IEEE Conf. Decision and Control (CDC)*, 2006, pp. 5826–5831.

[20] Y. Choo, "On the property of discrete impulse response gramian with application to model reduction," *IEICE Trans. Fundamentals*, vol. E88-A, no. 12, pp. 3658–3660, 2005.

[21] V. Sreeram and P. Agathoklis, "Discrete-system reduction via impulse response gramians and its relation to q-markov covers," *IEEE Trans. Automat. Control*, vol. 37, no. 5, pp. 653–658, 1992.