

# Are anti-aliasing filters really necessary for sampled-data control?

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**Abstract**—In the paper the role of anti-aliasing filters is revised based on control quality assessment of various setups of analog filters used prior to sampling, or their lack. Numerical results show that contrary to common belief possible benefits gained from anti-aliasing filters are very restricted.

## I. INTRODUCTION

In the scientific literature [5], [8], [2] strong belief is expressed that additional analog elements are necessary prior to sampling to guarantee correct digital signal processing, and control. Although various solutions are possible, these elements called anti-aliasing filters usually take the form of Butterworth filters whose cutoff frequency equals to the so called Nyquist frequency  $\omega_N = \pi/h$  depending solely on sampling period  $h$ . As an alternative [5], [6], [3] so called integrating or averaging samplers are considered.

This belief is usually supported by heuristic speculations based on Shannon-Kotelnikov Reconstruction Theorem, e.g. [7], which states that in order to reconstruct the signal  $s(t)$  from its samples  $s(ih)$ ,  $-\infty < i < \infty$ , the sampling frequency should be twice the highest frequency component in the signal. Since the spectra of physical signals often stretch on infinite frequency range, this gives rise to the idea of filters that cut off the portion of frequency spectrum lying outside the region determined by that theorem.

The model of a realistic control system studied in the paper is presented in Fig. 1 where  $K_c(s)$  is the transfer function of control path of the plant, while  $K_d(s)$  and  $K_n(s)$  represent filters forming stochastic disturbance and noise, respectively. It is also assumed that the controller consists of a linear state feedback with a discrete-time Kalman filter in series estimating state from scalar measurements which results from the LQG theory.

The aim of the paper is to develop tools for control quality assessment for various configurations of filters or their lack. This extends the results of papers [3], [4] from pure signal processing to control problems.

Conclusions drawn from numerical examples show that the common belief about necessity of anti-aliasing filters is very often not true, and that continuous-time Kalman filters that depend on disturbance and noise characteristics are always better than anti-aliasing filters that depend only on the sampling period.

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## II. MODELING OF THE ANALOG PART

### A. Plant and noise model

To analyse the properties of sampling we will use state-space models of the system control in Fig. 1 consisting of

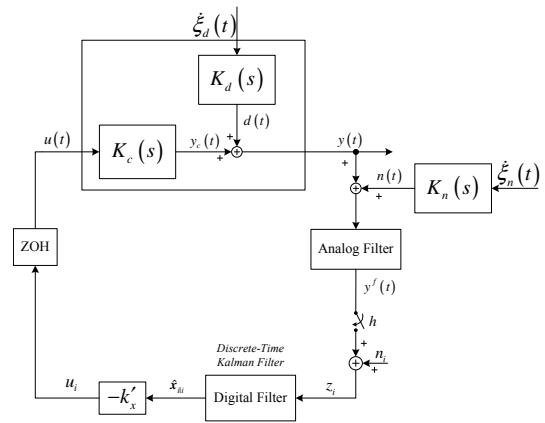


Fig. 1. Control system

control path:

$$\dot{\mathbf{x}}_c(t) = \mathbf{A}_c \mathbf{x}_c(t) + \mathbf{b}_c u(t), \quad \mathbf{x}_c(0) = \mathbf{0}, \quad (1)$$

$$y_c(t) = \mathbf{d}'_c \mathbf{x}_c(t), \quad (2)$$

disturbance signal

$$\dot{\mathbf{x}}_d(t) = \mathbf{A}_d \mathbf{x}_d(t) + \mathbf{c}_d \dot{\xi}_d(t), \quad \mathbf{x}_d(0) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{d,0}), \quad (3)$$

$$d(t) = \mathbf{d}'_d \mathbf{x}_d(t), \quad (4)$$

and noise model

$$\dot{\mathbf{x}}_n(t) = \mathbf{A}_n \mathbf{x}_n(t) + \mathbf{c}_n \dot{\xi}_n(t), \quad \mathbf{x}_n(0) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{n,0}), \quad (5)$$

$$n(t) = \mathbf{d}'_n \mathbf{x}_n(t), \quad (6)$$

where  $\dim \mathbf{x}_c = n_c$ ,  $\dim \mathbf{x}_d = n_d$ ,  $\dim \mathbf{x}_n = n_n$ ,  $\mathbf{x}_c(t)$ ,  $\mathbf{x}_d(t)$ ,  $\mathbf{x}_n(t)$  are state vectors,  $\mathbf{A}_c$ ,  $\mathbf{A}_d$ ,  $\mathbf{A}_n$  are matrices,  $\mathbf{c}_d$ ,  $\mathbf{c}_n$ ,  $\mathbf{d}_d$ ,  $\mathbf{d}_n$  and  $\mathbf{d}_c$  are vectors of appropriate dimensions. Processes  $\xi_d(t)$  and  $\xi_n(t)$  are independent continuous-time white noises with zero means and covariance functions defined as unit Dirac pulse functions, i.e.:

$$\mathbb{E}[\dot{\xi}_d(t)] = 0, \quad \mathbb{E}[\dot{\xi}_d(t)\dot{\xi}_d(\tau)] = \delta(t - \tau); \quad (7)$$

$$\mathbb{E}[\dot{\xi}_n(t)] = 0, \quad \mathbb{E}[\dot{\xi}_n(t)\dot{\xi}_n(\tau)] = \delta(t - \tau). \quad (8)$$

The disturbed signal  $y(t)$  is the sum of the signal of interest  $y_c(t)$  and disturbance  $d(t)$ :

$$y(t) = y_c(t) + d(t). \quad (9)$$

Measured signal  $y_2(t)$  is the sum of the signal of interest  $y(t)$  and noise  $n(t)$ :

$$y_2(t) = y(t) + n(t). \quad (10)$$

### B. Continuous-Time Filters

In the paper we consider Butterworth and averaging filters as anti-aliasing filters as well as a continuous-time Kalman filter.

1) *Continuous-time Butterworth filter*: Transfer function of Butterworth filter has the form:

$$K^f(s) = \frac{1}{B_n\left(\frac{s}{\omega_o}\right)}, \quad (11)$$

where  $B_n(*)$  is the  $n$ -degree Butterworth's polynomial and  $\omega_o$  is called the cutoff frequency. In this paper  $\omega_o$  will be assumed as Nyquist frequency  $\omega_o = \omega_N = \frac{\pi}{h}$ . The first Butterworth's polynomials are defined as follows:

$$B_1(x) = x + 1; \quad B_2(x) = x^2 + \sqrt{2} \cdot x + 1.$$

2) *Averaging filter*: It has the following transfer function:

$$K(s) = \frac{1 - e^{-sh}}{sh} \quad (12)$$

Sampling the output of this filter can be replaced by so called averaging sampling described further in sec. III.C.

3) *Continuous-time Kalman filter*: Since there is no white noise added to the measured output, the classical Kalman filter for system in (3)–(6) becomes singular. One way to overcome the problem is to replace the continuous-time filter with a discrete-time one working at sampling frequency  $1/h_f$  high enough. The output of such filter could be resampled at lower frequency if necessary. An alternative solution is to use its continuous-time approximation obtained by expanding  $F_{dn} = I + A_{dn}h_f$  which leads to:

$$\dot{\hat{x}}_{dn}(t) = (I - \mathbf{k}_{dn}^f \mathbf{d}'_{dn}) \mathbf{A}_{dn} \hat{x}_{dn}(t) + \mathbf{k}_{c,dn}^f [y_{dn}(t) - \mathbf{d}'_{dn} \hat{x}_{dn}(t)] \quad (13)$$

with  $\mathbf{k}_{c,dn}^f = \mathbf{k}_{dn}^f/h_f$ , where  $\mathbf{k}_{dn}^f$  is the Kalman gain of the discrete-time filter working with period  $h_f$ .

### C. State-space model of system with analog filter

Either filter, anti-aliasing or Kalman, can be expressed in the following form

$$\dot{\mathbf{x}}^f(t) = \mathbf{A}^f \mathbf{x}^f(t) + \mathbf{b}^f y_2(t), \quad (14)$$

$$y^f(t) = \mathbf{d}'^f \mathbf{x}^f(t). \quad (15)$$

Then the system consisting of a filter in (14)–(15), together with plant of (1)–(6) can be aggregated to the following

$$\dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \mathbf{b} u(t) + \mathbf{C} \dot{\boldsymbol{\xi}}(t), \quad (16)$$

$$y(t) = \mathbf{d}'_y \mathbf{x}(t), \quad (17)$$

$$y_2(t) = \mathbf{d}'_{y_2} \mathbf{x}(t), \quad (18)$$

$$y^f(t) = \mathbf{d}'^f \mathbf{x}(t), \quad (19)$$

where:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_d & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_n & \mathbf{0} \\ \mathbf{b}^f \mathbf{d}'_c & \mathbf{b}^f \mathbf{d}'_d & \mathbf{b}^f \mathbf{d}'_n & \mathbf{A}^f \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{c}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_n \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_c \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{d}_y = \begin{bmatrix} \mathbf{d}_c \\ \mathbf{d}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{d}_{y_2} = \begin{bmatrix} \mathbf{d}_c \\ \mathbf{d}_d \\ \mathbf{d}_n \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{d}^f \end{bmatrix},$$

$$\mathbf{x}(t) = \begin{bmatrix} \hat{\mathbf{x}}_c(t) \\ \hat{\mathbf{x}}_d(t) \\ \hat{\mathbf{x}}_n(t) \\ \hat{\mathbf{x}}^f(t) \end{bmatrix}, \quad \dot{\boldsymbol{\xi}}(t) = \begin{bmatrix} \dot{\boldsymbol{\xi}}_d(t) \\ \dot{\boldsymbol{\xi}}_n(t) \end{bmatrix}.$$

## III. SAMPLING AND DISCRETE-TIME KALMAN FILTERING

### A. Instantaneous sampling

Simple instantaneous sampling with sampling period  $h$  consists in taking the values of the sampled signal at discrete time instants  $t_i = ih, i = 0, 1, \dots$ . Available measurements  $z_i$  are expressed as

$$z_i = y_2(t_i). \quad (20)$$

Then the problem defined by measurement equation (20) and state equation (16) is equivalent with the following discrete-time system:

$$\mathbf{x}_{i+1} = \mathbf{F} \mathbf{x}_i + \mathbf{g} u_i + \mathbf{w}_i, \quad (21)$$

$$z_i = \mathbf{d}' \mathbf{x}_i, \quad (22)$$

where:

$$\mathbf{F}(\tau) = e^{\mathbf{A}\tau}, \quad \mathbf{F} = \mathbf{F}(h), \quad (23)$$

$$\mathbf{g}(\tau) = \int_0^\tau e^{\mathbf{A}\nu} \mathbf{b} d\nu, \quad \mathbf{g} = \mathbf{g}(h) \quad (24)$$

and  $\mathbf{w}_i$  is a zero mean vector Gaussian noise with  $E\{\mathbf{w}_i \mathbf{w}_i'\} = \mathbf{W}$ , and

$$\mathbf{W} = \int_0^h e^{\mathbf{A}s} \mathbf{C} \mathbf{C}' e^{\mathbf{A}'s} ds. \quad (25)$$

Vectors  $\mathbf{x}_0$  and  $\mathbf{w}_i$  are independent for all  $i \geq 0$ .

### B. Discrete-time Kalman filter

The limiting Kalman filter, [1], that provides  $(\hat{\mathbf{x}}_{i|i} = E[\mathbf{x}_i | \mathbf{z}_i])$  for the discrete-time system in (21)–(22) as  $i \rightarrow \infty$  has the form:

$$\hat{\mathbf{x}}_{i+1|i+1} = \hat{\mathbf{x}}_{i+1|i} + \mathbf{k}^f (z_{i+1} - \mathbf{d}' \hat{\mathbf{x}}_{i+1|i}), \quad (26)$$

$$\hat{\mathbf{x}}_{i+1|i} = \mathbf{F} \hat{\mathbf{x}}_{i|i} + \mathbf{g} u_i, \quad \mathbf{x}_{0|-1} = \mathbf{0}, \quad (27)$$

where

$$\mathbf{k}^f = \frac{\boldsymbol{\Sigma} \mathbf{d}}{\mathbf{d}' \boldsymbol{\Sigma} \mathbf{d}}, \quad \boldsymbol{\Sigma} = \mathbf{W} + \mathbf{F} \left( \boldsymbol{\Sigma} - \frac{\boldsymbol{\Sigma} \mathbf{d} \mathbf{d}' \boldsymbol{\Sigma}'}{\mathbf{d}' \boldsymbol{\Sigma} \mathbf{d}} \right) \mathbf{F}'. \quad (28)$$

### C. Averaging sampling [3], [5]

Let us define the mean value of  $y(t)$  over the sampling interval  $h$  between the sampling times  $t_i$  and  $t_{i+1}$  as

$$z_{i+1} = \frac{1}{h} \int_{t_i}^{t_{i+1}} y_2(t) dt. \quad (29)$$

Then dividing the output equation by  $h$  we get

$$\frac{dz(t)}{dt} = \frac{1}{h} y_2(t) = \frac{1}{h} \mathbf{d}' \mathbf{x}(t). \quad (30)$$

As a result, the state equation can be extended as follows

$$\frac{d}{dt} \begin{bmatrix} \mathbf{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{d}' & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \mathbf{0} \end{bmatrix}.$$

where:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_c & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{b}_c \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{c}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_n \end{bmatrix},$$

$$\mathbf{d}_0 = \begin{bmatrix} \mathbf{d}_c \\ \mathbf{d}_d \\ \mathbf{0} \end{bmatrix}, \mathbf{d} = \begin{bmatrix} \mathbf{d}_c \\ \mathbf{d}_d \\ \mathbf{d}_n \end{bmatrix}, \mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_c(t) \\ \mathbf{x}_d(t) \\ \mathbf{x}_n(t) \end{bmatrix}, \dot{\xi}(t) = \begin{bmatrix} \dot{\xi}_d(t) \\ \dot{\xi}_n(t) \end{bmatrix}.$$

Integrating it between the  $i$ -th and  $(i+1)$ -th sampling instants yields

$$\mathbf{x}_{i+1} = \mathbf{F} \mathbf{x}_i + \mathbf{g} u_i + \mathbf{w}_i, \quad (31)$$

$$z_{i+1} = \mathbf{f}' \mathbf{x}_i + \mathbf{g}_* u_i + v_i, \quad (32)$$

with

$$\mathbf{F} = e^{\mathbf{A}h}, \quad \mathbf{g} = \int_0^h e^{\mathbf{A}\nu} \mathbf{b} d\nu, \quad \mathbf{f}' = \frac{1}{h} \mathbf{d}' \int_0^h e^{\mathbf{A}s} ds,$$

$$\mathbf{g}_* = \frac{1}{h} \mathbf{d}' \int_0^h \int_0^s e^{\mathbf{A}\nu} d\nu ds \mathbf{b} =$$

$$= \frac{1}{h} \mathbf{d}' (\mathbf{A})^{-1} [(\mathbf{A})^{-1} (e^{\mathbf{A}h} - \mathbf{I}) - \mathbf{I}h] \mathbf{b},$$

and

$$\mathbf{E} \begin{bmatrix} \mathbf{w}_i \mathbf{w}_j' & \mathbf{w}_i v_j \\ v_i \mathbf{w}_j' & v_i v_j \end{bmatrix} = \begin{bmatrix} \mathbf{W} & \gamma \\ \gamma' & \rho^2 \end{bmatrix} \delta_{ij},$$

where

$$\begin{bmatrix} \mathbf{W} & \gamma \\ \gamma' & \rho^2 \end{bmatrix} = \int_0^h e^{\bar{\mathbf{A}}s} \begin{bmatrix} \mathbf{C} \mathbf{C}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} e^{\bar{\mathbf{A}}'s} ds, \quad \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \frac{1}{h} \mathbf{d}' & \mathbf{0} \end{bmatrix}.$$

### D. Discrete-time Kalman filter for averaging sampling

The results of averaging sampling can further be improved by using a discrete-time Kalman filter. We have the following:

*Lemma 1:* Denote

$$\bar{\mathbf{d}} = \frac{\gamma}{\rho^2}, \quad \bar{\mathbf{F}} = \mathbf{F} - \bar{\mathbf{d}} \mathbf{f}', \quad \bar{\mathbf{g}} = \mathbf{g} - \bar{\mathbf{d}} \mathbf{g}_*, \quad \bar{\mathbf{W}} = \mathbf{W} - \frac{\gamma \gamma'}{\rho^2}.$$

Then the Kalman filter for (31)-(32) that provides  $(\hat{\mathbf{x}}_{i|i} = \mathbf{E}[\mathbf{x}_i | \bar{\mathbf{z}}_i])$  has the following form

$$\hat{\mathbf{x}}_{i|i+1} = \hat{\mathbf{x}}_{i|i} + \mathbf{k}^f (z_{i+1} - \mathbf{f}' \hat{\mathbf{x}}_{i|i} - \mathbf{g}_* u_i), \quad (33)$$

$$\hat{\mathbf{x}}_{i+1|i+1} = \bar{\mathbf{F}} \hat{\mathbf{x}}_{i|i+1} + \bar{\mathbf{g}} u_i + \bar{\mathbf{d}} z_{i+1}, \quad \hat{\mathbf{x}}_{0|0} = \mathbf{0}, \quad (34)$$

where

$$\mathbf{k}^f = \Sigma \mathbf{f} (\mathbf{f}' \Sigma \mathbf{f} + \rho^2)^{-1}, \quad (35)$$

and  $\Sigma$  is a solution of the matrix Riccati equation

$$\Sigma = \bar{\mathbf{W}} + \bar{\mathbf{F}} \left( \Sigma + \frac{\Sigma \mathbf{f} \mathbf{f}' \Sigma}{\mathbf{f}' \Sigma \mathbf{f} + \rho^2} \right) \bar{\mathbf{F}}'. \quad (36)$$

*Proof* Since  $\mathbf{w}_i$  and  $v_i$  are correlated, we can introduce  $\bar{\mathbf{w}}_i$  defined as

$$\bar{\mathbf{w}}_i = \mathbf{w}_i - \frac{\gamma}{\rho^2} v_i, \quad (37)$$

such that  $\bar{\mathbf{w}}_i$  and  $v_i$  are independent, and

$$\text{cov} \{ \bar{\mathbf{w}}_i, v_j \} = \mathbf{E} \begin{bmatrix} \bar{\mathbf{w}}_i \bar{\mathbf{w}}_j' & \bar{\mathbf{w}}_i v_j \\ v_i \bar{\mathbf{w}}_j' & v_i v_j \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{W}} & \mathbf{0} \\ \mathbf{0}' & \rho^2 \end{bmatrix} \delta_{ij}. \quad (38)$$

Inserting

$$\mathbf{w}_i = \bar{\mathbf{w}}_i + \frac{\gamma}{\rho^2} v_i, \quad (39)$$

from (37), and

$$v_i = z_{i+1} - \mathbf{f}' \mathbf{x}_i - \mathbf{g}_* u_i, \quad (40)$$

from (32) into (31) results in

$$\mathbf{x}_{i+1} = \bar{\mathbf{F}} \mathbf{x}_i + \bar{\mathbf{g}} u_i + \bar{\mathbf{d}} z_{i+1} + \bar{\mathbf{w}}_i. \quad (41)$$

From (41), Kalman filter equations (33)-(34) follow.

Equation (34) together with (33) give:

$$\hat{\mathbf{x}}_{i+1|i+1} = \bar{\mathbf{F}} (\mathbf{I} - \mathbf{k}^f \mathbf{f}') \hat{\mathbf{x}}_{i|i} + (\bar{\mathbf{g}} - \bar{\mathbf{F}} \mathbf{k}^f \mathbf{g}_*) u_i + (\bar{\mathbf{d}} + \bar{\mathbf{F}} \mathbf{k}^f) z_{i+1}, \quad \hat{\mathbf{x}}_{0|0} = \mathbf{0}. \quad (42)$$

## IV. CONTROL ALGORITHMS

### A. Performance index and control law

The aim of the system is to keep the output of the system close to the reference value  $y^r(t) = 0$  based on noisy sampled measurements defined in (10), i.e. to make the error  $e(t) = y^r(t) - y(t)$  small.

To this end, a LQG control problem with a continuous performance index  $J$  is formulated, where

$$J = \lim_{N \rightarrow \infty} \mathbf{E} \frac{1}{Nh} \int_0^{Nh} \{ y^2(t) + \lambda u^2(t) \} dt. \quad (43)$$

Since noise influences only state estimate  $\hat{\mathbf{x}}_{i|i}$  and not the control law being a linear function of  $\hat{\mathbf{x}}_{i|i}$  the above sampled data control problem can be reformulated as follows.

The problem defined by modulation equation

$$u(t) = u_i, \quad \text{for } t \in (ih, ih + h], \quad i = 0, 1, \dots, \quad (44)$$

measurement equation (20), state equation (16) with:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_d \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_c \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = \mathbf{c} = \begin{bmatrix} \mathbf{0} \\ \mathbf{c}_d \end{bmatrix},$$

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_c \\ \mathbf{d}_d \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_c(t) \\ \mathbf{x}_d(t) \end{bmatrix}, \quad \dot{\boldsymbol{\xi}}(t) = \dot{\boldsymbol{\xi}}_d(t),$$

and performance index (43) is equivalent with the following discrete-time problem with (21)-(22) defined as (23)-(24) and:

$$J = \lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{N} \sum_{i=0}^{N-1} \{ \mathbf{x}'_i \mathbf{Q}_1 \mathbf{x}_i + 2\mathbf{x}'_i \mathbf{q}_{12} u_i + q_2 u_i^2 + q_w \}, \quad (45)$$

where

$$\mathbf{Q}_1 = \frac{1}{h} \int_0^h \mathbf{F}'(\tau) \mathbf{M} \mathbf{F}(\tau) d\tau, \quad \mathbf{M} = \mathbf{d} \mathbf{d}'$$

$$\mathbf{q}_{12} = \frac{1}{h} \int_0^h \mathbf{F}'(\tau) \mathbf{M} \mathbf{g}(\tau) d\tau,$$

$$q_2 = \frac{1}{h} \int_0^h \mathbf{g}'(\tau) \mathbf{M} \mathbf{g}(\tau) d\tau + \lambda,$$

$$q_w = \mathbf{d}' \left\{ \int_0^h \int_0^\tau \mathbf{F}(\tau-s) \mathbf{c} \mathbf{c}' \mathbf{F}'(\tau-s) ds d\tau \right\} \mathbf{d},$$

The optimal control law minimizing the performance index (45) for the discrete stochastic system (21) is a linear function

$$u_i = -\mathbf{k}'_x \hat{\mathbf{x}}_{i|i}, \quad \mathbf{k}'_x = \frac{\mathbf{q}_{12} + \mathbf{F}' \mathbf{K} \mathbf{g}}{q_2 + \mathbf{g}' \mathbf{K} \mathbf{g}}, \quad (46)$$

where  $\hat{\mathbf{x}}_{i|i}$  denotes the Kalman filter estimate of the state  $\mathbf{x}_i$  based on available information up to and including  $i$  from (26)-(27) or (33)-(34). The feedback gain  $\mathbf{k}_x$  depends on the positive definite solution  $\mathbf{K}$  of the following algebraic Riccati equation:

$$\mathbf{K} = \mathbf{Q}_1 + \mathbf{F}' \mathbf{K} \mathbf{F} - \frac{(\mathbf{q}_{12} + \mathbf{F}' \mathbf{K} \mathbf{g})(\mathbf{q}_{12} + \mathbf{F}' \mathbf{K} \mathbf{g})'}{q_2 + \mathbf{g}' \mathbf{K} \mathbf{g}}. \quad (47)$$

## V. CONTROL SYSTEM ASSESSMENT

The quality of the control systems will be assessed based on plots depicting standard deviation of the output variable versus standard deviation of the control signal. They allow the weighting factor  $\lambda$  of the performance index to be chosen such that the standard deviation of the control signal does not exceed certain value. Standard deviation is a good measure of expected magnitudes of signals. To this end appropriate variations should be calculated.

### A. Output and control variances for systems with continuous-time filters

The following formulae express the variances of interest:

$$\sigma_y^2 = \text{var} \{y_i\} = \mathbf{d}'_0 \mathbf{V}^o \mathbf{d}_0, \quad (48)$$

$$\sigma_u^2 = \text{var} \{u_i\} = \mathbf{k}'_x \mathbf{V}^f \mathbf{k}_x, \quad (49)$$

where  $\mathbf{V}^o$ ,  $\mathbf{V}^f$ , and  $\mathbf{V}^{fo}$  are submatrices of matrix  $\mathbf{V}$

$$\mathbf{V} = \mathbb{E} \left\{ \begin{bmatrix} \mathbf{x}_i \\ \hat{\mathbf{x}}_{i|i} \end{bmatrix} \begin{bmatrix} \mathbf{x}'_i & \hat{\mathbf{x}}'_{i|i} \end{bmatrix} \right\} = \begin{bmatrix} \mathbf{V}^o & \mathbf{V}^{of} \\ \mathbf{V}^{fo} & \mathbf{V}^f \end{bmatrix} \quad (50)$$

which is a solution of the following matrix Lyapunov equation:

$$\mathbf{V} = \boldsymbol{\Phi} \mathbf{V} \boldsymbol{\Phi}' + \boldsymbol{\Omega} \mathbf{W} \boldsymbol{\Omega}', \quad (51)$$

with:

$$\boldsymbol{\Lambda} = (\mathbf{I} - \mathbf{k}^f \mathbf{d}')(\mathbf{F} + \mathbf{g} \mathbf{k}'_x), \quad \boldsymbol{\Psi} = (\boldsymbol{\Lambda} + \mathbf{k}^f \mathbf{d}' \mathbf{g} \mathbf{k}'_x),$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{F} & \mathbf{g} \mathbf{k}'_x \\ \mathbf{k}^f \mathbf{d}' \mathbf{F} & \boldsymbol{\Psi} \end{bmatrix}, \quad \boldsymbol{\Omega} = \begin{bmatrix} \mathbf{I} \\ \mathbf{k}^f \mathbf{d}' \end{bmatrix}.$$

### B. Output and control variances for averaging sampling

In the case of averaging sampling we have:

$$\sigma_y^2 = \text{var} \{y_i\} = \mathbf{d}'_0 \mathbf{V}^o \mathbf{d}_0, \quad (52)$$

$$\sigma_u^2 = \text{var} \{u_i\} = \mathbf{k}'_x \mathbf{V}^f \mathbf{k}_x, \quad (53)$$

where the covariance matrix  $\mathbf{V}$  has the form as in eq. (50) and:

$$\mathbf{V} = \boldsymbol{\Phi} \mathbf{V} \boldsymbol{\Phi}' + \boldsymbol{\Gamma} \mathbf{W} \boldsymbol{\Gamma}' + \boldsymbol{\Gamma} \boldsymbol{\gamma} \mathbf{E}' + \mathbf{E} \boldsymbol{\gamma}' \boldsymbol{\Gamma}' + \mathbf{E} \mathbf{E}' \rho^2, \quad (54)$$

where:

$$\boldsymbol{\Lambda} = \left[ \bar{\mathbf{F}} - \bar{\mathbf{F}} \mathbf{k}^f \mathbf{f}' + (\bar{\mathbf{g}} - \bar{\mathbf{F}} \mathbf{k}^f \mathbf{g}_* + \boldsymbol{\Upsilon} \mathbf{g}_*) \mathbf{k}'_x \right];$$

$$\boldsymbol{\Psi} = \boldsymbol{\Upsilon} \mathbf{f}', \quad \boldsymbol{\Upsilon} = (\bar{\mathbf{d}} + \bar{\mathbf{F}} \mathbf{k}^f)$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{F} & \mathbf{g} \mathbf{k}'_x \\ \boldsymbol{\Psi} & \boldsymbol{\Lambda} \end{bmatrix}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\Upsilon} \end{bmatrix}.$$

## VI. EXAMPLES

We will study the properties of control systems for a plant having control path

$$K_c(s) = \frac{1}{(1 + 0.5s)^2}, \quad (55)$$

with disturbance and noise modeled by:

$$K_d(s) = \frac{k_d}{(1 + T_d s)^2}, \quad K_n(s) = \frac{k_n}{T_n^2 s^2 + 2\zeta_n T_n s + 1}, \quad (56)$$

with  $T_d = 2$ ,  $T_n = 0.05$ ,  $\zeta_n = 1$  and  $k_d$  and  $k_n$  chosen such that  $\text{var} d(t) = 1$ ,  $\text{var} n(t) = 0.5^2$ .

Spectral characteristics of disturbance and noise signals along with sampling frequencies corresponding to values of sampling intervals  $h$  studied further are displayed in Fig.2.

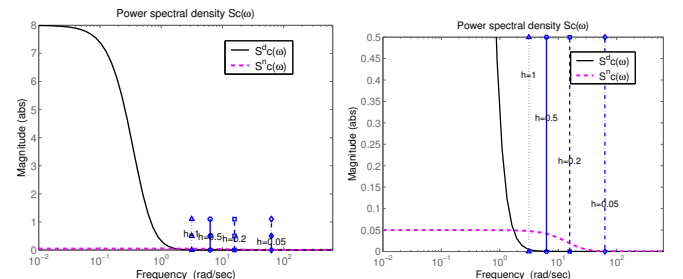


Fig. 2. Spectral densities:  $S_c^d(\omega)$  and  $S_c^n(\omega)$  for  $d(t)$  and  $n(t)$

Properties of control systems with various structures of continuous-time filters are displayed in Fig.4 for different values of sampling period  $h$ , and compared with systems without any continuous-time filter. Discrete-time Kalman filters were designed assuming system structures depicted in Fig.3. General observation is that the smaller  $h$  the better the results for all possible configurations, and that both anti-aliasing filters give a marginal improvement with respect to the system without any filter for  $h = 0.5$  while for both smaller and larger values of  $h$  the improvement is negligible. The best results are attained for all values of  $h$  when the continuous-time Kalman filter of eq.(13) with  $\mathbf{k}_{c,dn}^f = \mathbf{k}_{c,dn}^f/h$  is employed. Classical approach to noisy systems fitted with an anti-aliasing filter is forget about noise, add the dynamics of the filter to the plant dynamics and solve the problem as though the noise never existed. The result of such approach is depicted in Fig.5 and compared with systems where the discrete-time model takes the noise characteristics into account. It is clear that neglecting noise leads to bad results, resulting mainly in large control magnitudes.

Influence of noise characteristics characterized by the value of damping parameter  $\zeta_n$  determining the shape of spectral density in Fig.6a) on the control quality is presented in Fig.6b)-d) for  $h = 0.2$ . The main observation is that the smaller is the value of  $\zeta_n$  the better is disturbance attenuation. Similarly, the influence of the parameter  $T_n$  determining noise band is presented in Fig.7. The main observation is that the smaller  $T_n$  the better disturbance attenuation irrespectively of the type of filtering used. In all cases the quality of systems with both types of antialiasing filter is only marginally better.

## VII. SIMPLIFIED MODELS

In the paper we assume a broadband noise, whose model might not be available. Moreover, the noise model contributes to Kalman filter complexity. It is likely that simplified discrete-time system model with discrete-time white noise with appropriately chosen variance can be good alternative to the exact one.

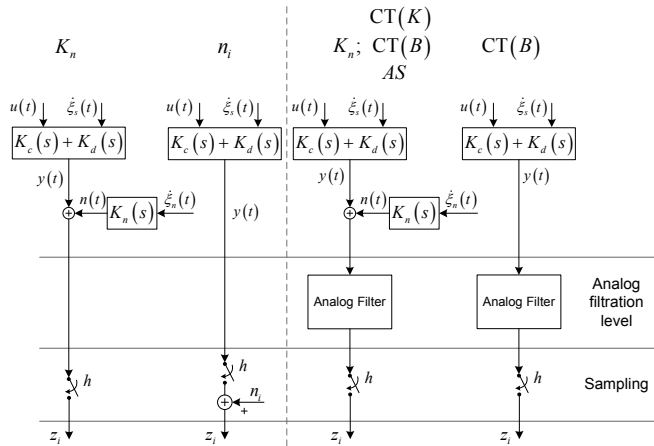


Fig. 3. Models of signals and CT filters assumed for DT Kalman filter design

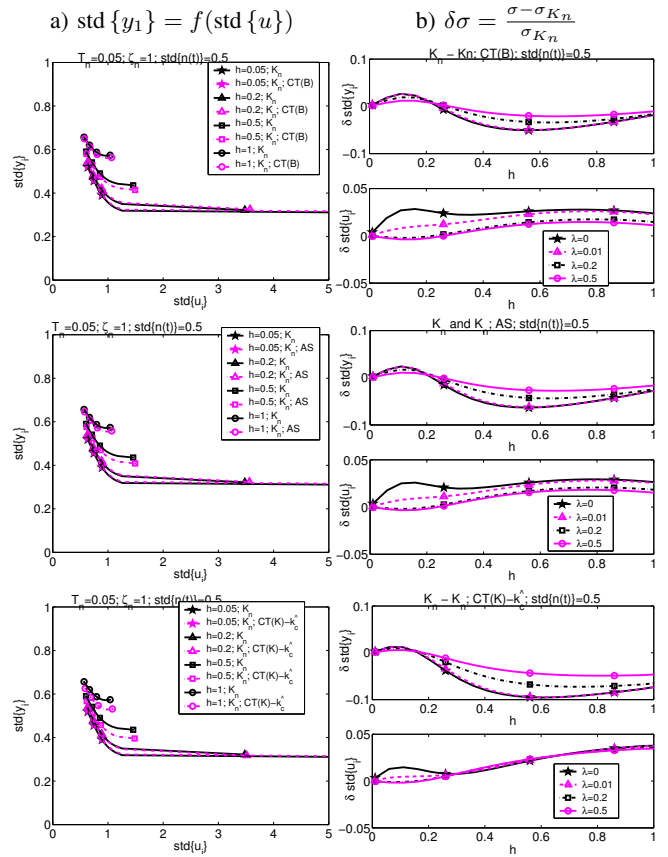


Fig. 4. Results for various types of CT filters compared with system without any CT filter labeled  $K_n$

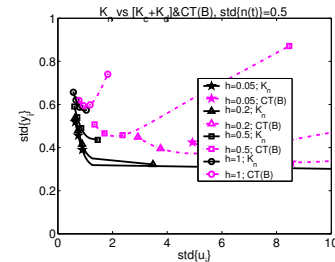


Fig. 5. Classical approach: Butterworth filter and DT Kalman filter designed assuming no noise [CT(B)] vs DT Kalman filter with noise model [ $K_n$ ]

### A. Discrete Simplified models

We propose a discrete-time model of instantaneously sampled noisy signal

$$\mathbf{x}_{i+1}^p = \mathbf{F}^p \mathbf{x}_i^p + \mathbf{g}^p u_i + \mathbf{w}_i^p, \quad (57)$$

$$z_i = \mathbf{d}_p' \mathbf{x}_i^p + n_i, \quad (58)$$

$$y_i = \mathbf{d}_p' \mathbf{x}_i^p, \quad (59)$$

with

$$\mathbf{F}^p = e^{\mathbf{A}^p h}, \quad \mathbf{g}^p = \int_0^h e^{\mathbf{A}^p \nu} \mathbf{b}^p d\nu,$$

$$\mathbf{W}^p = \int_0^h e^{\mathbf{A}^p \nu} \mathbf{c}^p \mathbf{c}^{p'} e^{\mathbf{A}^p \nu} d\nu,$$

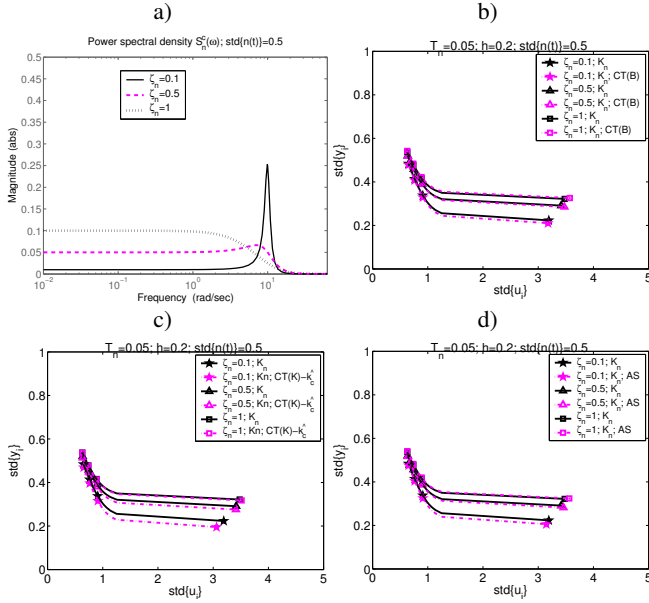


Fig. 6. a) Power spectral density for different values of  $\zeta_n$ ; b)-d) Results for different values of  $\zeta_n$ .  $T_n = 0.05$

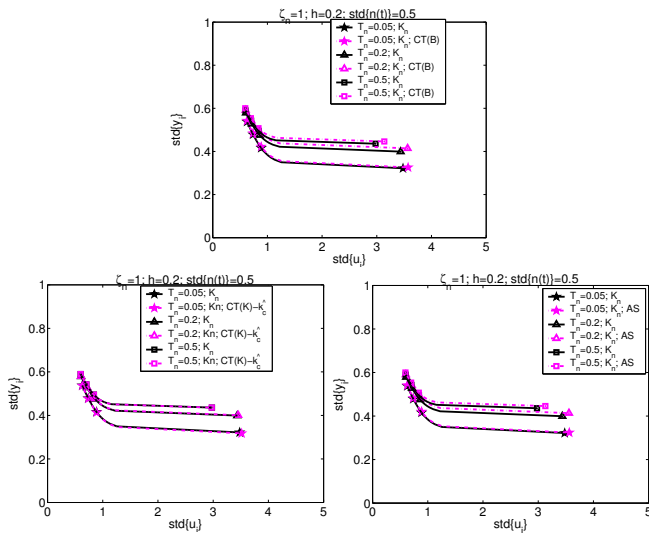


Fig. 7. Results for different values of  $T_n$ .  $\zeta_n = 1$

with continuous-time description:

$$\mathbf{A}^p = \begin{bmatrix} \mathbf{A}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_d \end{bmatrix}, \mathbf{b}^p = \begin{bmatrix} \mathbf{b}_c \\ \mathbf{0} \end{bmatrix}, \mathbf{c}^p = \begin{bmatrix} \mathbf{0} \\ \mathbf{c}_d \end{bmatrix}, \quad (60)$$

$$\mathbf{d}^p = \begin{bmatrix} \mathbf{d}_c \\ \mathbf{d}_d \end{bmatrix}, \mathbf{x}^p(t) = \begin{bmatrix} \mathbf{x}_c(t) \\ \mathbf{x}_d(t) \end{bmatrix}, \quad (61)$$

in which noise is presented as discrete-time white noise  $n_i$  whose variance  $\rho^2$  equals to the variance of  $n(t)$  of the original system, i.e.  $\rho^2 = \text{var}\{n_i\} = \text{var}\{n(t)\}$ , and can be calculated as

$$\rho^2 = \mathbf{d}'_n \mathbf{Q}_n \mathbf{d}_n, \quad (62)$$

where  $\mathbf{Q}_n$  fulfills the following Lyapunov equation:

$$\mathbf{A}_n \mathbf{Q}_n + \mathbf{Q}_n \mathbf{A}'_n = -\mathbf{d}_n \mathbf{d}'_n. \quad (63)$$

## B. Discrete-time Kalman filter for discrete simplified model

Kalman filter equations for system in (57)–(58) have formally the same for as in (26)–(27), except for  $\dim \mathbf{x}_i^m = n_m$ , and

$$\mathbf{k}^f = \Sigma \mathbf{d}^p (\mathbf{d}^{p'} \Sigma \mathbf{d}^p + \rho^2)^{-1}, \quad (64)$$

where  $\Sigma$  is a solution of

$$\Sigma = \mathbf{W}^p + \mathbf{F}^p \left( \Sigma - \frac{\Sigma \mathbf{d}^p \mathbf{d}^{p'} \Sigma'}{\mathbf{d}^{p'} \Sigma \mathbf{d}^p + \rho^2} \right) \mathbf{F}^{p'}. \quad (65)$$

## C. Examples

It is interesting that even a simplified discrete-time noise model in the form of discrete-time white noise  $n_i$  with appropriately chosen variance leads to very good results. A comparison of an exact and approximate systems is depicted in Fig.8.

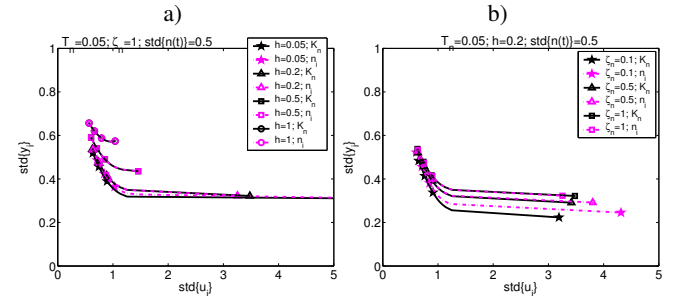


Fig. 8. Full model DT Kalman filter vs. simplified DT with  $n_i$

## VIII. CONCLUSION

The results of the paper show that the common belief about necessity of using anti-aliasing filters in sampled data control systems is not justified. Much more important is the knowledge of the noise characteristics. The main tool to improve the control system performance is increasing the sampling and control signal modulation frequencies. No additional continuous-time filters are then necessary to arrive at good control quality.

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