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$\mathcal{L}_1$ Adaptive Output Feedback Controller for Nonlinear Systems in the Presence of Unmodeled Dynamics

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Abstract— **This paper presents the $\mathcal{L}_1$ adaptive output feedback controller for a class of uncertain nonlinear systems in the presence of time and state dependent unknown nonlinearities, and multiplicative unmodeled dynamics. The $\mathcal{L}_1$ adaptive controller ensures uniformly bounded transient and asymptotic tracking for system's both signals, input and output, simultaneously. The performance bounds can be systematically improved by increasing the adaptation rate.**

I. Introduction

This paper considers a class of uncertain nonlinear systems, and develops an adaptive output feedback control architecture that ensures uniformly bounded transient response for system's input and output signals simultaneously. We notice that improvement of the transient performance of adaptive controllers has been addressed from various perspectives in numerous publications [1]–[11], to name a few. This paper builds on previous work by the authors [12]–[19], and extends the $\mathcal{L}_1$ adaptive output feedback control architecture to a class of uncertain nonlinear systems in the presence of time and state dependent unknown nonlinearities, as well as multiplicative unmodeled dynamics. We prove that subject to a set of mild assumptions, the system can be transformed into an equivalent linear system with time-varying unknown parameters and disturbances. For the latter, we extend the output feedback controller initially proposed in [12], which yields semiglobal performance results for the original uncertain nonlinear system. The main benefit of the $\mathcal{L}_1$ adaptive controller is its ability of fast adaptation with guaranteed robustness, as proven in [14]–[16]. The $\mathcal{L}_\infty$-norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation gain.

The paper is organized as follows: Section II gives the problem formulation. Section III presents the $\mathcal{L}_1$ adaptive output feedback control architecture. Stability and performance bounds are derived in Section IV. Section V presents simulation results. Section VI concludes the paper.

II. Problem Formulation

Consider the following system dynamics:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\left(\mu_u(t) + f(x(t), z(t), t)\right)\,,\ x(0) = x_0\,,\\ z(t) &= g_o\left(x_z(t), t\right)\,,\\ \dot{x}_z(t) &= g\left(x_z(t), x(t), t\right)\,, \qquad x_z(0) = x_{z0}\,, \qquad (1)\\ \mu_u(s) &= F(s)u(s)\,, \qquad y(t) \;=\; c^\top x(t)\,, \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, which is not measured; $u(t) \in \mathbb{R}$ is the control signal; $y(t) \in \mathbb{R}$ is the only measured output; $b, c \in \mathbb{R}^n$ are known constant vectors; A is a known Hurwitz $n \times n$ matrix; $z(t)$ and $x_z(t)$ are the output and the state vector of unmodeled dynamics; f, g_o, and g are unknown nonlinear functions; $F(s)$ is an unknown stable proper transfer function that represents multiplicative unmodeled dynamics at the input of the system; and $H(s) = c^\top(s\mathbb{I} - A)^{-1}b$ is a stable minimum-phase system with relative degree 1.

Assumption 1: There exists L_F such that $\|F(s)\|_{\mathcal{L}_1} \leq L_F$, where $\|F(s)\|_{\mathcal{L}_1}$ is the $\mathcal{L}_1$-norm of the transfer function.

Assumption 2: [Stability of internal dynamics] The z-dynamics are bounded-input-bounded-output (BIBO) stable, i.e. there exist $L_{z1} > 0$ and $L_{z2} > 0$ such that

$$\|z_t\|_{\mathcal{L}_\infty} \;=\; L_{z1}\,\|x_t\|_{\mathcal{L}_\infty} \;+\; L_{z2}\,. \qquad (2)$$

Further, let $X(t) \triangleq [\; x^\top(t) \quad z^\top(t) \;]^\top$.

Assumption 3: [Semiglobal Lipschitz condition] For any $\delta > 0$, there exist positive K_δ and B such that

$$\begin{aligned} |f(X_1, t) - f(X_2, t)| &\leq K_\delta\,\|X_1(t) - X_2(t)\|_\infty\,, \qquad (3)\\ |f(0, t)| &\leq B\,, \qquad (4) \end{aligned}$$

for all $\|X_i(t)\|_\infty \leq \delta$, $i = 1, 2$, uniformly in t.

Assumption 4: [Semiglobal uniform boundedness of partial derivatives] For any $\delta > 0$, there exist $d_{f_x}(\delta) > 0$, and $d_{f_t}(\delta) > 0$ such that for any $\|x(t)\|_\infty \leq \delta$, the partial derivatives of $f(X, t)$ are piece-wise continuous and bounded

$$\left\|\frac{\partial f(X,t)}{\partial X}\right\| \;\leq\; d_{f_x}(\delta)\,, \qquad \left|\frac{\partial f(X,t)}{\partial t}\right| \;\leq\; d_{f_t}(\delta)\,. \qquad (5)$$

The control objective is to design an adaptive output feedback controller to ensure that $y(t)$ tracks the output response of a *desired system* to a given bounded reference signal $r(t)$ *both in transient and steady-state*, while all other signals remain bounded.

III. $\mathcal{L}_1$ Adaptive Controller

A. Definitions

For every $\delta > 0$, let $L_\delta \;\triangleq\; (\bar{\delta}/\delta)K_{\bar{\delta}}$, where $K_{\bar{\delta}}$ is the Lipschitz constant defined in (3), while $\bar{\delta}$ is defined as $\bar{\delta} \;\triangleq\; \max\{\delta,\ L_{z1}\delta + L_{z2}\}$. Let

$$H_x(s) = (s\mathbb{I} - A)^{-1}b = \frac{A_T\,[\; 1 \quad s \quad \cdots \quad s^{n-1} \;]^\top}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} \qquad (6)$$

$$H(s) = c^\top H_x(s) \;\triangleq\; H_n(s)/H_d(s)\,. \qquad (7)$$

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Also, let

$$A_m = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

and $b_m = [0 \ \cdots \ 0 \ 1]^\top$. Since A_m is Hurwitz, for any $Q > 0$ there exists a $P = P^\top > 0$ that solves the algebraic Lyapunov equation $A_m^\top P + PA_m = -Q$. Further, let

$$H_{xm}(s) = (s\mathbb{I} - A_m)^{-1} b_m = \frac{\begin{bmatrix} 1 & s & \cdots & s^{n-1} \end{bmatrix}^\top}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}.$$

It follows from (6) that $H_x(s) = A_T H_{xm}(s)$. Letting $c_m = Pb_m$, it follows from Kalman-Yakubovich-Popov lemma that

$$H_m(s) = c_m^\top (s\mathbb{I} - A_m)^{-1} b_m \triangleq H_p(s)/H_d(s)$$

is strictly positive real and has relative degree 1. Also, let $T(s) \triangleq \frac{H_p(s)}{H_n(s)}$, which is a stable minimum-phase proper transfer function, and notice that $H_m(s) = H(s)T(s)$.

Let $r_0(t)$ and $r_{m0}(t)$ be the signals with corresponding Laplace transforms $(s\mathbb{I} - A)^{-1}x_0$ and $(s\mathbb{I} - A_m)^{-1}x_{m0}$ respectively, where x_{m0} is such that

$$c_m^\top x_{m0} = c^\top x_0 . \quad (8)$$

The design of the $\mathcal{L}_1$ adaptive controller involves a strictly proper transfer function $D(s)$ and a gain $k \in \mathbb{R}^+$, which leads to a strictly proper stable system

$$C(s) = (kF(s)D(s)) / (1 + kF(s)D(s)) \quad (9)$$

with DC gain $C(0) = 1$. Let

$$G(s) \triangleq H_{xm}(s)(1 - C(s)), \qquad k_g \triangleq -1/\left(c_m^\top A_m^{-1} b_m\right),$$

and define

$$G_m \triangleq \max_{F(s)} \|G(s)\|_{\mathcal{L}_1}, \quad C_a \triangleq \max_{F(s)} \left\| \frac{C(s)}{F(s)} \frac{1}{H_m(s)} c_m^\top \right\|_{\mathcal{L}_1},$$
$$C_m \triangleq \max_{F(s)} \|C(s)\|_{\mathcal{L}_1}, \quad C_f \triangleq \max_{F(s)} \|C(s)/F(s)\|_{\mathcal{L}_1}.$$

For the proofs of stability and performance bounds, the choice of $D(s)$ and k needs to ensure that there exists $\rho_r > 0$ such that

$$\|G(s)\|_{\mathcal{L}_1} < \frac{\rho_r - \|k_g C(s) H_{xm}(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty} - \|r_{m0}\|_{\mathcal{L}_\infty}}{L(\rho_r)\rho_r + \Delta_{\sigma_m}(\rho_r)} \quad (10)$$

where

$$L(\rho_r) \triangleq \left\|T^{-1}(s)\right\|_{\mathcal{L}_1} \|T(s)A_T\|_{\mathcal{L}_1} L_\rho \quad (11)$$
$$\rho \triangleq \|T(s)A_T\|_{\mathcal{L}_1} \left(\rho_r + \gamma_x + \|H_{xm}(s)\|_{\mathcal{L}_1} \Delta_{\sigma_{m2}} + \|r_{m0}\|_{\mathcal{L}_\infty}\right) + \|r_0\|_{\mathcal{L}_\infty}, \quad (12)$$

and

$$\Delta_{\sigma_m}(\rho_r) \geq \Delta_{\sigma_{m1}} + \Delta_{\sigma_{m2}} + L(\rho_r)\Big(\|r_{m0}\|_{\mathcal{L}_\infty} + \|H_{xm}(s)\|_{\mathcal{L}_1} \Delta_{\sigma_{m2}}\Big) \quad (13)$$
$$\Delta_{\sigma_{m1}} \geq \left\|T^{-1}(s)\right\|_{\mathcal{L}_1} \left(L_\rho \|r_0\|_{\mathcal{L}_\infty} + L_\rho L_{z2} + B + \epsilon\right)$$
$$\Delta_{\sigma_{m2}} \geq \left\|\left(c^\top r_0 - c_m^\top r_{m0}\right)/H_m(s)\right\|_{\mathcal{L}_\infty},$$

with $\epsilon > 0$, and with γ_x being defined as

$$\gamma_x \triangleq C_m / (1 - G_m L(\rho_r))\gamma_0 + \beta \quad (14)$$

for some arbitrarily small positive constants γ_0 and β.

Finally, let

$$\rho_u \triangleq \|T(s)\|_{\mathcal{L}_1} (\rho_{v_r} + \gamma_v), \quad (15)$$

where ρ_{v_r} and γ_v are defined as

$$\rho_{v_r} \triangleq \left\|\frac{C(s)}{F(s)}\right\|_{\mathcal{L}_1} \left(L(\rho_r)\rho_r + \Delta_{\sigma_m}(\rho_r) + |k_g| \|r\|_{\mathcal{L}_\infty}\right) \quad (16)$$
$$\gamma_v \triangleq C_f L(\rho_r)\gamma_x + C_a \gamma_0 . \quad (17)$$

B. System Transformation

In this section we demonstrate that the nonlinear system with unmodeled dynamics in (1) can be transformed into a linear system with unknown time-varying parameters and the same multiplicative unmodeled dynamics at the input.

Lemma 1: For the system in (1), if

$$\|x_t\| \leq \rho, \qquad \|u_t\| \leq \rho_u, \quad (18)$$

then there exist differentiable $\theta_f(\tau)$ and $\sigma_f(\tau)$ with bounded derivatives over $\tau \in [0, t]$ such that

$$|\theta_f(\tau)| < \theta_{fb}(\rho_r), \qquad \left|\dot{\theta}_f(\tau)\right| < d_\theta(\rho_r), \quad (19)$$
$$|\sigma_f(\tau)| < \Delta_{\sigma_f}(\rho_r), \qquad |\dot{\sigma}_f(\tau)| < d_{\sigma_f}(\rho_r), \quad (20)$$
$$f(x(\tau), z(\tau), \tau) = \theta_f(\tau) \|x_\tau\|_{\mathcal{L}_\infty} + \sigma_f(\tau), \quad (21)$$

where θ_{fb} and Δ_{σ_f} are given by

$$\theta_{fb}(\rho_r) \triangleq L_\rho, \qquad \Delta_{\sigma_f}(\rho_r) \triangleq L_\rho L_{z2} + B + \epsilon .$$

Proof. The proof is similar to the proof of Lemma 2 in [18].

If (18) holds, Lemma 1 implies that the nonlinear system in (1) can be rewritten over $\tau \in [0, t]$ as

$$\dot{x}(\tau) = Ax(\tau) + b\left(\mu_u(\tau) + \theta_f(\tau)\|x_\tau\|_{\mathcal{L}_\infty} + \sigma_f(\tau)\right),$$
$$\mu_u(s) = F(s)u(s), \quad y(\tau) = c^\top x(\tau), \quad x(0) = x_0, \quad (22)$$

where $\theta_f(\tau)$ and $\sigma_f(\tau)$ are unknown bounded time-varying signals with bounded derivatives.

Let $w_\xi(t)$ to be the output of the system $\mathcal{W}_\xi$ driven by the input $\xi(t)$ and given by

$$\mathcal{W}_\xi : \quad \begin{cases} w_\xi(s) = T^{-1}(s) w_1(s) \\ w_1(t) = \theta_f(t) \|w_{2t}\|_{\mathcal{L}_\infty} \\ w_2(s) = T(s) A_T \xi(s) \end{cases} . \quad (23)$$

It follows from (19) and the definition of $L(\rho_r)$ in (11) that $\|w_{\xi t}\|_{\mathcal{L}_\infty} \leq L(\rho_r) \|\xi_t\|_{\mathcal{L}_\infty}$.

Further, define

$$\rho_m \triangleq \rho_r + \gamma_x, \qquad \rho_v \triangleq \rho_{v_r} + \gamma_v,$$

where γ_x and γ_v were defined in (14) and (17) respectively.

Lemma 2: For the system in (1), if $\|x_t\| \leq \rho$ and $\|u_t\| \leq \rho_u$, then there exists a bounded signal $\sigma_m(\tau)$ over the interval $\tau \in [0, t]$, whose derivative is also bounded, such that the output $y(\tau)$ of the system in (1) is equal to the output $y_m(\tau)$ of the following system

$$\begin{aligned} \dot{x}_m(\tau) &= A_m x_m(\tau) + b_m\left(\mu_v(\tau) + w_{x_m}(\tau) + \sigma_m(\tau)\right), \\ \mu_v(s) &= F(s)v(s), \quad x_m(0) = x_{m0}, \\ y_m(\tau) &= c_m^\top x_m(\tau), \end{aligned} \tag{24}$$

where $w_{x_m}(\tau)$ is the output of the system $\mathcal{W}_\xi$ in (23) driven by $x_m(\tau)$, and $v(s) = T^{-1}(s)u(s)$ is the new (virtual) control signal. Moreover, we have

$$|\sigma_m(\tau)| \le \Delta_{\sigma_m}(\rho_r), \qquad |\dot{\sigma}_m(\tau)| \le d_{\sigma_m}(\rho_r)$$

for all $\tau \in [0,t]$, where Δ_{σ_m} was defined in (13), and d_{σ_m} can be derived from the original bounds on $\theta_f(\tau)$ and $\sigma_f(\tau)$.

Proof. The proof is similar to the proof of Lemma 2 in [12].

C. $\mathcal{L}_1$ Adaptive Output Feedback Controller

Since for any $v(t)$ the output of the system in (24) is equivalent to the output of the system in (1) with $u(s) = T(s)v(s)$, we will design an adaptive output feedback controller $v(t)$ for the system in (24) and, using $T(s)$, we will implement it for the system in (1). The elements of the $\mathcal{L}_1$ adaptive output feedback control architecture are introduced below.

State predictor: We consider the following state predictor

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m\hat{x}(t) + b_m\left(\hat{\omega}(t)v(t) + \hat{\sigma}(t)\right), \\ \hat{y}(t) &= c_m^\top \hat{x}(t), \quad \hat{x}(0) = x_{m0}, \end{aligned} \tag{25}$$

where x_{m0} was introduced in (8), and the adaptive estimates $\hat{\omega}(t)$ and $\hat{\sigma}(t)$ are governed by the following adaptation laws.

Adaptive laws:

$$\begin{aligned} \dot{\hat{\omega}}(t) &= \Gamma_c \mathrm{Proj}\left(\hat{\omega}(t), -\tilde{y}(t)v(t)\right), \quad \hat{\omega}(0) = \hat{\omega}_0, \\ \dot{\hat{\sigma}}(t) &= \Gamma_c \mathrm{Proj}\left(\hat{\sigma}(t), -\tilde{y}(t)\right), \quad \hat{\sigma}(0) = \hat{\sigma}_0, \end{aligned} \tag{26}$$

where $\tilde{y}(t) = \hat{y}(t) - y(t)$, $\Gamma_c \in \mathbb{R}^+$ is the adaptation rate subject to a computable lower bound, and $\mathrm{Proj}(\cdot,\cdot)$ denotes the projection operator [20].

Control law: The control law is generated through feedback of the following system

$$v(s) = -k\chi(s), \qquad \chi(s) = D(s)\bar{r}(s), \tag{27}$$

where $\bar{r}(t) = \hat{\omega}(t)v(t) + \hat{\sigma}(t) - k_g r(t)$, while k and $D(s)$ were introduced before (9).

The complete $\mathcal{L}_1$ adaptive controller consists of (25)-(27), subject to the $\mathcal{L}_1$-norm condition in (10).

IV. Analysis of $\mathcal{L}_1$ Adaptive Controller

A. Closed-Loop Reference System

In this section, we characterize the closed-loop reference system that the $\mathcal{L}_1$ adaptive controller tracks both in transient and steady-state and prove its stability. Towards this end, we consider the ideal non-adaptive version of the adaptive controller and define the *closed-loop reference system* as

$$\begin{aligned} \dot{x}_{ref}(t) &= A_m x_{ref}(t) + b_m\left(\mu_{v_{ref}}(t) + w_{x_{ref}}(t) + \sigma_m(t)\right), \\ \mu_{v_{ref}}(s) &= F(s)v_{ref}(s), \quad x_{ref}(0) = x_{m0}, \\ v_{ref}(s) &= -kD(s)\bar{r}_{ref}(s), \quad y_{ref}(t) = c_m^\top x_{ref}(t), \end{aligned} \tag{28}$$

where $w_{x_{ref}}(t)$ is the output of the system $\mathcal{W}_\xi$ in (23) driven by $x_{ref}(t)$, and

$$\bar{r}_{ref}(t) = \mu_{v_{ref}}(t) + w_{x_{ref}}(t) + \sigma_m(t) - k_g r(t).$$

We note that the control law $\mu_{v_{ref}}(t)$, which will be used in the analysis of the performance bounds, is not implementable since its definition involves $F(s)$, $\theta_f(t)$, and $\sigma_f(t)$, which are unknown. This closed-loop reference system defines the achievable control objective. The next lemma proves stability of this system by the appropriate choice of k and $D(s)$.

Lemma 3: For the closed-loop reference system in (28), subject to the $\mathcal{L}_1$-norm condition in (10), if $\|x_{m0}\|_\infty < \rho_r$, and the bounds in (19) and (20) hold, then

$$\|x_{ref}\|_{\mathcal{L}_\infty} < \rho_r, \qquad \|v_{ref}\|_{\mathcal{L}_\infty} < \rho_{v_r}, \tag{29}$$

where ρ_r and ρ_{v_r} were defined in (10) and (16) respectively.

Proof. The proof is omitted due to space limitations.

B. Equivalent Linear Time-Varying System

In this section, we demonstrate that the linear time-varying system with multiplicative unmodeled dynamics at the input in (24) can be transformed into a new equivalent linear system with time-varying parameters.

In order to streamline the subsequent analysis, we need to introduce several notations. Define ω_ℓ and ω_u be two nonzero constants with the same sign, $\omega_\ell < \omega_u$. Also, let

$$\Delta_\sigma(\rho_r) \ge L(\rho_r)\rho_m + \Delta_{\sigma_m}(\rho_r) + \Delta_{\sigma_\omega}(\rho_r) \tag{30}$$

$$\Delta_{\sigma_\omega}(\rho_r) \ge \|F(s) - (\omega_\ell + \omega_u)/2\|_{\mathcal{L}_1}\rho_v, \tag{31}$$

and define $\rho_{\dot{v}}$ as

$$\rho_{\dot{v}} \triangleq \|ksD(s)\|_{\mathcal{L}_1}\Big(\rho_v \max\{|\omega_\ell|, |\omega_u|\} + \Delta_\sigma(\rho_r) + |k_g|\,\|r\|_{\mathcal{L}_\infty}\Big). \tag{32}$$

It can be checked easily that $sD(s)$ is a stable and proper transfer function, and hence $\|ksD(s)\|_{\mathcal{L}_1}$ is finite.

Lemma 4: For the system in (24), if

$$\|x_{mt}\| \le \rho_m, \qquad \|v_t\| \le \rho_v, \tag{33}$$

then there exist ω and $\sigma_\omega(\tau)$ over $\tau \in [0,t]$ such that for any $0 \le \tau \le t$, we have

$$\begin{aligned} \omega_\ell < \omega < \omega_u, \quad |\sigma_\omega(\tau)| \le \Delta_{\sigma_\omega}(\rho_r), \\ \mu_v(\tau) = \omega v(\tau) + \sigma_\omega(\tau), \end{aligned} \tag{34}$$

and the system in (1) can be rewritten over $\tau \in [0,t]$ as

$$\begin{aligned} \dot{x}_m(\tau) &= A_m x_m(\tau) + b_m\left(\omega v(\tau) + \sigma(\tau)\right), \\ y_m(\tau) &= c_m^\top x_m(\tau), \quad x_m(0) = x_{m0}, \end{aligned} \tag{35}$$

where $\sigma(\tau) = w_{x_m}(\tau) + \sigma_m(\tau) + \sigma_\omega(\tau)$. If, in addition to (33), we have $\|\dot{v}_t\|_{\mathcal{L}_\infty} \le \rho_{\dot{v}}$, then $\sigma_\omega(\tau)$ is differentiable and for any $0 \le \tau \le t$

$$|\dot{\sigma}_\omega(\tau)| \le d_{\sigma_\omega}(\rho_r) \triangleq \left\|F(s) - \frac{\omega_\ell + \omega_u}{2}\right\|_{\mathcal{L}_1}\rho_{\dot{v}}. \tag{36}$$

Proof. The proof is similar to the proof of Lemma 2 in [13].

C. Transient and Steady-State Performance

We introduce the following notations

$$\beta_{01}(\rho_r) \triangleq 4\Delta_\sigma(\rho_r)L(\rho_r)\Big(d_\theta(\rho_r)/L_\rho + \|A_m\|_{\mathcal{L}_1} + L(\rho_r)\|b_m\|_{\mathcal{L}_1}\Big)$$

$$\beta_{02}(\rho_r) \triangleq 4\Delta_\sigma(\rho_r)\Big(L(\rho_r)\|b_m\|_{\mathcal{L}_1}(L_F\rho_v + \Delta_{\sigma_m}(\rho_r)) + d_{\sigma_m}(\rho_r) + d_{\bar{\sigma}}(\rho_r)\Big)$$

$$\beta_1(\rho_r) \triangleq \beta_{01}(\rho_r)\|C(s)\|_{\mathcal{L}_1}/\left(1 - \|G(s)\|_{\mathcal{L}_1} L(\rho_r)\right)$$

$$\beta_2(\rho_r) \triangleq \beta_{01}(\rho_r)\rho_r + \beta_{02}(\rho_r)$$

$$\theta_m(\rho_r) \geq 4\Delta_\sigma^2(\rho_r) + (\omega_u - \omega_\ell)^2 + \frac{\lambda_{\max}(P)}{\lambda_{\max}(Q)}(\beta_1(\rho_r)\gamma_0 + \beta_2(\rho_r)) .$$

Let the adaptive gain Γ_c be lower bounded bounded as

$$\Gamma_c > \frac{\theta_m(\rho_r)}{\lambda_{\min}(P)\gamma_0^2}, \tag{37}$$

and the projection of $\hat{\omega}(t)$ and $\hat{\sigma}(t)$ be confined to the bounds

$$\omega_\ell \leq \hat{\omega}(t) \leq \omega_u , \quad |\hat{\sigma}(t)| \leq \Delta_\sigma(\rho_r) . \tag{38}$$

Lemma 5: Given the system in (24) and the $\mathcal{L}_1$ adaptive controller defined via (25), (26) and (27) subject to (10), (37) and (38), if

$$\|x_{mt}\| \leq \rho_m , \quad \|v_t\| \leq \rho_v , \quad \|\dot{v}_t\| \leq \rho_{\dot{v}} , \tag{39}$$

then

$$\|\tilde{x}_t\|_{\mathcal{L}_\infty} < \gamma_0 , \tag{40}$$

where γ_0 was introduced in (14).

Proof. The proof is omitted due to space limitations.

Theorem 1: Given the system in (24) and the $\mathcal{L}_1$ adaptive controller defined via (25), (26), and (27) subject to (10), (37) and (38), if $\|x_{m0}\|_\infty < \rho_r$, then

$$\|x_m\|_{\mathcal{L}_\infty} < \rho_m , \quad \|v\|_{\mathcal{L}_\infty} < \rho_v , \tag{41}$$

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \gamma_0, \tag{42}$$

$$\|x_m - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_x, \tag{43}$$

$$\|y_m - y_{ref}\|_{\mathcal{L}_\infty} \leq \left\|c_m^\top\right\|_{\mathcal{L}_1}\gamma_x, \tag{44}$$

$$\|v - v_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_v , \tag{45}$$

where γ_x and γ_v were defined in (14) and (17) respectively.

Proof. The proof is given in the Appendix.

Thus, the tracking error between $y(t)$ and $y_{ref}(t)$, as well as between $v(t)$ and $v_{ref}(t)$, is uniformly bounded by a constant inverse proportional to Γ_c. This implies that both in transient and steady-state one can achieve arbitrarily close tracking performance for both signals simultaneously by increasing Γ_c. To understand how these bounds can be used for ensuring transient response with *desired* specifications, we consider the *ideal* control signal for the system in (24)

$$\mu_{v_{id}} = k_g r(t) - w_{x_{id}}(t) - \sigma_m(t) , \tag{46}$$

which leads to the desired system response

$$\dot{x}_{id}(t) = A_m x_{id}(t) + b_m k_g r(t), \quad y_{id}(t) = c_m^\top x_{id}(t) \tag{47}$$

by canceling the uncertainties exactly. In the closed-loop reference system (28), $\mu_{v_{id}}(t)$ is further low-pass filtered by $C(s)$ to have guaranteed low-frequency range. Thus, the closed-loop reference system in (28) has a different response as compared to (47) achieved with (46). Similar to [15], the response of $y_{ref}(t)$ can be made as close as possible to (47) by reducing $\|G(s)\|_{\mathcal{L}_1}$ arbitrarily. For constant $F(s) = F$, we can make $\|G(s)\|_{\mathcal{L}_1}$ arbitrarily small. However, for the general case of unknown $F(s)$, the design of k and $D(s)$ which satisfy (10), is an open problem, and depends on the available knowledge of $F(s)$.

V. Simulations

As an illustrative example, consider the system in (1) with

$$A = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where the input multiplicative unmodeled dynamics are $F(s) = (-s+40)/(s+30)$, and $f(x(t), z(t), t) = x^\top(t)x(t) + z^2(t)$, while $z(t)$ is the output of the unmodeled dynamics

$$z(s) = \frac{-s+1}{s^2+3s+2}z_u(s) , \quad z_u(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) . \tag{48}$$

The control objective is to design a control $u(t)$ to achieve tracking of bounded reference input $r(t)$ by $y(t)$, where $\|r\|_{\mathcal{L}_\infty} \leq 1$.

In the implementation of the $\mathcal{L}_1$ controller, we set

$$A_m = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad b_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $Q = \mathbb{I}$, which leads to

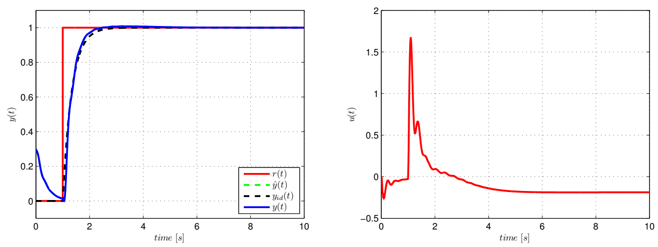
$$P = \begin{bmatrix} 1.1667 & 0.1667 \\ 0.1667 & 0.1667 \end{bmatrix}, \quad c_m = \begin{bmatrix} 0.1667 \\ 0.1667 \end{bmatrix}.$$

Also, we set $D(s) = \frac{1}{s}$, $k = 15$, and $\Gamma_c = 50,000$. It follows that $k_g = 18$, $T(s) = 0.1667(s+1)/(2s+3)$, and the compact sets can be conservatively chosen according to the following bounds $\omega_u = 8$, $\omega_\ell = 0.5$, and $\Delta_\sigma = 35$.

The simulation results of the $\mathcal{L}_1$ adaptive controller for a unit step reference command are shown in Figs 1a-1b. We see that $y(t)$ converges to $r(t)$ asymptotically. In Figs 2a-2b, we consider the performance of the $\mathcal{L}_1$ adaptive controller for a unit step input in the presence of the following input to the unmodeled dynamics in (48)

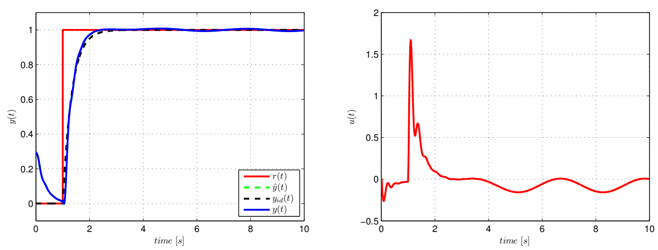
$$z_u(t) = \begin{bmatrix} \cos(t) & 1 \end{bmatrix} x(t) . \tag{49}$$

Figures 3a-3b show the response of the closed-loop system to a time-varying reference command $r(t) = \sin(0.3t)$ in the presence of the same input of the unmodeled dynamics as in (49). Finally, Figs 4a-4b consider higher frequencies in the input of the unmodeled dynamics, like $z_u(t) = \begin{bmatrix} \cos(5t) & 1 \end{bmatrix} x(t) + \sin(5t)$. We observe that the fast adaptation ability of $\mathcal{L}_1$ adaptive controller guarantees uniform transient performance to different reference inputs, independent of the unmodeled dynamics, without any retuning.



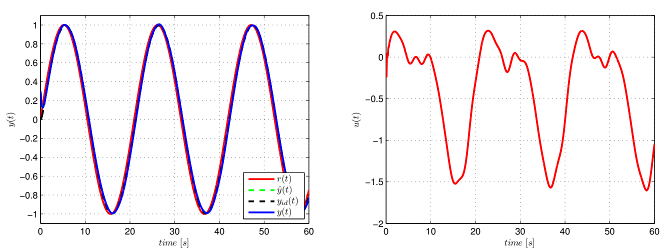
(a) Output response (b) Control action

Fig. 1: Performance of the $\mathcal{L}_1$ adaptive controller for a unit step reference command and $z_u(t) = [1\ 1]x(t)$.



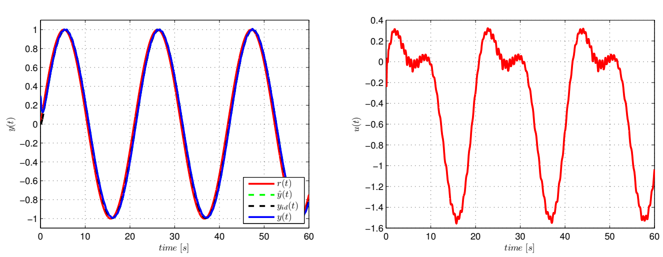
(a) Output response (b) Control action

Fig. 2: Performance of the $\mathcal{L}_1$ adaptive controller for a unit step reference command and $z_u(t) = [\cos(t)\ 1]x(t)$.



(a) Output response (b) Control action

Fig. 3: Performance of the $\mathcal{L}_1$ adaptive controller for $r(t) = \sin(0.3t)$ and $z_u(t) = [\cos(t)\ 1]x(t)$.



(a) Output response (b) Control action

Fig. 4: Performance of the $\mathcal{L}_1$ adaptive controller for $r(t) = \sin(0.3t)$ and $z_u(t) = [\cos(5t)\ 1]x(t) + \sin(5t)$.

VI. Conclusion

An $\mathcal{L}_1$ adaptive output feedback control architecture is presented that has guaranteed transient response in addition to stable tracking for a class of uncertain nonlinear systems in the presence of time and state dependent unknown nonlinearities, as well as linear multiplicative unmodeled dynamics. The control signal and the system response approximate the same signals of a closed-loop reference system, which can be designed to achieve desired specifications.

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Appendix

Proof of Theorem 1. We prove this Theorem by contradiction. Assume that (41) is not true. Then, since $\|x_{m0}\|_\infty < \rho_r$ by assumption, $v(0) = 0$, and $x_m(t)$ and $v(t)$ are continuous and differentiable, there exists a time $\tau \geq 0$ such that

$$\|x_m(\tau)\|_\infty = \rho_m\,, \quad \text{or} \tag{50}$$
$$\|v(\tau)\|_\infty = \rho_v\,, \tag{51}$$

while

$$\|x_{m\tau}\|_{\mathcal{L}_\infty} \leq \rho_m\,, \qquad \|v_\tau\|_{\mathcal{L}_\infty} \leq \rho_v\,, \tag{52}$$

which implies that

$$\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho\,, \qquad \|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u\,. \tag{53}$$

Hence, Lemmas 1 and 2 hold over $t \in [0, \tau]$, and the original nonlinear system in (1) can be rewritten as the linear system with unknown time-varying parameters in (24) for any $0 \leq t \leq \tau$.

Moreover, since $\theta_f(t)$ and $\sigma_f(t)$ are bounded as in (19) and (20) for any $0 \leq t \leq \tau$, it follows from Lemma 3 that the closed-loop reference system is BIBO stable and also that

$$\|x_{ref\,\tau}\|_{\mathcal{L}_\infty} \leq \rho_r \tag{54}$$
$$\|v_{ref\,\tau}\|_{\mathcal{L}_\infty} \leq \rho_{v_r}\,. \tag{55}$$

Since the projection operator ensures that for any $t \geq 0$ $|\hat{\sigma}(t)| \leq \Delta_\sigma$ and $|\hat{\omega}(t)| \leq \max\{|\omega_\ell|, |\omega_u|\}$, it follows from (52) that

$$\|\bar{r}_\tau\|_{\mathcal{L}_\infty} \leq \rho_v \max\{|\omega_\ell|, |\omega_u|\} + \Delta_\sigma(\rho_r) + |k_g|\, \|r\|_{\mathcal{L}_\infty}\,. \tag{56}$$

The control law in (27) implies $v(s) = -kD(s)\bar{r}(s)$, and hence $sv(s) = -skD(s)\bar{r}(s)$. Using (56) and the definition of $\rho_{\dot{v}}$ in (32), it follows that

$$\|\dot{v}_\tau\|_{\mathcal{L}_\infty} \leq \rho_{\dot{v}}\,, \tag{57}$$

and Lemma 4 implies that, for any $t \in [0, \tau]$, the system in (24) can be rewritten as the linear system in (35) and the upper bound in (36) holds. Hence, if we choose Γ_c according to (37), Lemma 5 implies that

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \leq \gamma_0\,. \tag{58}$$

Next, we define $\eta_m(t) = w_{x_m}(t) + \sigma_m(t)$ and $\tilde{r}(t) = \tilde{\omega}(t)v(t) + \tilde{\sigma}$, where $\tilde{\omega}(t) = \hat{\omega}(t) - \omega$ and $\tilde{\sigma}(t) = \hat{\sigma}(t) - \eta_m(t) - \sigma_\omega(t)$. Then, it follows from (27) that $\chi(s) = D(s)\left(\omega v(s) + \tilde{r}(s) + \eta_m(s) + \sigma_\omega(s) - k_g r(s)\right)$. Consequently

$$\chi(s) = \frac{D(s)}{1 + \omega k D(s)}\left(\eta_m(s) + \sigma_\omega(s) - k_g r(s) + \tilde{r}(s)\right) \tag{59}$$
$$v(s) = -\frac{kD(s)}{1 + \omega k D(s)}\left(\eta_m(s) + \sigma_\omega(s) - k_g r(s) + \tilde{r}(s)\right), \tag{60}$$

and hence one can write that

$$v(s) + kD(s)\left(\omega v(s) + \sigma_\omega(s)\right) = -kD(s)\left(\eta_m(s) - k_g r(s) + \tilde{r}(s)\right), \tag{61}$$

which along with (34) implies that

$$v(s) + kD(s)\mu_v(s) = -kD(s)\left(\eta_m(s) - k_g r(s) + \tilde{r}(s)\right). \tag{62}$$

Since $\mu_v(s) = F(s)v(s)$, it follows from (62) and the definition of $C(s)$ in (9) that

$$v(s) = -\frac{kD(s)}{1 + kD(s)F(s)}\left(\eta_m(s) - k_g r(s) + \tilde{r}(s)\right) \tag{63}$$
$$\mu_v(s) = C(s)\left(\eta_m(s) - k_g r(s) + \tilde{r}(s)\right), \tag{64}$$

and the system in (35) consequently takes the form

$$x_m(s) = G(s)\eta_m(s) - H_{xm}(s)C(s)\tilde{r}(s) + H_{xm}(s)C(s)k_g r(s) + r_{m0}(s)\,. \tag{65}$$

It follows from the definition of the reference system in (28) that

$$x_{ref}(s) = G(s)\eta_{ref}(s) + H_{xm}(s)C(s)k_g r(s) + r_{m0}(s)\,, \tag{66}$$

where $\eta_{ref}(t) = w_{x_{ref}}(t) + \sigma_m(t)$.

Let $e(t) = x_m(t) - x_{ref}(t)$. Then, using (65) and (66), we get

$$e(s) = G(s)\eta_e(s) - H_{xm}(s)C(s)\tilde{r}(s)\,, \quad e(0) = 0\,, \tag{67}$$

where $\eta_e(s)$ is defined as

$$\eta_e(s) = w_{x_m}(s) - w_{x_{ref}}(s) = T^{-1}(s)w_{1e}(s)\,, \tag{68}$$

with $w_{1e}(t)$ being the signal

$$w_{1e}(t) = \theta_f(t)(\|w_{2x_m\tau}\|_{\mathcal{L}_\infty} - \|w_{2x_{ref}\tau}\|_{\mathcal{L}_\infty})\,, \tag{69}$$

and $w_{2x_m}(t)$ and $w_{2x_{ref}}(t)$ are the signals with Laplace transformation $w_{2x_m}(s) = T(s)A_T x_m(s)$ and $w_{2x_{ref}}(s) = T(s)A_T x_{ref}(s)$ respectively. Since we have

$$\|w_{2x_m\tau}\|_{\mathcal{L}_\infty} - \|w_{2x_{ref}\tau}\|_{\mathcal{L}_\infty} \leq \|(w_{2x_m} - w_{2x_{ref}})_\tau\|_{\mathcal{L}_\infty}$$
$$-\|(w_{2x_m} - w_{2x_{ref}})_\tau\|_{\mathcal{L}_\infty} \leq \|w_{2x_m\tau}\|_{\mathcal{L}_\infty} - \|w_{2x_{ref}\tau}\|_{\mathcal{L}_\infty}\,,$$

then $\|\eta_{e\tau}\|_{\mathcal{L}_\infty} \leq \left\|T^{-1}(s)\right\|_{\mathcal{L}_1} L_\rho \left\|\left(w_{2x_m} - w_{2x_{ref}}\right)_\tau\right\|_{\mathcal{L}_\infty} \leq \left\|T^{-1}(s)\right\|_{\mathcal{L}_1} \|T(s)A_T\|_{\mathcal{L}_1} L_\rho \|e_\tau\|_{\mathcal{L}_\infty}$.

It follows from the definition of state predictor (25) and the system in (35) that over $t \in [0, \tau]$ the error dynamics can be written as $\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b_m\tilde{r}(t)\,, \tilde{x}(0) = 0$, which leads to

$$\tilde{x}(s) = H_{xm}(s)\tilde{r}(s)\,. \tag{70}$$

Hence, it follows from (67) and definition of $L(\rho_r)$ in (11) that $\|e_\tau\|_{\mathcal{L}_\infty} \leq \|G(s)\|_{\mathcal{L}_1} L(\rho_r)\, \|e_\tau\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1} \|(\tilde{x}_\tau)\|_{\mathcal{L}_\infty}$, which leads to $\|e_\tau\|_{\mathcal{L}_\infty} \leq \dfrac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L(\rho_r)} \|\tilde{x}_\tau\|_{\mathcal{L}_\infty}$. The condition in (10) ensures that $\|G(s)\|_{\mathcal{L}_1} L(\rho_r) < 1$, and hence it follows from (58) that $\|e_\tau\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L(\rho_r)}\gamma_0$, which along with the definition of γ_x in (14) leads to

$$\|e_\tau\|_{\mathcal{L}_\infty} \leq \gamma_x - \beta < \gamma_x\,. \tag{71}$$

We notice that from (54) and (71), we can conclude

$$\|x_{m\tau}\|_{\mathcal{L}_\infty} \leq \rho_r + \gamma_x - \beta = \rho_m - \beta < \rho_m\,. \tag{72}$$

On the other hand, it follows from (27) and (28) that

$$v(s) - v_{ref}(s) = -\frac{C(s)}{F(s)}\eta_e(s) - \frac{C(s)}{F(s)}\tilde{r}(s)\,, \tag{73}$$

and equation (70) implies that (73) can re written as $v(s) - v_{ref}(s) = -\frac{C(s)}{F(s)}\eta_e(s) - \frac{C(s)}{F(s)}\frac{1}{c_m^\top H_{xm}(s)}c_m^\top\tilde{x}(s)$. Since the choice of k and $D(s)$ guarantees that $C(s)$ is stable and strictly proper, the system $\frac{C(s)}{F(s)}\frac{1}{c_m^\top H_{xm}(s)}$ is stable and (at least) proper, which implies that its $\mathcal{L}_1$-norm exists and is bounded. Thus, we have $\left\|(v - v_{ref})_\tau\right\|_{\mathcal{L}_\infty} \leq \|C(s)/F(s)\|_{\mathcal{L}_1} L(\rho_r)\, \|e_\tau\|_{\mathcal{L}_\infty} + \left\|\frac{C(s)}{F(s)}\frac{1}{c_m^\top H_{xm}(s)}c_m^\top\right\|_{\mathcal{L}_1} \|\tilde{x}_\tau\|_{\mathcal{L}_\infty}$, and (58) and (71) along with the definition of γ_v in (17) lead to

$$\left\|(v - v_{ref})_\tau\right\|_{\mathcal{L}_\infty} < \gamma_v\,. \tag{74}$$

We notice that from (55) and (74), we can conclude

$$\|v_\tau\|_{\mathcal{L}_\infty} < \rho_{v_r} + \gamma_v = \rho_v\,. \tag{75}$$

Finally, we note that the upper bounds in (72) and (75) contradict the equalities in (50) and (51), which proves (41). The results in (42)-(45) follow directly from the bounds in (41), (57), (58), (71), and (74), and from the fact that $y_m(t) - y_{ref}(t) = c_m^\top\left(x_m(t) - x_{ref}(t)\right)$.