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Artificial Vector Fields for Robot Convergence and Circulation of Time-Varying Curves in N-dimensional Spaces

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Abstract— **This paper addresses the problem of controlling a single mobile robot to converge smoothly to a pre-specified closed curve. Once in the curve, the robot remains circulating along it. The main motivation for this is the control of unmanned airplanes, where the robot cannot converge to a single point. Our control law is based on an artificial vector field that allows for the generalization to time-varying curves defined in n-dimensional spaces. We also present results that may be used to control mobile robots moving with constant speed. We devise convergence proofs and present simulations that verify the proposed approach.**

I. INTRODUCTION

Artificial vector field based approaches have been extensively used to control mobile robots in the execution of different tasks. This is mainly due to the robustness of such methods to localization and actuator errors, which allows for real world applications. Given a domain, $\Omega \subset \mathbb{R}^n$, a vector field, $\mathbf{g} : \Omega \to T_{\mathbf{q}}(\Omega)$, where $T_{\mathbf{q}}(\Omega)$ is the tangent space of Ω, is defined. The desired task is then accomplished by enforcing the robots to use such vectors as velocity or acceleration inputs.

A classical problem in the robotics literature is the problem of driving a single robot from an initial configuration, $\mathbf{q}_0$, to a final configuration, $\mathbf{q}_f$. In this case, the domain, Ω, is the so-called robot's configuration space. This problem was solved by means of several artificial vector fields such as the ones proposed in [1], [2] and [3].

Another important problem recently considered is the problem of pattern generation. Different tasks such as, surveillance, manipulation, and boundary monitoring can be executed by using solutions of the pattern generation problem. Given a team of mobile robots, this problem consists of controlling such a team to converge to and form a pre-specified static geometrical pattern. In [4], [5], and [6], vector fields are computed to solve this problem for static two-dimensional patterns.

In [7], a dynamic vector field is computed to control a team of mobile robots to converge to and circulate along the boundary of a desired static two-dimensional geometric pattern. The basic idea is that the vector field generates a limit cycle which attracts the system. The vector field is dynamic in the sense that it changes according to the robots relative positions. This is necessary to guarantee collision avoidance. Other works such as [8], [9], [10], and [11] also propose to generate limit cycles that attract the system.

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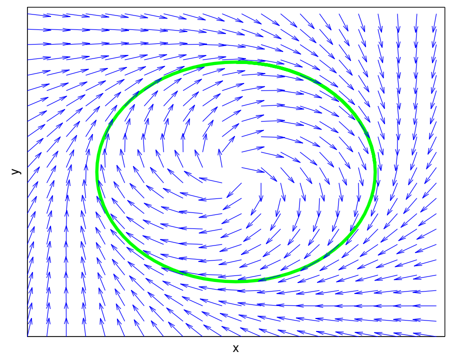

Fig. 1. *Example of curve and vector field.*

In the present work we address the problem of controlling a single robot, which may be a unmanned airplane, to converge and circulate along a given curve. This is an important problem in the case of monitoring or surveillance tasks performed by a single unmanned airplane. Since only one agent is considered, one could argue that this problem seems to be simpler than the one addressed in [7]. However, differently from previous works, we deal with the general problem of time-varying boundaries in n-dimensional spaces. This is interesting in the case of unmanned airplanes because we can define dynamic curves in $\mathbb{R}^3$. This extension was possible due to the particular form of the construction of the target curve: an intersection of level sets. The problem of convergence is then translated to the problem of driving a set of functions to zero. The problem statement is given bellow:

Problem Statement 1 *Let Γ be an curve (in this paper, mainly closed) , which may be static or time-varying, defined in a n-dimensional space. Compute a static or time-varying vector field, $\mathbf{h}$, such that its integral curves converge to and circulate Γ.*

Just to clarify the idea in Problem 1, we present in Figure 1 an example of a curve, Γ, in a two-dimensional space and a vector field $\mathbf{h}$ that solves the problem. For simplicity in the illustration, in this example both the curve and the vector field are static. In the next section we present our main mathematical result which solves the stated problem.

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In Section III we discuss some applications in robot control. Finally, we present our conclusions and discuss future directions in Section IV.

II. Methodology

In this section we devise a mathematical result which will be used later in robotic applications. As stated before, the problem consists of creating a vector field in $\mathbb{R}^n$ that guides the state variable to a curve that changes with time, and maintains it circulating along such a curve in a given fixed direction.

The method relies on our ability to find functions $\alpha_i(x_1, x_2, ..., x_n, t)$ $(i = 1, 2, 3, ..., n-1)$ such that the desired curve is obtained by the intersection of the level sets $\{[x_1, x_2, ..., x_n]^T \in \mathbb{R}^n | \alpha_i(x_1, x_2, ..., x_n, t) = 0\}$. From now on, the variable t will be a non-negative variable that represents time. So, all the assumptions (such as differentiability) are required to hold for (and only for) $t \geq 0$. All the vectors are column vectors unless mentioned otherwise, also, $\nabla_{\mathbf{q}}$ is defined to be the vector $[\frac{\partial}{\partial x_1} \;\; \frac{\partial}{\partial x_2} \;\; ... \frac{\partial}{\partial x_n}]^T$, $\nabla_\alpha = [\frac{\partial}{\partial \alpha_1} \;\; \frac{\partial}{\partial \alpha_2} \;\; ... \;\; \frac{\partial}{\partial \alpha_{n-1}}]^T$ and $\mathbf{q} = [x_1(t) \;\; x_2(t) \;\; ... \;\; x_n(t)]^T$. Now, we present some major definitions.

Definition 1 *A time-varying set of points, $\mathcal{C}(t)$, of a dynamical system $\dot{\mathbf{q}} = \mathbf{h}(\mathbf{q}, t)$ is said to be a repulsive set if there exists a neighborhood $\mathcal{N}(\mathcal{C})$ such that for all $\mathbf{q} \in \mathcal{N}$ we have $\dot{D} > 0$, where D is the distance between $\mathbf{q}$ and $\mathcal{C}$.*

Definition 2 *Let $\mathbf{v}_i$ $(i = 1, 2, 3, ..., n-1)$ $n \geq 2$ be vectors in $\mathbb{R}^n$. The wedge product [12] $\wedge_{i=1}^{n-1} \mathbf{v}_i$ is the vector in $\mathbb{R}^n$ such that the $i-th$ element is given by the cofactor of $n-th$ row and $i-th$ column of the matrix such that the $i-th$ row is given by the vector $\mathbf{v}_i$ for $i = 1, 2, 3, ..., n-1$ (the last row is unnecessary, so it is left undefined).*

The above definition may be seen as an extension of the cross product in $\mathbb{R}^3$ to $\mathbb{R}^n$ (the reader can note that if $n = 3$ we reduce to the usual cross product formula). The resulting vector is orthogonal (*i.e.*, the scalar product is null) to each $\mathbf{v}_i$. The function is an alternating multilinear form, so, it is linear in each argument and changes sign by swapping two vectors (this comes directly from determinant properties). As an implication of their properties, the vector is null whenever two (or more) vectors are linearly dependent.

In this paper, if $n = 3$ we will write $\mathbf{v}_1 \times \mathbf{v}_2$ instead of $\wedge_{i=1}^{2} \mathbf{v}_i$.

Definition 3 *Let $\alpha_i(x_1, x_2, ..., x_n, t) : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ $(i = 1, 2, 3, ..., n-1)$ $n \geq 2$ be differentiable functions in all of their arguments. We denote here by $M(\alpha)$ the matrix in $\mathbb{R}^{n \times n}$ such that its $i-th$ row is given by the vector $\nabla_{\mathbf{q}} \alpha_i$ for $i = 1, 2, ..., n-1$ and $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i$ in the n-th row. Also, we denote here by $\mathbf{a}(\alpha)$ the column vector such that the $i-th$ row is given by $\frac{\partial \alpha_i}{\partial t}$ for $i = 1, 2, ..., n-1$ and 0 in the $n-th$ row. Also, $M_*(\alpha)$ and $\mathbf{a}_*(\alpha)$ are matrices obtained from $M(\alpha)$ and $\mathbf{a}(\alpha)$ respectively by removing the last row.*

By the definition of $M(\alpha)$, we note that the matrix is invertible if and only if the vectors $\nabla_{\mathbf{q}} \alpha_i$, $i = 1, 2, 3, ..., n-1$ are linearly independent.

Definition 4 *We denote by $\mathcal{C}(t)$ the set $\{\mathbf{q} \in \mathbb{R}^n | \wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i = \mathbf{0}\}$ (*i.e.*, points such that $\nabla_{\mathbf{q}} \alpha_i$ are linearly dependent) and $\mathcal{D}(t) = \{\mathbf{q} \in \mathbb{R}^n | \alpha_i(\mathbf{q}, t) = 0, \forall i \leq n-1\}$ (*i.e.*, points that lie in the intersection of the level sets $\alpha_i = 0$).*

Definition 5 *We define four classes of functions:*

- *$\alpha_i(x_1, x_2, ..., x_n, t) : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ $i = 1, 2, 3, ..., n-1$ and $n \geq 2$ functions with continuous partial derivatives and bounded second partial derivatives in all of their arguments such that $\mathcal{D}(t)$ is a connected set.*
- *$V : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ a negative definite function with continuous partial derivatives and bounded second partial derivatives in all of its arguments such that its gradient is null only in the origin.*
- *$H(x_1, x_2, ..., x_n, t) : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ a function that is continuous in all its arguments such that it is not null for all points of $\mathcal{D}(t)$.*
- *$G(x_1, x_2, ..., x_n, t) : \mathbb{R}^{n+1} \mapsto \mathbb{R}$ a non-negative function with bounded partial derivatives in all its arguments, except maybe in points such that $\nabla_{\mathbf{q}} V = \mathbf{0}$. Besides, it is only null, possibly but not obligatory, in points such that $\nabla_{\mathbf{q}} V = \mathbf{0}$. Furthermore, $\lim_{\mathbf{q} \to \mathcal{D}(t)} G \nabla_{\mathbf{q}} V = \mathbf{0}$ and both $\lim_{\mathbf{q} \to \mathcal{D}(t)} \nabla_{\mathbf{q}} G \|\nabla_{\mathbf{q}} V\|^2$ and $\lim_{\mathbf{q} \to \mathcal{D}(t)} \frac{\partial G}{\partial t} \|\nabla_{\mathbf{q}} V\|^2$ are bounded values.*

At this point, it is important to remark that the definition of the function G, which seems to be a little bit complex, will be useful in the next section, when we will use the methodology to solve a robotic guidance problem where a constant value of $\|\dot{\mathbf{q}}\|$ is desired.

We will use the following Lemma presented in [13] to prove our main result:

Lemma 1 ("Lyapunov-Like Lemma"[13]) *If a function $f(\mathbf{q}, t)$ satisfies the following properties:*

- *$f(\mathbf{q}, t)$ is lower bounded*
- *$\frac{d}{dt} f(\mathbf{q}, t)$ is negative semidefinite*
- *$\frac{d}{dt} f(\mathbf{q}, t)$ is uniformly continuous (ensured if $\frac{d^2}{dt^2} f(\mathbf{q}, t)$ is bounded)*

then $\lim_{t \to \infty} \frac{df}{dt} = 0$.

Proof: This Lemma follows directly from the well-known Barbalat's Lemma [14]. The Barbalat's Lemma states that if a differentiable function $w(t)$ has a finite limit as $t \to \infty$, and if $\dot{w}(t)$ is uniformly continuous, then $\dot{w}(t) \to 0$ as $t \to \infty$. ■

The main result of this paper is summarized in the form of the next Theorem.

Theorem 1 *Consider the nonautonomous dynamical system:*

$$\dot{\mathbf{q}}(t) = G \nabla_{\mathbf{q}} V + H \wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i - M(\alpha)^{-1} \mathbf{a}(\alpha) , \quad (1)$$

where $V = V(\alpha_1, \alpha_2, ..., \alpha_{n-1})$. Assume that the set $\mathcal{C}(t)$ is a repulsive set, and that $\mathcal{C}(t) \cap \mathcal{D}(t) = \emptyset$. Therefore, the system is such that the state variable $\mathbf{q}$ asymptotically converges to the set $\mathcal{D}(t)$ and maintains circulating along it with a given fixed direction for any initial condition ($t = 0$) $\mathbf{q}_0$ that is not in $\mathcal{C}(0)$. Moreover, $\dot{\mathbf{q}}$ is continuous.

Proof: We note that $\dot{\mathbf{q}}$ is continuous for any t because of the differentiability of the involved functions, the fact that $\lim_{\mathbf{q}\to\mathcal{D}(t)} G\nabla_{\mathbf{q}}V = \mathbf{0}$ (so, the possibly discontinuity of G on $\mathcal{D}(t)$ is not a problem) and that, by hypothesis, the points on the set $\mathcal{C}(t)$ are repulsive points (that guarantees the existence of $M(\alpha)^{-1}$). For simplification, we will provide the proof in two steps: first we will ensure that a given function f satisfies Lemma 1 hypotheses, and second we will use this result in the second step to ensure convergence and circulation.

Step 1 *$f(\mathbf{q}, t) = -V$ satisfies Lemma 1 hypotheses*

The function f is positive definite (so, lower bounded by 0). Developing the expression we note that $\nabla_{\mathbf{q}}V^T = \nabla_\alpha V^T M_*(\alpha)$ which implies that $\frac{df}{dt} = -\nabla_\alpha V^T (M_*(\alpha)\dot{\mathbf{q}} + \mathbf{a}_*(\alpha))$. Substituting the expression of $\dot{\mathbf{q}}$ and using the fact that $\wedge_{i=1}^{n-1}\nabla_{\mathbf{q}}\alpha_i$ is orthogonal to $\nabla_{\mathbf{q}}\alpha_i$ for all the i's (and so $M_*(\alpha)(\wedge_{i=1}^{n-1}\nabla_{\mathbf{q}}\alpha_i) = \mathbf{0}$) we have that $\frac{df}{dt} = -\nabla_\alpha V^T (GM_*(\alpha_i)\nabla_{\mathbf{q}}V - M_*(\alpha)M(\alpha)^{-1}\mathbf{a}(\alpha) + \mathbf{a}_*(\alpha))$. Developing the expression, we note that $M_*(\alpha)M(\alpha)^{-1}\mathbf{a}(\alpha) = \mathbf{a}_*(\alpha)$ and therefore finally we have that $\frac{df}{dt} = -G\nabla_\alpha V^T M_*(\alpha)\nabla_{\mathbf{q}}V = -G\|\nabla_{\mathbf{q}}V\|^2$, which is negative semidefinite by the hypotheses on G and $\nabla_{\mathbf{q}}V$.

We will not develop $\frac{d^2 f}{dt^2}$ but, the reader may note that it is bounded for any t since the points which causes the term $M(\alpha)^{-1}$ to be unbounded are by hypothesis in a repulsive set, the derivatives of the involved functions exist, and the second partial derivatives that appear are bounded by hypothesis. Furthermore, the problem caused by the possible absence of bound of the partial derivatives of G on $\mathcal{D}(t)$ is avoided by the two last hypotheses on G concerning the existence of limits, since $\frac{dG}{dt}\|\nabla_{\mathbf{q}}V\|^2 = (\nabla_{\mathbf{q}}G\|\nabla_{\mathbf{q}}V\|^2)^T\dot{\mathbf{q}} + \frac{\partial G}{\partial t}\|\nabla_{\mathbf{q}}V\|^2$. Since the three requirements are satisfied, Lemma 1 ensures that $\frac{df}{dt} \to 0$.

Step 2 *The convergence and circulation is ensured*

By the first step $-G\|\nabla_{\mathbf{q}}V\|^2$ approaches zero. That implies that $\nabla_{\mathbf{q}}V^T = \nabla_\alpha V^T M_*(\alpha)$ or G approaches zero. We will prove now that in order to this to happen, it is necessary that $\mathbf{q}$ is on $\mathcal{D}(t)$. If it is the first case suppose that there is any non null vector $\nabla_\alpha V^T$ that nulls $\nabla_{\mathbf{q}}V^T$. So, as the system is overdetermined for the variable $\nabla_\alpha V^T$ (we have $n-1$ variables and n equations) that implies that the vectors of the matrix $M_*(\alpha)$ are linearly dependent, and this doesn't happen since these points are in a repulsive set by hypothesis. So, we conclude that $\nabla_\alpha V^T$ is null, but this implies that the α_i's (the function arguments) are all null for $i = 1, 2, 3, ..., n-1$, so we must be on $\mathcal{D}(t)$. If the latter , $\mathbf{q}$ must be on $\mathcal{D}(t)$ by the same argument since G is only possibly null for points such that $\nabla_{\mathbf{q}}V = \mathbf{0}$. Sufficiency that $-G\|\nabla_{\mathbf{q}}V\|^2 \to 0$ if either (or both) of the cases happen comes from the fact that $\lim_{\mathbf{q}\to\mathcal{D}(t)} G\nabla_{\mathbf{q}}V = \mathbf{0}$ by hypothesis on G, and that implies also that $\lim_{\mathbf{q}\to\mathcal{D}(t)} G\|\nabla_{\mathbf{q}}V\|^2 = 0$. So, asymptotical convergence to $\mathcal{D}(t)$ is ensured.

Once in $\mathcal{D}(t)$, we have only $\dot{\mathbf{q}} = -M(\alpha)^{-1}\mathbf{a}(\alpha) + H\wedge_{i=1}^{n-1}\nabla_{\mathbf{q}}\alpha_i$, that implies that $\frac{dV}{dt} = 0$ (which means that the point continues in $\mathcal{D}(t)$). First, it is necessary to note that $\dot{\mathbf{q}}$ is never null in $\mathcal{D}(t)$. To see this, as $\wedge_{i=1}^{n-1}\nabla_{\mathbf{q}}\alpha_i$ is non null in $\mathcal{D}(t)$ (by the hypothesis $\mathcal{C}(t) \cap \mathcal{D}(t) = \emptyset$) it is sufficient to note that the term $\mathbf{w} = -M(\alpha)^{-1}\mathbf{a}(\alpha)$ obviously satisfies $M(\alpha)\mathbf{w} = -\mathbf{a}(\alpha)$, and the $n-th$ row of this equation says that $\mathbf{w}$ is orthogonal to $\wedge_{i=1}^{n-1}\nabla_{\mathbf{q}}\alpha_i$ (therefore, the sum is null only if both terms are null, which does not happen with at least one of the terms). In fact, only the term $\wedge_{i=1}^{n-1}\nabla_{\mathbf{q}}\alpha_i$ is responsible to keep the state circulating along the curve, and the term $\mathbf{w}$ is responsible for the instantaneous response to the shape variation of the curve (so it continues going to the set $\mathcal{D}(t)$).

The state circulates $\mathcal{D}(t)$ in the same direction (because $\dot{\mathbf{q}}$ is continuous which implies that changing direction needs first to become null, and this does not happen). The sign of H on $\mathcal{D}(t)$ defines the motion direction.

Connectedness of $\mathcal{D}(t)$ is required so $\mathcal{D}(t)$ will be a single well defined curve (instead of, for example, two disjoint curves). ■

For $n = 2$, except for the last term that provides support to time-varying boundaries, Equation (1) gives similar results to those presented in [7] for a single robot.

It should be clear that if the boundary is static (set $\mathcal{D}(t)$ is fixed in time), the term $M(\alpha)^{-1}\mathbf{a}(\alpha)$ in (1) vanishes. However, even if the set is not static, can we use the system presented in Theorem 1 without this term and ensure that it works? It turns out that this is possible in a particular condition, as we show in the following Corollary.

Corollary 1 *If*

- $\lim_{t\to\infty}\|\mathbf{a}\| = 0$ *for any trajectory* $\mathbf{q}(t)$
- *In the system obtained via (1), with term $M^{-1}\mathbf{a}$, the set $\mathcal{C}(t)$ is repulsive*
- *In the system obtained via (1), without term $M^{-1}\mathbf{a}$, the set of points such that $\nabla_{\mathbf{q}}V = \mathbf{0}$ that is not on $\mathcal{D}(t)$ is repulsive*

then convergence and circulation is ensured to the system without the term $M^{-1}\mathbf{a}$.

First consider the system with the term, so, we note that $\|M^{-1}\mathbf{a}\| \leq \|M^{-1}\|\|\mathbf{a}\|$. We also note that $\|M^{-1}\|$ is bounded since otherwise it would imply that $\mathbf{q}$ approaches a point that makes M^{-1} be singular (linearly dependent $\nabla\alpha_i$'s), which does not happen by hypothesis. As a result, the term $M^{-1}\mathbf{a}$ approaches to zero as time increases since $\lim_{t\to\infty}\|\mathbf{a}\| = 0$. Therefore, the system with the term approaches the one without the term since the differing term will be negligible with time (and the system without it does not stop in the equilibrium points, *i.e.* points such that $\nabla_{\mathbf{q}} V = \mathbf{0}$ which are not in $\mathcal{D}(t)$, by hypothesis). Thus, as convergence is ensured to the system with the term, the same applies to the system without it.

The term $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i$ maintains the point circulating the curve.

Qualitatively, the hypothesis of the corollary means that the curve varies slowly with time. Using this, $-V$ is not strictly decreasing, but $-V$ goes to 0 as time goes to ∞ in the same way.

In the next section we will use the obtained results in robotic applications.

III. Applications to Robot Control

In this section we will discuss some practical applications of the main result in robot control. It is clear that the majority of applications are those where $n = 3$ or $n = 2$. Therefore, we will discuss only applications in three and two dimensional environments. From now on, we consider that our robot is a holonomic point robot with dynamics given by

$$\dot{\mathbf{q}}(t) = \mathbf{u}\,, \tag{2}$$

where $\mathbf{q}(t) = [x_1(t) \quad x_2(t) \quad x_3(t)]^T$ and $\mathbf{u}$ is the control input.

The basic idea is that the right-hand side of Equation (1) gives us a vector field $\mathbf{h}(\mathbf{q}, t)$ which solves Problem 1. Therefore, we use:

$$\mathbf{u} = \mathbf{h}(\mathbf{q}, t) \tag{3}$$

as our control law.

Next, we will illustrate the methodology with some examples.

A. Static Boundaries

At first, we consider time fixed boundaries on $\mathbb{R}^3$. In this case, the last term of Equation (1) vanishes. The hypothesis concerning the set $\mathcal{C}(t)$ can be then relaxed. First, the constraint that the points where $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i = \mathbf{0}$ lie on a repulsive set are due to the fact that the matrix $M(\alpha)$ must be invertible. Since $\mathbf{a}(\alpha)$ vanishes in time fixed boundaries, this is no longer necessary.

But one can note that now the points such that $\nabla_{\mathbf{q}} V = \sum_{i=1}^{n-1} \frac{\partial V}{\partial \alpha_i} \nabla_{\mathbf{q}} \alpha_i = \mathbf{0}$ out of $\mathcal{D}$ are also equilibria points of the system (1): when $\nabla_{\mathbf{q}} V$ vanishes, then $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i = \mathbf{0}$. To see that, we note that $\nabla_{\mathbf{q}} V = \sum_{i=1}^{n-1} \frac{\partial V}{\partial \alpha_i} \nabla_{\mathbf{q}} \alpha_i = \mathbf{0}$ vanishing out of $\mathcal{D}$ implies that at least one of the $\frac{\partial V}{\partial \alpha_i}$ is non null (by hypothesis on V). This implies that the vectors $\nabla_{\mathbf{q}} \alpha_i$ are linearly dependent, so $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i$ vanishes. Then we need $\mathcal{C}$ to be a unstable set. For an autonomous dynamical system this is equivalent to say that those points are unstable points of equilibria.

A desired property in some robotic tasks is to have constant speed, so the vector field must have a constant norm. Provided that the target set $\mathcal{D}$ is static this is always possible. We then need to choose G and H such that $\|\dot{\mathbf{q}}\|$ is constant (say, unity).

To this end, choose $G = J/\|\nabla_{\mathbf{q}} V\|$, with $J(x_1, x_2, ..., t)$ such that $0 \leq J \leq 1$, J is differentiable in all its arguments and $\lim_{\mathbf{q} \to \mathcal{D}(t)} G = 0$ (one can see, by developing the expressions for $\nabla_{\mathbf{q}} G$ and $\frac{\partial G}{\partial t}$, that this ensures that the three limit constraints on G are achieved). Choose also $H = \sqrt{1 - J^2}/\| \wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i \|$. Now, it is clear that the points where $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i$ vanishes give us discontinuities in the vector field. So, one must ensure that these points (that in general are *not* equilibria points of the system (1)) lie on a repulsive set. Then, one can easily check that the norm is equal to one (by the orthogonality of the vectors $\nabla_{\mathbf{q}} V$ and $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i$) and that $\dot{\mathbf{q}}$ is continuous since both $\nabla_{\mathbf{q}} V$ and $\wedge_{i=1}^{n-1} \nabla_{\mathbf{q}} \alpha_i$ vanishing points are in a repulsive set. The choice of G above justifies the troublesome G requirements: note that G, as defined, is not differentiable in $\mathcal{D}(t)$, in general, since $\|\nabla_{\mathbf{q}} V\|$ is null in this set.

To illustrate the methodology, we will create a three-dimensional limit cycle to the curve parameterized as $[x_1(\tau) \quad x_2(\tau) \quad x_3(\tau)]^T = [cos(\tau) \quad sin(2\tau) \quad -sin(\tau)]^T$ for $0 \leq \tau \leq 2\pi$. This is a closed curve and seen from above on the $x_1 x_2$ plane it forms an *eight* shape. One can note that this curve is the intersection of the surfaces:

$$\alpha_1(x_1, x_2, x_3, t) = \frac{1}{2}(x_1^2 + x_3^2 - 1) = 0\,, \tag{4}$$

$$\alpha_2(x_1, x_2, x_3, t) = (x_1^2 + x_3^2)(2x_1 x_3 + x_2) = 0. \tag{5}$$

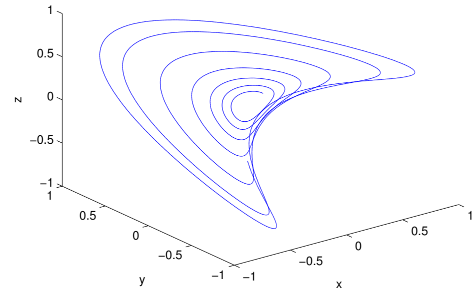

Fig. 2. *Convergence to the target curve with the initial condition $x_1(0) = 0.1$, $x_2(0) = 0.1$ and $x_3(0) = 0.1$.*

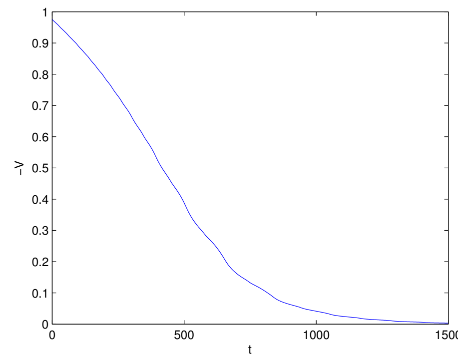

Fig. 3. *Plot of $-V$ versus time for Figure 2: as $t \to \infty$ $-V$ (and so α_1 and α_2) goes to 0.*

Therefore, we have

$$\nabla_{\mathbf{q}}\alpha_1 = [x_1 \;\; 0 \;\; x_3]^T , \quad (6)$$

$$\nabla_{\mathbf{q}}\alpha_2 = \begin{bmatrix} 6x_1^2x_3 + 2x_3^3 + 2x_1x_2 \\ x_1^2 + x_3^2 \\ 6x_3^2x_1 + 2x_1^3 + 2x_3x_2 \end{bmatrix} , \quad (7)$$

$$\nabla_{\mathbf{q}}\alpha_1 \times \nabla_{\mathbf{q}}\alpha_2 = \begin{bmatrix} -x_3(x_1^2 + x_3^2) \\ 2(x_3^4 - x_1^4) \\ x_1(x_1^2 + x_3^2) \end{bmatrix} . \quad (8)$$

We can verify that $\nabla_{\mathbf{q}}\alpha_1 \times \nabla_{\mathbf{q}}\alpha_2$ is only null if $x_1 = x_3 = 0$. We choose $V = -\frac{1}{2}(\alpha_1^2 + \alpha_2^2)$ and $J = \|\nabla_{\mathbf{q}}V\|^2/(1 + \|\nabla_{\mathbf{q}}V\|^2)$, then clearly all the conditions on G holds.

To show that the set $\mathcal{C}$ (set of points such that $x_1 = x_3 = 0$) is repulsive, we use the fact that if $D(x_1, x_2, x_3)$ is the Euclidean distance between $\mathbf{q}$ and the set $\mathcal{C}$ ($x_1 = x_3 = 0$) then $D^2 = 2\alpha_1 + 1$. So, $\frac{dD^2}{dt} = 2\frac{d\alpha_1}{dt} = 2G(\frac{\partial V}{\partial \alpha_1}\|\nabla_{\mathbf{q}}\alpha_1\|^2 + \frac{\partial V}{\partial \alpha_2}\nabla_{\mathbf{q}}\alpha_1^T\nabla_{\mathbf{q}}\alpha_2)$. If we perform a second order Taylor expansion on $\frac{\partial V}{\partial \alpha_1}\|\nabla_{\mathbf{q}}\alpha_1\|^2 + \frac{\partial V}{\partial \alpha_2}\nabla_{\mathbf{q}}\alpha_1^T\nabla_{\mathbf{q}}\alpha_2$ near $x_1 = x_3 = 0$ we obtain $(x_1^2 + x_3^2)/2$. Since G is positive by hypothesis, then near $\mathcal{C}$ $\frac{dD^2}{dt} > 0$. Thus, the set is repulsive. Figure 2 shows the simulation for the initial condition $x_1(0) = x_2(0) = x_3(0) = 0.1$. Figure 3 plots $-V$ versus time, so one can check that in fact $-V \to 0$.

B. Time-Varying Boundaries

A simple example of application of Theorem 1 with time-varying boundaries is to guide a robot so it can stays in a circle of radius R parallel to the plane x_1x_2 with a moving center given by $\tilde{\mathbf{q}}(t) = [\tilde{x}_1(t) \;\; \tilde{x}_2(t) \;\; \tilde{x}_3(t)]^T$ (we suppose that $\tilde{\mathbf{q}}(t)$ is differentiable). In real world applications, this moving center could represent a moving target or intruder that could be, for example, an enemy robot. We assumed above that these two robots are point robots, but in real world applications this assumption may not be valid. So, one could claim that the robots could collide to each other. However, we will show that we can use our approach in such a way that such collisions will be avoided. Assume that the robots can be modeled as spheres with radius R_{robot} and R_{target}, so, we necessarily have $R > R_{robot} + R_{target}$ (one should choose R with an certain safe margin).

We choose, strategically, that

$$\alpha_1(x_1, x_2, x_3, t) = \frac{1}{2}((x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 \quad (9)$$

$$-R^2) , \quad (10)$$

$$\alpha_2(x_1, x_2, x_3, t) = x_3 - \tilde{x}_3. \quad (11)$$

The set $\mathcal{D}(t)$ is clearly a circle parallel to the plane x_1x_2 ($x_3 = 0$) , radius R and center $\tilde{\mathbf{q}}$. Consider in addition that $\frac{\partial V}{\partial \alpha_1} > 0$ for $\alpha_1 < 0$.

Thus, we have that $\nabla_{\mathbf{q}}\alpha_1 = [x_1 - \tilde{x}_1 \;\; x_2 - \tilde{x}_2 \;\; 0]^T$ and $\nabla_{\mathbf{q}}\alpha_2 = [0 \;\; 0 \;\; 1]^T$. Therefore, the gradients are orthogonal. The set $\mathcal{C}(t)$ is given by the line with $x_1 = \tilde{x}_1(t)$ and $x_2 = \tilde{x}_2(t)$ for all x_3. We suppose that the initial condition $\mathbf{q}_0$ is not in the cylinder $(x_1 - \tilde{x}_1(0))^2 + (x_2 - \tilde{x}_2(0))^2 \leq (R_{target} + R_{robot})^2$ (therefore, not in $\mathcal{C}(0)$).

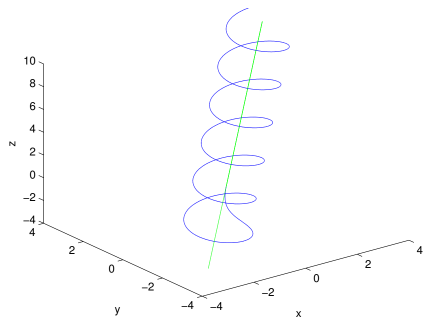

Fig. 4. *Convergence to the target curve with the initial condition $x_1(0) = x_2(0) = x_3(0) = 0$.*

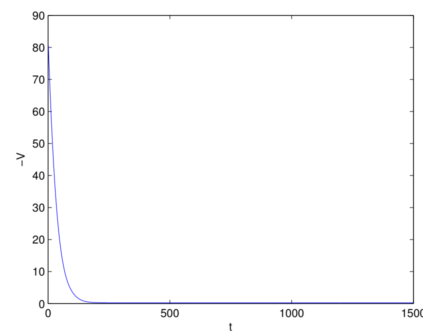

Fig. 5. *Plot of $-V$ versus time for Figure 4: as $t \to \infty$ $-V$ (and so α_1 and α_2) goes to 0.*

To show that $\mathcal{C}(t)$ is a repulsive set, we use again the fact that if $D(x_1, x_2, x_3, t)$ is the Euclidean distance from $\mathbf{q}(t)$ to $\mathcal{C}(t)$, then $D^2 = 2\alpha_1 + R^2$. We proceed developing $\frac{dD^2}{dt} = \frac{d}{dt}\left(2\alpha_1 + R^2\right) = 2\nabla_{\mathbf{q}}\alpha_1^T(\dot{\mathbf{q}} - \dot{\tilde{\mathbf{q}}}) = 2(G\frac{\partial V}{\partial \alpha_1}\|\nabla_{\mathbf{q}}\alpha_1\|^2 - \nabla_{\mathbf{q}}\alpha_1^T M(\alpha)^{-1}\mathbf{a}(\alpha) - \nabla_{\mathbf{q}}\alpha_1^T\dot{\tilde{\mathbf{q}}})$ (using the fact that $\nabla_{\mathbf{q}}\alpha_1$ is orthogonal to $\nabla_{\mathbf{q}}\alpha_2$ and $\nabla_{\mathbf{q}}\alpha_1 \times \nabla_{\mathbf{q}}\alpha_2$).

Now, as $\mathbf{w} = M(\alpha)^{-1}\mathbf{a}(\alpha)$ satisfies the equation $M(\alpha)\mathbf{w} = \mathbf{a}(\alpha)$ for all the points that are not in $\mathcal{C}(t)$ (otherwise the matrix $M(\alpha)^{-1}$ does not exist), we have that $\nabla_{\mathbf{q}}\alpha_1^T\mathbf{w} = \frac{\partial \alpha_1}{\partial t} = -\nabla_{\mathbf{q}}\alpha_1^T\dot{\tilde{\mathbf{q}}}$. So, $\frac{dD^2}{dt} = 2G\frac{\partial V}{\partial \alpha_1}\|\nabla_{\mathbf{q}}\alpha_1\|^2$.

If the point is at a distance smaller than R from $\mathcal{C}(t)$, $\alpha_1 < 0$ and D^2 grows (by the hypothesis on $\frac{\partial V}{\partial \alpha_1}$). Therefore, $\mathcal{C}(t)$ is a repulsive set. Since the initial condition is not in the cylinder described above, the collision of the two robots will never occur. If, for any value of t, the robot approaches very near to the cylinder boundary, an evasive maneuver is necessary.

Figure 4 shows a simulation with $\tilde{x}_1(t) = \tilde{x}_2(t) = \tilde{x}_3(t) = 0.2t - 3$, initial condition $\mathbf{q}_0 = \mathbf{0}$, $V = -\frac{1}{2}(\alpha_1^2 + \alpha_2^2)$, $G = J/\|\nabla_{\mathbf{q}}V\|$, $J = \|\nabla_{\mathbf{q}}V\|^2/(1 + \|\nabla_{\mathbf{q}}V\|^2)$, $H = \sqrt{1 - J^2}/\|\nabla_{\mathbf{q}}\alpha_1 \times \nabla_{\mathbf{q}}\alpha_2\|$ and $R = 1$. Figure 5 plots $-V$ versus time, so one can check that in fact $-V \to 0$.

Our last example is an illustration of Corollary 1: a time-varying boundary without the term $M(\alpha)^{-1}\mathbf{a}(\alpha)$. The target curve is

$$\alpha_1(x_1, x_2, t) = \frac{1}{2}(x_1^2 + x_2^2 - \sqrt{0.2^2 + t}). \quad (12)$$

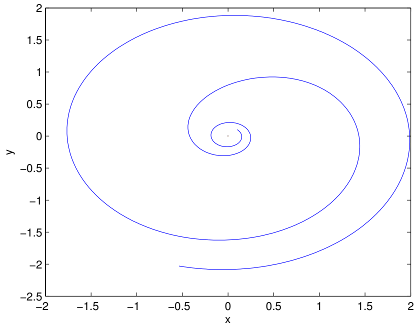

Fig. 6. *Convergence to the target curve with the initial condition $x_1(0) = x_2(0) = 0.1$.*

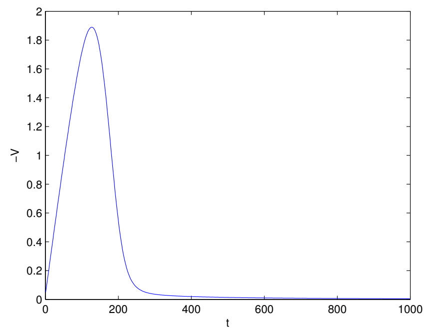

Fig. 7. *Plot of $-V$ versus time for Figure 7: as $t \to \infty$ $-V$ (and so α_1) goes to 0.*

We use $V = -\frac{1}{2}\alpha_1^2$, $G = J/\|\nabla_{\mathbf{q}} V\|$, $J = \|\nabla_{\mathbf{q}} V\|^2/(1+\|\nabla_{\mathbf{q}} V\|^2)$ and $H = \sqrt{1 - J^2}/\| \wedge_{i=1}^{1} \nabla_{\mathbf{q}} \alpha_1\|$. The set $\mathcal{C}(t)$ is the same set of points such that $\nabla_{\mathbf{q}} V = \mathbf{0}$ and not in $\mathcal{D}(t) : x_1 = x_2 = 0$, which can be shown to be repulsive in both systems (with and without the term $M(\alpha)^{-1}\mathbf{a}(\alpha)$) in an analogous way to the others using the time derivative of the square of the Euclidean distance $D(x_1, x_2)^2 = x_1^2 + x_2^2$. Since

$$\lim_{t\to\infty} \|\mathbf{a}\| = \lim_{t\to\infty} \frac{1}{2\sqrt{0.2^2 + t}} = 0 \tag{13}$$

for all $\mathbf{q}$, the Corollary can be used. Figure 6 shows a simulation with $x_1(0) = x_2(0) = 0.1$. Figure 7 plots $-V$ versus time, so one can check that in fact $-V \to 0$, but note that in this case $-V$ is not strictly decreasing.

IV. Conclusions and Future Work

We propose a vector field based controller that generates an attractive limit cycle to the system. Our vector field is defined in a n-dimensional space and may be time-varying. Therefore, we present a solution to the general problem of driving a single kinematically controlled robot to converge to and circulate along a time-varying boundary in a n-dimensional space. The main motivation for our work is the control of unmanned airplanes and we present simulations of curves in three-dimensional spaces. We also present some derivations that allow for robots with constant speed. In the case of static boundaries it is always possible to use constant speed. We also present a condition in which even in the dynamic case we can have constant speed.

Our approach relies on the computation of functions that are equal to zero exactly at the desired shape and also attend some further constraints. An important future direction is an automatic strategy to compute such functions.

Future work includes the use of the proposed approach to control an actual unmanned airplane. This would require the consideration of the airplane dynamics and also trajectories with bounded curvature. It would be also interesting to extend this work to consider obstacles. We would then be able to control robots in generic free configuration spaces. We also intend to extend the proposed approach to devise decentralized controllers for swarms of unmanned airplanes. In this case we need to include inter-agent collision avoidance strategies such as in [7].

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