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Average consensus for networks of continuous-time agents with delayed information and jointly-connected topologies

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***Abstract*— This paper investigates average consensus problem in networks of continuous-time agents with delayed information and jointly-connected topologies. A sufficient condition in terms of linear matrix inequalities (LMIs) is given under which all agents asymptotically reach average consensus, where the communication structures vary over time and the corresponding graphs may not be connected. Finally, simulation results are provided to demonstrate the effectiveness of our theoretical results.**

Keywords—-average consensus, time-delay, switching jointly-connected topologies, multi-agent systems

I. INTRODUCTION

Distributed coordinated control of multiple agents has attracted a great deal of attention from many fields such as biology, physics, robotics and control engineering [1]-[19]. One critical problem in distributed coordinated control of multiple agents is to find control laws to make all agents reach an agreement regarding a certain quantity of interest that depends on the states of all agents. This problem is usually called consensus problem.

One difficulty of consensus problems is how to investigate the effects of communication topology among the agents. This is important because the topologies heavily influence the stability of the multi-agent systems, especially when the communication topology is switching jointly-connected. In the past decade, numerous studies have been conducted on this problem [2]-[5]. For example, Jadbabaie *et al.* studied the consensus problems in discrete-time multi-agent systems with jointly-connected topologies [2]. Moreau reported a simple network model of agents interacting via time-dependent communication links based on graph theory and set-valued Lyapunov theory [3]. Ren *et al.* extended the results of [2] and gave some more relaxable conditions [4].

Another difficulty is how to analyze the stability of the multi-agent systems when the communication time-delays are involved. In practical situation, the disturbance of time-delays is usually unavoidable which might make the multi-agent system to oscillate or diverge, and therefore it is important to investigate its effects on the behavior of the multi-agent system. Compared with conventional control systems, to deal with delay-related problems in multi-agent systems is much more difficult and complex, since the closed-loop system matrices are usually singular. In order to solve such problems, many researches have been also carried out [6]-[9]. In [6], Olfati-Saber *et al.* investigated a systematical framework of consensus problems in networks of agents with a simple continuous-time integrator and gave a sufficient and necessary condition for average consensus of the system with time-delay and fixed topology. Bliman *et al.* focused on the average consensus problem and extended the results of [6] to the nonuniform time-delay case [7]. Moreover, Liu *et al.* addressed an asynchronous discrete-time formulation with fixed topology and derived conditions under which a multi-agent system achieves cohesiveness in the presence of sensing delays, sensing errors and sensing topology [9].

Recently, much attention has been paid to combining both difficulties. The authors of [10]-[13] studied the networks of agents where time-delays affect only the information that is being transmitted and showed that arbitrary bounded time-delays can safely be tolerated. Also, Sun *et al.* investigated the average consensus problem and gave sufficient conditions for state consensus of the system under the assumption that each possible communication topology is connected [14].

In this paper, we investigate the average consensus problem in networks of continuous-time agents with delayed information. The communication topologies considered here are jointly-connected and coupled with time-delays, different from [14], where each possible communication topology is required to be connected. As commonly known, it is much more hard to study the consensus problem on jointly-connected topologies than on connected topologies, especially when time-delays are involved. The approach adopted is to construct a common Lyapunov function whose derivative is negative semi-definite and use a contradiction method to show that the Lypuanov function eventually converges to zero. The obtained condition is given in terms of LMIs and each LMI corresponds to a possible connected component of the communication topology.

The following notations will be used throughout this paper. $\mathbb{R}^m$ denotes the set of all m dimensional real column vectors; I_m denotes the m dimensional unit matrix; $\otimes$ denotes the kronecker product; $\mathbf{1}$ represents $[1, 1, \cdots, 1]^T$ with compatible dimensions (sometimes, we use $\mathbf{1}_n$ to denote $\mathbf{1}$ with dimension n); 0 denotes zero value or zero matrix with appropriate dimensions; the symbol $*$ denotes the symmetric term of

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a symmetric matrix; $\|\cdot\|$ refers to the standard Euclidean norm for vectors; $C([a,b],\mathbb{R}^n)$ represents the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$.

II. Problem Description

A. Graph Theory

Let $G(\mathscr{V},\varepsilon,\mathscr{A})$ be an undirected graph of order n, where $\mathscr{V}=\{\mathrm{v}_1,\cdots,\mathrm{v}_n\}$ is the set of nodes, $\varepsilon\subseteq\mathscr{V}\times\mathscr{V}$ is the set of edges, and $\mathscr{A}=[a_{ij}]$ is a weighted adjacency matrix. The node indexes belong to a finite index set $\mathscr{I}=\{1,2,\cdots,n\}$. An edge of G is denoted by $e_{ij}=(\mathrm{v}_i,\mathrm{v}_j)$. The adjacency matrix is defined as $a_{ii}=0$ and $a_{ij}=a_{ji}\geq 0$. $a_{ij}>0$ if and only if there is an edge between node v_i and node v_j. The set of neighbors of node v_i is denoted by $N_i=\{\mathrm{v}_j\in\mathscr{V}:(\mathrm{v}_i,\mathrm{v}_j)\in\varepsilon\}$. The in-degree and out-degree of node v_i are defined respectively as $d_{in}(\mathrm{v}_i)=\sum_{j=1}^n a_{ji}$ and $d_o(\mathrm{v}_i)=\sum_{j=1}^n a_{ij}$. Then, the Laplacian corresponding to the undirected graph is defined as $L=[l_{ij}]$, where $l_{ii}=d_o(\mathrm{v}_i)$ and $l_{ij}=-a_{ij},\ i\neq j$. A path is a sequence of ordered edges of the form $(\mathrm{v}_{i_1},\mathrm{v}_{i_2}),(\mathrm{v}_{i_2},\mathrm{v}_{i_3}),\cdots$, where $\mathrm{v}_{i_j}\in\mathscr{V}$. If there is a path from every node to every other node, the graph is said to be connected. The union of a collection of graphs $\bar{G}_1,\cdots,\bar{G}_m$, with the same node set $\mathscr{V}$, is defined as the graph $\bar{G}_{1-m}$ with the node set $\mathscr{V}$ and edge set equaling the union of the edge sets of all of the graphs in the collection. Moreover, this collection, $\bar{G}_1,\cdots,\bar{G}_m$ is jointly-connected if its union graph $\bar{G}_{1-m}$ is connected.

Lemma 1: [20] If the graph G is connected, then its Laplacian L satisfies:

1) Zero is a simple eigenvalue of L, and $\mathbf{1}_n$ is the corresponding eigenvector, i.e., $L\mathbf{1}_n=0$,

2) The rest $n-1$ eigenvalues are all positive and real.

B. Model

Suppose that the multi-agent system under consideration consists of n agents. Each agent is regarded as a node in an undirected graph, G. Each edge $(\mathrm{v}_j,\mathrm{v}_i)\in\varepsilon(G(t))$ or $(\mathrm{v}_i,\mathrm{v}_j)\in\varepsilon(G(t))$ corresponds to an available information channel between agent v_i and agent v_j at time t. Moreover, each agent updates its current state based upon the information received from its neighbors. And the set of the neighbors of the ith agent at time t is denoted by $N_i(t)$. The Laplacian of the graph $G(t)$ is denoted by L_σ.

Let x_i be the state of the ith agent. Suppose that each agent has the dynamics as follows:

$$\dot{x}_i(t)=u_i(t),$$

with the initial condition $x_i(s)=x_i(0),\ s\in(-\infty,0]$, where $u_i(t)$ is the control input (or protocol) at time t.

We say protocol u_i asymptotically solves the consensus problem, if and only if the states of agents satisfy

$$\lim_{t\to+\infty}[x_i(t)-x_j(t)]=0,$$

for all $i,j\in\mathscr{I}$. Furthermore, if

$$\lim_{t\to+\infty}x_i(t)=\frac{1}{n}\sum_j x_j(0),$$

we say protocol u_i asymptotically solves the average-consensus problem.

In this paper, we are interested in discussing the average-consensus problem for networks of agents with switching jointly-connected topologies and time-delays. To solve this problem, we use the following linear consensus protocol,

$$u_i(t)=\sum_{\mathrm{v}_j\in N_i}a_{ij}(t)(x_j(t-\tau_{ij})-x_i(t-\tau_{ij})),\tag{1}$$

where $\tau_{ij}=\tau_{ji}$, i.e., the delays in transmission between the ith agent and the jth agent are identical.

Then, the network dynamics is summarized as

$$\dot{x}(t)=-\sum_{m=1}^{M}L_{\sigma m}x(t-\tau_m)\tag{2}$$

with the initial condition $x(s)=x(0),\ s\in(-\infty,0]$, where $L_{\sigma m}$ is the coefficient matrix associated with the time-delay τ_m, $\sum_{m=1}^{M}L_{\sigma m}=L_\sigma$, $M\leq n(n-1)/2$, $\tau_m\in\{\tau_{ij},i,j\in\mathscr{I}\}$ for $m=1,\cdots,M$ and $\sigma(t):[0,+\infty)\to P=\{1,\cdots,N\}$($N\in\mathbb{Z}^+$ denotes the total number of all possible graphs) is a switching signal that determines the communication topology $G(t)$. Note here that $L_{\sigma m}$ is a symmetric matrix, since the graph $G(t)$ is undirected and $\tau_{ij}=\tau_{ji}$. Thus, $\mathbf{1}_n^T L_{\sigma m}=0$, which implies $\sum_i\dot{x}_i=0$. Consequently, $\beta=\frac{1}{n}\sum_i x_i(t)=\frac{1}{n}\sum_i x_i(0)$ is an invariant quantity, and $x(t)$ can be decomposed into $x(t)=\beta\mathbf{1}_n+\delta(t)$ with $\sum_i\delta_i(t)=0$. Then, the system (2) is equivalent to

$$\dot{\delta}(t)=-\sum_{m=1}^{M}L_{\sigma m}\delta(t-\tau_m).\tag{3}$$

It is clear that if the zero solution of system (3) is asymptotically stable, then all agents will converge to the common value $\frac{1}{n}\sum_i x_i(0)$.

Consider an infinite sequence of nonempty, bounded and contiguous time-intervals $[t_r,t_{r+1})$, $(r=0,1,\cdots)$ with $t_0=0$ and $t_{r+1}-t_r\leq T_1$ for some constant $T_1>0$. In each interval $[t_r,t_{r+1})$ there is a sequence of subintervals:

$$[t_{r_0},t_{r_1}),[t_{r_1},t_{r_2}),\cdots,[t_{r_{m_r-1}},t_{r_{m_r}})\tag{4}$$

with $t_{r_0}=t_r$ and $t_{r_{m_r}}=t_{r+1}$ satisfying $t_{r_{j+1}}-t_{r_j}\geq T_2, 0\leq j\leq m_r-1$ for some integer $m_r\geq 0$ and given constant $T_2>0$ such that the communication topology described by $G(t)$ switches at t_{r_j} and it does not change during each subinterval $[t_{r_j},t_{r_{j+1}})$. Evidently, there are at most $m_*=\lfloor\frac{T_1}{T_2}\rfloor$ subintervals in each interval $[t_r,t_{r+1})$, where $\lfloor\frac{T_1}{T_2}\rfloor$ denotes the maximum integer no larger than $\frac{T_1}{T_2}$.

With the switching topologies defined above, $-\sum_{m=1}^{M}L_{\sigma m}\delta(t-\tau_m)$ is piecewise continuous in t for any fixed $\delta(s)=\phi(s)\in C([t-\tau_{max},t],\mathbb{R}^n)$, $s\in[t-\tau_{max},t]$, where τ_{max} is the largest time-delay. Thus, we need to work with a weaker concept of solution, i.e.,

$$\delta(t)=\delta(0)-\int_0^t\sum_{m=1}^{M}L_{\sigma m}\delta(s-\tau_m)ds.\tag{5}$$

Under the initial condition $\delta(s) = \delta(0)$ ($s \in (-\infty, 0]$), this function is piecewise differentiable and satisfies (3) almost everywhere. It is absolutely continuous and provide a solution of (3) in the sense of *Carathéodory* according to [22] and [23] (see p.55 of [22] and p.10 of [23]). The solution of (3) will be discussed in this way.

III. Main results

In this section, we will analyze the stability for networks of agents with time-delays and switching jointly-connected topologies.

Before presenting the main result, we need first introduce some lemmas.

Lemma 2: (Schur Complement) [24] For a given symmetric matrix S with the form $S = [S_{ij}]$, $S_{11} \in \mathbb{R}^{r\times r}, S_{12} \in \mathbb{R}^{r\times(n-r)}, S_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, then, $S < 0$ if and only if $S_{11} < 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$ or $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{21} < 0$.

Lemma 3: [14] For any real differentiable vector function $x(t) \in \mathbb{R}^n$ and any constant matrix $0 < W = W^T \in \mathbb{R}^{n\times n}$, we have the following inequality

$$\begin{aligned} &1/\tau[x(t)-x(t-\tau)]^T W[x(t)-x(t-\tau)] \\ &\leq \int_{t-\tau}^{t} \dot{x}^T(s)W\dot{x}(s)ds, t \geq 0, \end{aligned}$$

where τ denotes the time-delay.

Lemma 4: [15] Write

$$\Psi_n = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

The following statements hold.

(1) The eigenvalues of Ψ_n are n with multiplicity $n-1$ and 0 with multiplicity 1. The vectors $\mathbf{1}_n^T$ and $\mathbf{1}_n$ are the left and the right eigenvectors of Ψ_n associated with the zero eigenvalue, respectively.

(2) There exists an orthogonal matrix $U_n \in \mathbb{R}^{n\times n}$ such that $U_n^T \Psi_n U_n = \begin{bmatrix} nI_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$ and the last column is $\frac{\mathbf{1}_n}{\sqrt{n}}$.

(3) Let $\Xi \in \mathbb{R}^{n\times n}$ be the Laplacian of any undirected graph, then $U_n^T \Xi U_n = \begin{bmatrix} \bar{U}_n^T \Xi \bar{U}_n & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{\Xi} & 0 \\ 0 & 0 \end{bmatrix}$, where $\bar{\Xi} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\bar{U}_n$ denotes the first $n-1$ columns of U_n.

Suppose that the (time-invariant) communication graph $G(t)$ on subinterval $[t_{r_j}, t_{r_{j+1}})$ has $l_\sigma \geq 1$ connected components with the corresponding sets of nodes denoted by $\varphi_{r_j}^1, \varphi_{r_j}^2, \cdots, \varphi_{r_j}^{l_\sigma}$. Then there exists a permutation matrix $E_\sigma \in \mathbb{R}^{n\times n}$ such that

$$\tilde{L}_\sigma \triangleq E_\sigma^T L_\sigma E_\sigma = diag\{L_\sigma^1, L_\sigma^2, \cdots, L_\sigma^{l_\sigma}\},$$

and

$$\delta^T(t)E_\sigma = [\delta_\sigma^{1\,T}, \delta_\sigma^{2\,T}, \cdots, \delta_\sigma^{l_\sigma\,T}], \tag{6}$$

where each block matrix $L_\sigma^i \in \mathbb{R}^{d_\sigma^i \times d_\sigma^i}$ is also a Laplacian of the corresponding connected component with d_σ^i denoting the number of nodes in $\varphi_{r_j}^i$. Then in each subinterval $[t_{r_j}, t_{r_{j+1}})$ system (3) can be decomposed into the following l_σ subsystems:

$$\dot{\delta}_\sigma^i(t) = -\sum_{m=1}^{M} L_{\sigma m}^i \delta_\sigma^i(t-\tau_m), \quad i = 1, 2, \cdots, l_\sigma, \tag{7}$$

where $\delta_\sigma^i(t) = \begin{bmatrix} \delta_{\sigma 1}^{i\,T}(t) & \cdots & \delta_{\sigma d_\sigma^i}^{i\,T}(t) \end{bmatrix}^T \in \mathbb{R}^{d_\sigma^i}$ and $L_\sigma^i = \sum_{m=1}^{M} L_{\sigma m}^i$. Noting that $(L_{\sigma m}^i)^T = L_{\sigma m}^i$, it is easy to see that $\mathbf{1}^T \dot{\delta}_\sigma^i(t) = 0$, $t \in [t_{r_j}, t_{r_{j+1}})$. Hence, $\frac{1}{d_\sigma^i}\sum_{k=1}^{d_\sigma^i} \delta_{\sigma k}^i(t)$ is an invariant quantity in subinterval $[t_{r_j}, t_{r_{j+1}})$. Denote $\phi_{r_j}^i = \frac{1}{d_\sigma^i}\sum_{k=1}^{d_\sigma^i} \delta_{\sigma k}^i(t_{r_j})$. Then $\delta_\sigma^i(t)$ can be decomposed into $\delta_\sigma^i(t) = (\delta_\sigma^i(t) - \phi_{r_j}^i \mathbf{1}) + \phi_{r_j}^i \mathbf{1}$, where $\mathbf{1}^T(\delta_\sigma^i(t) - \phi_{r_j}^i \mathbf{1}) = 0$.

Consider the following $(M+2)d_\sigma^i \times (M+2)d_\sigma^i$ symmetric matrix

$$\Phi_\sigma^i(t) = \begin{bmatrix} -2\gamma L_\sigma^i & \gamma L_{\sigma 1}^i & \gamma L_{\sigma 2}^i & \cdots & \gamma L_{\sigma M}^i & -L_\sigma^i \\ * & -\frac{1}{\tau_1}I & 0 & \cdots & 0 & L_{\sigma 1}^i \\ * & * & -\frac{1}{\tau_2}I & \ddots & \vdots & L_{\sigma 2}^i \\ * & * & * & \ddots & 0 & \vdots \\ * & * & * & * & -\frac{1}{\tau_M}I & L_{\sigma M}^i \\ * & * & * & * & * & -\frac{1}{\sum_{m=1}^{M}\tau_m}I \end{bmatrix}.$$

It is clear that $\Phi_\sigma^i(t)[\mathbf{1}_{d_\sigma^i}^T, \mathbf{0}_{(M+1)d_\sigma^i}^T]^T = 0$. Let $H_\sigma^i = \mathrm{diag}\{U_{d_\sigma^i}, I_{(M+1)d_\sigma^i}\}$ and $\bar{H}_\sigma^i = \mathrm{diag}\{\bar{U}_{d_\sigma^i}, I_{(M+1)d_\sigma^i}\}$ with $U_{d_\sigma^i}$ and $\bar{U}_{d_\sigma^i}$ as defined in Lemma 4. Then by Lemma 4, it is easy to see that $H_\sigma^{i\,T}\Phi_\sigma^i H_\sigma^i$ has the following form:

$$\begin{bmatrix} \Theta_{11} & 0 & \Theta_{12} \\ * & 0 & 0 \\ * & * & \Theta_{22} \end{bmatrix}.$$

Hence, $\Phi_\sigma^i \leq 0$ and $\mathrm{rank}(\Phi_\sigma^i) = (M+2)d_\sigma^i - 1$ hold if and only if $\bar{H}_\sigma^{i\,T}\Phi_\sigma^i \bar{H}_\sigma^i < 0$ which is a LMI and can be easily solved by using available numerical software.

On the other hand, by Lemma 2, $\Phi_\sigma^i(t) \leq 0$ with $rank(\Phi_\sigma^i(t)) = (M+2)d_\sigma^i - 1$ if and only if $\Xi_\sigma^i(t) \leq 0$ with $rank(\Xi_\sigma^i(t)) = (M+1)d_\sigma^i - 1$ where $\Xi_\sigma^i(t) = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}$ with

$$\Xi_{11} = -2\gamma L_\sigma^i + \sum_{m=1}^{M} \tau_m (L_\sigma^i)^2,$$

$$\Xi_{12} = \begin{bmatrix} \gamma L_{\sigma 1}^i - \sum_{m=1}^{M} \tau_m L_\sigma^i L_{\sigma 1}^i & \cdots & \gamma L_{\sigma M}^i - \sum_{m=1}^{M} \tau_m L_\sigma^i L_{\sigma M}^i \end{bmatrix},$$

$$\Xi_{22} = \begin{bmatrix} \sum_{m=1}^{M} \tau_m (L_{\sigma 1}^i)^2 - 1/\tau_1 I & \cdots & \sum_{m=1}^{M} \tau_m L_{\sigma 1}^i L_{\sigma M}^i \\ \vdots & \ddots & \vdots \\ \sum_{m=1}^{M} \tau_m L_{\sigma M}^i L_{\sigma 1}^i & \cdots & \sum_{m=1}^{M} \tau_m (L_{\sigma M}^i)^2 - 1/\tau_M I \end{bmatrix}.$$

Clearly, $\Xi_\sigma^i(t)[\mathbf{1}_{d_\sigma^i}^T, \mathbf{0}_{Md_\sigma^i}^T]^T = 0$. Denote $y_1 = [\delta_\sigma^i(t)^T, z_1^T]^T \in \mathbb{R}^{(M+1)d_\sigma^i}$ and $y_2 = [(\delta_\sigma^i(t) - \phi_{r_j}^i \mathbf{1})^T, z_1^T]^T \in \mathbb{R}^{(M+1)d_\sigma^i}$, where $z_1 \in \mathbb{R}^{Md_\sigma^i}$. Then if $rank(\Xi_\sigma^i) = (M+1)d_\sigma^i - 1$ and $\Xi_\sigma^i(t) \leq 0$, then y_2 is orthogonal to the null space of Ξ^i_σ, spanned by the vector $[\mathbf{1}^T_{d^i_\sigma}, \mathbf{0}^T_{Md^i_\sigma}]^T$. Therefore we have

$$y_1^T \Xi^i_\sigma y_1 = y_2^T \Xi^i_\sigma y_2 \leq \lambda_{\Xi^i_\sigma} \|y_2\|^2 \leq \lambda_{\Xi^i_\sigma} \|\delta^i_\sigma(t) - \phi^i_{r_j}\mathbf{1}\|^2, \quad (8)$$

where $\lambda_{\Xi^i_\sigma} < 0$ denotes the largest nonzero eigenvalue of Ξ^i_σ.

Theorem 1: Consider a network of agents with time-delays and switching topologies where the collection of graphs in each interval $[t_r, t_{r+1})$ is jointly-connected. For each subinterval $[t_{r_j}, t_{r_{j+1}})$, if there exists a common constant $\gamma > 0$ such that

$$\bar{H^i_\sigma}^T \Phi^i_\sigma \bar{H}^i_\sigma < 0, \quad (9)$$

the protocol (1) solves the average consensus problem.

Proof: To prove this theorem, we take two steps. First, we construct a common Lyapunov function $V(t)$ and prove $\dot{V}(t) \leq 0$ under condition (9). Then we verify the asymptotical convergence of the multi-agent system to a common value by contradiction.

Step 1) Define a common Lyapunov function for system (3) as follows

$$V(t) = \gamma \delta^T(t)\delta(t) + \sum_{m=1}^{M} \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \dot{\delta}^T(s)\dot{\delta}(s) ds d\theta, \ (\gamma > 0).$$

From (6), $V(t)$ can be rewritten as

$$V(t) = \gamma \sum_{i=1}^{l_\sigma} {\delta^i_\sigma}^T(t)\delta^i_\sigma(t) + \sum_{i=1}^{l_\sigma}\sum_{m=1}^{M} \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \dot{\delta^i_\sigma}^T(s)\dot{\delta^i_\sigma}(s) ds d\theta.$$

Calculating $\dot{V}$ along the trajectories of (7), we get

$$\begin{aligned}
\dot{V}(t) =& \sum_{i=1}^{l_\sigma} [-2\gamma {\delta^i_\sigma}^T(t) \sum_{m=1}^{M} L^i_{\sigma m}\delta^i_\sigma(t-\tau_m) \\
&+ \sum_{m=1}^{M} \tau_m \dot{\delta^i_\sigma}^T(t)\dot{\delta^i_\sigma}(t) - \sum_{m=1}^{M} \int_{t-\tau_m}^{t} \dot{\delta^i_\sigma}^T(s)\dot{\delta^i_\sigma}(s) ds] \\
=& \sum_{i=1}^{l_\sigma} [-2\gamma {\delta^i_\sigma}^T(t) \sum_{m=1}^{M} L^i_{\sigma m}\delta^i_\sigma(t-\tau_m) \\
&+ (\sum_{m=1}^{M}\tau_m)(\sum_{m=1}^{M} L^i_{\sigma m}\delta^i_\sigma(t-\tau_m))^T(\sum_{m=1}^{M} L^i_{\sigma m}\delta^i_\sigma(t-\tau_m)) \\
&- \sum_{m=1}^{M} \int_{t-\tau_m}^{t} \dot{\delta^i_\sigma}^T(s)\dot{\delta^i_\sigma}(s) ds]
\end{aligned}$$

Let $\zeta^i_{\sigma m}(t) = \delta^i_\sigma(t) - \delta^i_\sigma(t-\tau_m)$. Then, by Lemma 3, we have

$$\begin{aligned}
&\dot{V}(t) \\
\leq& \sum_{i=1}^{l_\sigma} \{-2\gamma {\delta^i_\sigma}^T(t) \sum_{m=1}^{M} L^i_{\sigma m}\delta^i_\sigma(t) + 2\gamma {\delta^i_\sigma}^T(t) \sum_{m=1}^{M} L^i_{\sigma m}\zeta^i_{\sigma m}(t) \\
&+ (\sum_{m=1}^{M}\tau_m) \sum_{m=1}^{M} [L^i_{\sigma m}(\delta^i_\sigma(t) - \zeta^i_{\sigma m}(t))]^T \\
&\times \sum_{m=1}^{M} [L^i_{\sigma m}(\delta^i_\sigma(t) - \zeta^i_{\sigma m}(t))] - \sum_{m=1}^{M} 1/\tau_m \zeta^i_{\sigma m}(t)^T \zeta^i_{\sigma m}(t)\} \\
=& \sum_{i=1}^{l_\sigma} \eta_i^T \Xi^i_\sigma(t) \eta_i
\end{aligned}$$

where $\eta_i = [\delta^i_\sigma(t)^T, \zeta^i_{\sigma 1}(t)^T, \cdots, \zeta^i_{\sigma M}(t)^T]^T$. As previously discussed, it is easy to see that $\Xi^i_\sigma(t) \leq 0$ with $rank(\Xi^i_\sigma(t)) = (M+1)d^i_\sigma - 1$ under the condition (9). Then it follows from (8) that

$$\dot{V}(t) \leq \lambda_{max} \sum_{i=1}^{l_\sigma} \|\delta^i_\sigma(t) - \phi^i_{r_j}\mathbf{1}\|^2 \leq 0, \quad (10)$$

where $\lambda_{max} = \max_{i,\sigma}\{\lambda_{\Xi^i_\sigma}\} < 0$.

Since $\dot{V}(t) \leq 0$, then the system (3) is uniformly stable. Hence, $\delta(t)$ and $\delta(t-\tau_m)$ $(m = 1, 2, \cdots, M)$ are bounded. From (3), $\dot{\delta}(t)$ is also bounded. Suppose that $|\dot{\delta}_k(t)| < \omega < +\infty$ $(k = 1, 2, \cdots, n)$. From (6), the absolute value of each component of $\dot{\delta^i_\sigma}$ $(i = 1, \cdots, l_\sigma)$ is also less than ω.

Step 2) Since $\dot{V}(t) \leq 0$ and $V(t) \geq 0$, then $V(t)$ tends to a nonnegative constant value, denoted by V_0, as $t \to +\infty$, i.e., for any $\varepsilon > 0$ there exists $T(\varepsilon) < +\infty$ such that

$$|V(t) - V_0| < \varepsilon \quad \text{when} \quad t > T(\varepsilon). \quad (11)$$

Suppose that $V_0 > 0$. Since $V(t) \geq V_0$, we have, for any interval $[t - 2\tau_{max}, t]$, there exist at least a $\delta_k(t_c)$ $(t_c \in [t - 2\tau_{max}, t],\ k \in \{1, \cdots, n\})$ satisfying that $|\delta_k(t_c)| \geq c = \sqrt{\frac{2V_0}{n(2\gamma + M(\sum_{m=1}^{M}\tau_m^2)\lambda_F)}}$, where $\lambda_F = \max_\sigma\{F_\sigma\}$ with

$$F_\sigma = \begin{bmatrix} L_{\sigma 1}L_{\sigma 1} & L_{\sigma 1}L_{\sigma 2} & \cdots & L_{\sigma 1}L_{\sigma M} \\ * & L_{\sigma 2}L_{\sigma 2} & \ddots & \vdots \\ * & * & \ddots & \vdots \\ * & * & * & L_{\sigma M}L_{\sigma M} \end{bmatrix}.$$

If not, we have

$$\begin{aligned}
V(t) &= \gamma \delta^T(t)\delta(t) + \sum_{m=1}^{M} \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \dot{\delta}^T(s)\dot{\delta}(s) ds d\theta \\
&< \gamma n c^2 + \sum_{m=1}^{M} \int_{-\tau_m}^{0} \int_{t+\theta}^{t} [\sum_{m=1}^{M} L_{\sigma m}\delta(t-\tau_m)]^T \\
&\times [\sum_{m=1}^{M} L_{\sigma m}\delta(t-\tau_m)] ds d\theta \\
&< \gamma n c^2 + \sum_{m=1}^{M} \int_{-\tau_m}^{0} \int_{t+\theta}^{t} \lambda_F M n c^2 ds d\theta \leq V_0,
\end{aligned}$$

which is a contradiction.

From (6), without loss of generality, we assume that $|\delta^1_{\sigma 1}(t_c)| \geq c,\ t_c \in [t_{r_j}, t_{r_{j+1}})$ with $t_{r_j} > T(\varepsilon)$ and the corresponding node index is p.

Note that $\sum_{k=1}^{n} \delta_k(t) = 0$ and then a contradiction can be obtained if we show that $\sum_{k=1}^{n} \delta_k(t) \neq 0$. In the following, we will use the bounds ε and c to estimate $\delta(t)$. To achieve this, we take $\varepsilon < \min\{-\frac{\lambda_{max}\omega^2 T_2^3}{12}, -\frac{\lambda_{max}}{3\omega}\frac{c^3}{10^6 n^3 m_*^3}\}$.

It follows from (10)(11) that

$$\varepsilon > V(t_{r_j}) - V(t_{r_{j+1}}) \geq -\int_{t_{r_j}}^{t_{r_{j+1}}} \lambda_{max} \sum_{i=1}^{l_\sigma} \|\delta^i_\sigma(s) - \phi^i_{r_j}\mathbf{1}\|^2 ds \quad (12)$$

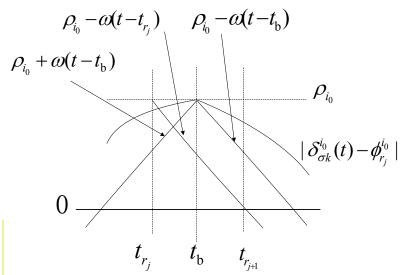

Fig.1

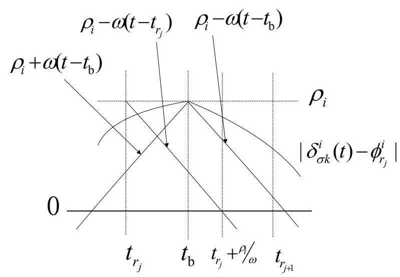

Fig.2

Let $\rho_i = \max\limits_k \max\limits_{s\in[t_{r_j},t_{r_{j+1}}]}(|\delta^i_{\sigma k}(s)-\phi^i_{r_j}|)$. (As previously discussed, $\delta(t)$ is continuous in t and hence this maximum also exists.)

Then we discuss the following cases:

Case 1): there exists $i_0\in\{1,\cdots,l_\sigma\}$ such that $\frac{\rho_{i_0}}{\omega}\geq t_{r_{j+1}}-t_{r_j}\geq T_2$. Suppose that $\rho_{i_0}=|\delta^{i_0}_{\sigma k}(t_b)-\phi^{i_0}_{r_j}|$ ($t_b\in[t_{r_j},t_{r_{j+1}}]$, $k\in\{1,\cdots,d^{i_0}_\sigma\}$). Since $|\dot{\delta}^{i_0}_{\sigma k}(t)|<\omega$, then it is easy to see that $|\delta^{i_0}_{\sigma k}(t)-\phi^{i_0}_{r_j}|>\rho_{i_0}+\omega(t-t_b)\geq 0$ for $t\in[t_{r_j},t_b]$, and $|\delta^{i_0}_{\sigma k}(t)-\phi^{i_0}_{r_j}|>\rho_{i_0}-\omega(t-t_b)\geq 0$ for $t\in[t_b,t_{r_{j+1}}]$. Also,

$$\begin{aligned}&\int_{t_{r_j}}^{t_b}[\rho_{i_0}+\omega(s-t_b)]^2\mathrm{d}s+\int_{t_b}^{t_{r_{j+1}}}[\rho_{i_0}-\omega(s-t_b)]^2\mathrm{d}s\\ &\geq\int_{t_{r_j}}^{t_{r_{j+1}}}[\rho_{i_0}-\omega(s-t_{r_j})]^2\mathrm{d}s.\end{aligned}$$

Thus, we have

$$\begin{aligned}\varepsilon &> -\int_{t_{r_j}}^{t_{r_{j+1}}}\lambda_{max}\sum_{i=1}^{l_\sigma}\|\delta^i_\sigma(s)-\phi^i_{r_j}\mathbf{1}\|^2\mathrm{d}s\\ &\geq -\int_{t_{r_j}}^{t_{r_{j+1}}}\lambda_{max}\|\delta^{i_0}_\sigma(s)-\phi^{i_0}_{r_j}\mathbf{1}\|^2\mathrm{d}s\\ &\geq -\int_{t_{r_j}}^{t_{r_{j+1}}}\lambda_{max}(\delta^{i_0}_{\sigma k}(s)-\phi^{i_0}_{r_j})^2\mathrm{d}s\\ &\geq -\int_{t_{r_j}}^{t_{r_{j+1}}}\lambda_{max}[\rho_{i_0}-\omega(s-t_{r_j})]^2\mathrm{d}s\\ &= -\lambda_{max}(t_{r_{j+1}}-t_{r_j})(\rho_{i_0}-\frac{\omega(t_{r_{j+1}}-t_{r_j})}{2})^2\\ &\quad-\frac{\lambda_{max}\omega^2(t_{r_{j+1}}-t_{r_j})^3}{12}\\ &\geq -\frac{\lambda_{max}\omega^2T_2^3}{12}>\varepsilon,\end{aligned}$$

which yields a contradiction. This means that this case cannot occur and all $\frac{\rho_i}{\omega}$ ($i=1,\cdots,l_\sigma$) must be less than $t_{r_{j+1}}-t_{r_j}$.

Case 2): $\frac{\rho_i}{\omega}<t_{r_{j+1}}-t_{r_j}$ for $i=1,\cdots,l_\sigma$. Similar to the analysis of case 1), we have

$$\varepsilon>-\int_0^{\frac{\rho_i}{\omega}}\lambda_{max}(\rho_i-\omega s)^2\mathrm{d}s\geq-\frac{\lambda_{max}}{3\omega}\rho_i^3.$$

Since $\varepsilon<\min\{-\frac{\lambda_{max}\omega^2T_2^3}{12},-\frac{\lambda_{max}}{3\omega}\frac{c^3}{10^6n^3m_*^3}\}$, then it follows that $\rho_i=\max\limits_k\max\limits_{t\in[t_{r_j},t_{r_{j+1}}]}(|\delta^i_{\sigma k}(t)-\phi^i_{r_j}|)<\frac{c}{10^2nm_*}$. Particularly, $|\delta^1_{\sigma 1}(t_c)-\phi^1_{r_j}|<\frac{c}{10^2nm_*}$. Hence, $\max\limits_k\max\limits_{t\in[t_{r_j},t_{r_{j+1}}]}(|\delta^1_{\sigma k}(t)-\delta^1_{\sigma 1}(t_c)|)<\frac{2c}{10^2nm_*}$. In other words, the states of agents that are connected to the pth agent (including the pth agent) all fall in the interval $(\delta^1_{\sigma 1}(t_c)-\frac{2c}{10^2nm_*},\delta^1_{\sigma 1}(t_c)+\frac{2c}{10^2nm_*})$. For the next subinterval $[t_{r_{j+1}},t_{r_{j+2}})$, it is easy to see that the states of agents which are jointly-connected to the pth agent during $[t_{r_j},t_{r_{j+2}})$ (including the pth agent) fall in the interval $(\delta^1_{\sigma 1}(t_c)-\frac{2c}{10^2nm_*}\times 2,\delta^1_{\sigma 1}(t_c)+\frac{2c}{10^2nm_*}\times 2)$. Further, since all agents are jointly-connected during $[t_{r_j},t_{r+2}]$, by induction, we have

$$\max_k\max_{t\in[t_{r+2},t_{r+3}]}(|\delta_k(t)-\delta^1_{\sigma 1}(t_c)|)<\frac{2c}{10^2nm_*}3m_*<\frac{c}{10n}.$$

Since $\delta^1_{\sigma 1}(t_c)\geq c$, it follows that $\sum\limits_{k=1}^n\delta_k(t)\neq 0$, $t\in[t_{r+2},t_{r+3}]$, which also yields a contradiction. Thus $\lim\limits_{t\to+\infty}V(t)=0$, and $\lim\limits_{t\to+\infty}\delta(t)=0$; that is, average consensus can be achieved under the condition (9). $\square$

Now, we will discuss the feasibility of (9). By Lemma 2, $\bar{H^i_\sigma}^T\Phi^i_\sigma\bar{H}^i_\sigma<0$ is equivalent to

$$\begin{bmatrix}-2\gamma(\bar{U}_{d^i_\sigma})^TL^i_\sigma\bar{U}_{d^i_\sigma} & \gamma(\bar{U}_{d^i_\sigma})^TL^i_{\sigma 1} & \cdots & \gamma(\bar{U}_{d^i_\sigma})^TL^i_{\sigma M}\\ * & -\frac{1}{\tau_1}I & 0 & 0\\ * & * & \ddots & 0\\ * & * & * & -\frac{1}{\tau_M}I\end{bmatrix}$$
$$+(\sum_{m=1}^M\tau_m)[-L^i_\sigma\bar{U}_{d^i_\sigma},L^i_{\sigma 1},\cdots,L^i_{\sigma M}]^T[-L^i_\sigma\bar{U}_{d^i_\sigma},L^i_{\sigma 1},\cdots,L^i_{\sigma M}]<0 \tag{13}$$

Noting that $(\bar{U}_{d^i_\sigma})^TL^i_\sigma\bar{U}_{d^i_\sigma}>0$, by choosing τ_{max} sufficiently small, it is easy to see that the inequality (13) holds. Hence $\bar{H^i_\sigma}^T\Phi^i_\sigma\bar{H}^i_\sigma<0$ is always feasible for sufficiently small τ_{max}.

IV. Simulations

Numerical simulations will be given to illustrate the theoretical results obtained in the previous section. Fig.3 shows four different graphs each with 6 nodes. All graphs in this figure are not connected and the weight of each edge is 1. Moreover, the time-delays corresponding to the edges $(1,2)$, $(1,6)$, $(2,3)$, $(3,4)$, $(4,5)$ and $(5,6)$ are 0.4 s, 0.4 s, 0.4 s, 0.4 s, 0.2 s, 0.3 s, respectively. In Fig.4, a finite state machine is shown with four states $\{G_a,G_b,G_c,G_d\}$ which denote the states of a network with switching topologies and time-delays, and it starts at G_a, and switches every 0.1 s to the next state.

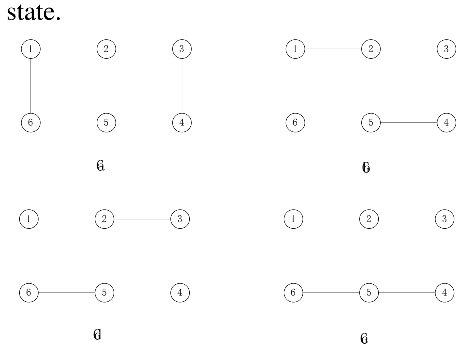

Fig.3 Four undirected graphs.

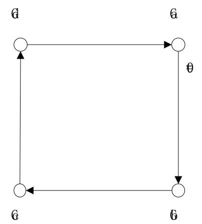

Fig. 4 Finite machine with four states denoting the states of a network with switching topologies and time-delay

It is solved that one solution for (9) is $\gamma = 1.1012$. Fig.5 and Fig.6 show the corresponding state trajectories of all agents with different initial conditions.

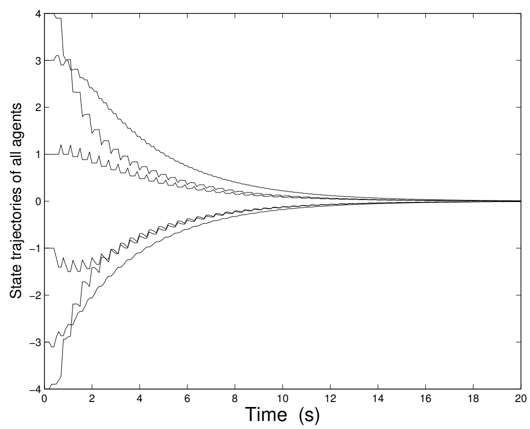

Fig.5 State trajectories of all agents.

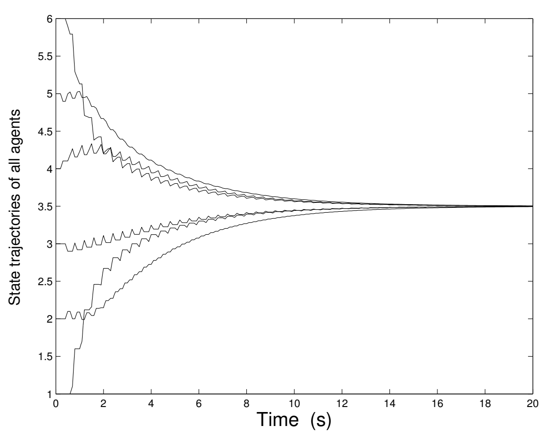

Fig.6 State trajectories of all agents.

It is clear that all agents asymptotically achieve average consensus, although the state trajectories are not quite smooth due to the switching of the network topology.

V. Conclusions

In this paper, we study the average consensus problem in networks of continuous-time agents with delayed information, where the communication structures vary over time and the corresponding graphs may not be connected. In the analysis, we first introduce a common Lyapunov function for the disagreement dynamics of the network. Then based on this Lyapunov function, we derive a sufficient condition in terms of LMIs under which all agents asymptotically reach average consensus. Finally, simulation results are provided to demonstrate the effectiveness of our theoretical results.

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