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State Estimation for Linear Impulsive Systems

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***Abstract*— In this paper, we treat the fundamental problem of state estimation for a class of linear impulsive systems with time-driven impulsive effects. We show that a *strong observability* property enables an impulsive observer to be constructed that yields uniformly exponentially stable estimation error dynamics. This approach accommodates impulsive systems with arbitrarily-spaced impulse times and singular state transition matrices in a manner reminiscent of well-known results for time-varying discrete-time linear systems. As an example, an observer is constructed for an impulsive system that produces general cubic spline signals.**

I. INTRODUCTION

Impulsive systems evolve according to continuous-time dynamics and are also subject to impulsive effects governed by discrete-time dynamics that yield instantaneous changes. These impulsive effects may occur at prescribed time instants and/or be triggered by specified events along a particular trajectory. In this paper, we focus on state estimation for linear impulsive systems with time-driven impulsive effects specified by a set of impulse times. The main contribution is an observer construction for such systems that satisfy a *strong observability* property.

It has been shown by several researchers that state estimation and filtering for continuous-time linear systems with discrete [1], [10], mixed continuous-discrete [7], or multirate [9] measurements leads naturally to observers with an impulsive structure. Moreover, it has been shown that filter/observer gains that yield optimal estimation performance (with respect to various optimality criteria) are derived from differential Riccati equations with jumps. More general measurement schemes considered in [8] yield observers and Riccati equations described by differential equations over distributions. Here we consider systems with impulsive dynamics as well as discrete-time measurements which potentially results in a singular state transition matrix. We also accommodate impulsive systems with arbitrarily-spaced impulse times. These system characteristics significantly complicate the computation of observer gains that yield uniformly exponentially stable error dynamics. Our approach is inspired by the work of Anderson and Moore in [6] for time-varying discrete-time linear systems with singular state transition matrices.

The remainder of the paper is organized as follows. In Section II, the class of linear impulsive systems is specified for which stability and observability properties are characterized that underpin the ensuing analysis. In Section III, an impulsive observer construction is derived that achieves uniformly exponentially stable error dynamics. In Section IV, this scheme is applied to an impulsive system that produces general cubic spline signals. Concluding remarks are offered in Section V.

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II. PRELIMINARIES

We consider linear impulsive systems described by

$$\begin{aligned} \dot{x}(t) &= A_{\mathcal{C}}x(t) + B_{\mathcal{C}}u(t) & t \in \mathbb{R}\setminus\mathcal{T} \\ x(\tau_k) &= A_{\mathcal{I}}x(\tau_k^-) + E_{\mathcal{I}}w[k] & \tau_k \in \mathcal{T} \\ y[k] &= C_{\mathcal{I}}x(\tau_k^-) \end{aligned} \tag{1}$$

where $\mathcal{T}$ is a countably infinite set of strictly increasing impulse times assumed to contain a finite number of elements on any finite time interval, $x(t)$ is the continuous-time state that undergoes instantaneous changes at the impulse times, $u(t)$ is a continuous-time input, $w[k]$ is a discrete-time input, and $y[k]$ is a discrete-time measurement. The state space for (1) is denoted by $\mathcal{X}$.

Given an initial time t_0 and final time $t_f > t_0$, for notational simplicity we denote the subset of impulse times $\mathcal{T}\cap(t_0, t_f)$ by $\{\tau_1, \tau_2, \ldots, \tau_k\}$. We then define $\delta_0 = \tau_1 - t_0$ $\delta_i = \tau_{i+1} - \tau_i$ for $i = 1, \ldots, k-1$, and $\delta_k = t_f - \tau_k$. In terms of this, the state transition matrix of (1) is given by

$$\Phi(t_f, t_0) = e^{A_{\mathcal{C}}\delta_k}A_{\mathcal{I}}e^{A_{\mathcal{C}}\delta_{k-1}}A_{\mathcal{I}}\cdots A_{\mathcal{I}}e^{A_{\mathcal{C}}\delta_1}A_{\mathcal{I}}e^{A_{\mathcal{C}}\delta_0} \tag{2}$$

The state transition matrix is invertible for all $t_f > t_0$ if and only if $A_{\mathcal{I}}$ is invertible. In this case, $\Phi(t_0, t_f)$ can be defined via $\Phi(t_0, t_f) = \Phi^{-1}(t_f, t_0)$. The state response of (1) on $[\tau_k, \tau_{k+1})$ given $x(t_0) = x_0$ can be compactly written as

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t}\Phi(t,\tau)B_{\mathcal{C}}u(\tau)\,d\tau + \sum_{j=1}^{k}\Phi(t,\tau_j)B_{\mathcal{I}}w[j]$$

In the remainder of the paper, we focus on the construction of an impulsive observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A_{\mathcal{C}}\hat{x}(t) + B_{\mathcal{C}}u(t) & t \in \mathbb{R}\setminus\mathcal{T} \\ \hat{x}(\tau_k) &= A_{\mathcal{I}}\hat{x}(\tau_k^-) + E_{\mathcal{I}}w[k] & \\ &\quad + L_{\mathcal{I}}[k]\left(C_{\mathcal{I}}\hat{x}(\tau_k^-) - y[k]\right) & \tau_k \in \mathcal{T} \end{aligned} \tag{3}$$

in which the observer gain $L_{\mathcal{I}}[k]$ is to be determined in order to yield uniformly exponentially stable error dynamics

$$\begin{aligned} \dot{\tilde{x}}(t) &= A_{\mathcal{C}}\tilde{x}(t) & t \in \mathbb{R}\setminus\mathcal{T} \\ \tilde{x}(\tau_k) &= \left(A_{\mathcal{I}} + L_{\mathcal{I}}[k]C_{\mathcal{I}}\right)\tilde{x}(\tau_k^-) & \tau_k \in \mathcal{T} \end{aligned} \tag{4}$$

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A. Stability

We cast our stability discussion in terms of a homogeneous linear impulsive state equation with non-constant coefficient matrices

$$\begin{aligned} \dot{z}(t) &= \hat{A}_{\mathcal{C}}(t)z(t) \qquad t \in \mathbb{R} \setminus \mathcal{T}, \quad z(t_0) = z_0, \\ z(\tau_k) &= \hat{A}_{\mathcal{I}}[k]z(\tau_k^-) \qquad \tau_k \in \mathcal{T} \end{aligned} \tag{5}$$

and, because the state of an impulsive system evolves in continuous time, we adopt the usual notion of uniform exponential stability. In order to characterize uniform exponential stability of (5), we bound the spacing between consecutive impulse times as follows.

Assumption 2.1: For a countably infinite impulse time set $\mathcal{T}$, the spacing between consecutive impulse times $\delta_k = \tau_{k+1} - \tau_k$ satisfies

$$\underline{\delta} := \inf_k \delta_k > 0 \quad \text{and} \quad \overline{\delta} := \sup_k \delta_k < \infty$$

We remark that the lower bound ensures that $\tau_k \to \infty$ as $k \to \infty$. The following theorem provides a Lyapunov-type criterion for uniform exponential stability of linear impulsive systems that is specifically tailored to our needs.

Theorem 2.2: The impulsive system (5) with $\mathcal{T}$ satisfying Assumption 2.1 is uniformly exponentially stable if there exists a symmetric, piecewise-continuously-differentiable matrix function $P(t)$ such that for finite, positive constants η and ρ

$$\eta I \leq P(t) \leq \rho I \tag{6}$$

$$\dot{P}(t) + \hat{A}_{\mathcal{C}}^T(t)P(t) + P(t)\hat{A}_{\mathcal{C}}(t) + Q_{\mathcal{C}}(t) \leq 0, \; t \in \mathbb{R} \setminus \mathcal{T} \tag{7}$$

$$\hat{A}_{\mathcal{I}}^T[k]P(\tau_k)\hat{A}_{\mathcal{I}}[k] - P(\tau_k^-) + Q_{\mathcal{I}}[k] \leq 0, \; \tau_k \in \mathcal{T}, \tag{8}$$

where $Q_{\mathcal{C}}(t)$ and $Q_{\mathcal{I}}[k]$ are bounded, positive semi-definite matrices, and either $Q_{\mathcal{C}}(t)$, $Q_{\mathcal{I}}[k]$, or both, are uniformly positive definite.

This condition is either similar to or a special case of other results in the literature [10], [11], [12] and so a proof is omitted here but is available in [2].

B. Observability

Observability with respect to a continuous-time output is considered in [4]. We adapt the analysis therein to the case of a discrete-time output and define the unobservable set on a finite interval with fixed impulse times as

$$\mathcal{Q}_{\text{fixed}}(t_0, t_f, \mathcal{T}) = \{x_0 \in \mathcal{X} \mid C_{\mathcal{I}}\Phi(\tau_k^-, t_0)x_0 = 0 \\ \text{for all } \tau_k \in \mathcal{T} \cap (t_0, t_f]\}$$

It is clear that $\mathcal{Q}_{\text{fixed}}(t_0, t_f, \mathcal{T})$ is a subspaces of $\mathcal{X}$. The observer construction we pursue makes use of the following observability property along with associated properties of an appropriately defined observability gramian.

Definition 2.3: (Strongly Observable System) An impulsive system (1) is said to be *strongly observable* if there exists a positive integer ℓ such that for all impulse time sets $\mathcal{T}$, $\mathcal{Q}_{\text{fixed}}(t_0, t_f, \mathcal{T}) = 0$ for any finite interval $(t_0, t_f]$ containing at least ℓ impulse times in $\mathcal{T}$.

Definition 2.4: For the impulsive system (1), given integers $k_f > k_0$, $t_0 \in [\tau_{k_0-1}, \tau_{k_0})$, and $t_f \in [\tau_{k_f-1}, \tau_{k_f})$ the observability gramian $M_{\mathcal{O}}(t_0, t_f)$ is defined by

$$M_{\mathcal{O}}(t_0, t_f) = \sum_{j=k_0}^{k_f-1} \Phi^T(\tau_j^-, t_0)\, C_{\mathcal{I}}^T C_{\mathcal{I}}\, \Phi(\tau_j^-, t_0)$$

For zero input and $x(t_0) = x_0$,

$$x_0^T M_{\mathcal{O}}(t_0, t_f)x_0 = \sum_{j=k_0}^{k_f-1} \|y[j]\|^2$$

from which it follows that for a strongly observable system, the observability gramian is positive definite for any impulse time set and any finite interval containing at least ℓ impulse times in that set. Conversely, if there exists an integer ℓ such that the observability gramian is positive definite for any impulse time set and any finite interval containing at least ℓ impulse times in that set, then the system is strongly observable.

Definition 2.5: For the impulsive system (1), given integers $k_f > k_0$, $t_0 \in [\tau_{k_0-1}, \tau_{k_0})$, and $t_f \in [\tau_{k_f-1}, \tau_{k_f})$ the weighted observability gramian $M_\alpha(t_0, t_f)$ is defined by

$$M_\alpha(t_0, t_f) = \sum_{j=k_0}^{k_f-1} \alpha^{2(j-k_0)} \Phi^T(\tau_j^-, t_0)\, C_{\mathcal{I}}^T C_{\mathcal{I}}\, \Phi(\tau_j^-, t_0) \tag{9}$$

for a finite constant $\alpha > 1$.

Similar conclusions can be drawn relating strong observability of (1) and positive definiteness of the weighted observability gramian. Additionally, under our assumption on the impulse time set, the following lemma establishes important uniform positive definiteness and boundedness properties of the observability gramian and its weighted variant.

Lemma 2.6: The weighted observability gramian of a strongly observable linear impulsive system (1) is such that, for any finite $\alpha > 1$, $M_\alpha(t_0, t_f)$ is uniformly positive definite and bounded on any interval $(t_0, t_f]$ containing ℓ impulse times for all impulse time sets $\mathcal{T}$ satisfying Assumption 2.1.

Proof. We first show that $M_\alpha(\tau_{k-\ell}^-, \tau_{k-1})$ is uniformly positive definite and bounded with respect to $k \in \mathbb{Z}$ for any impulse time set $\mathcal{T}$ satisfying Assumption 2.1. In terms of the indeterminates $\delta_1, \delta_2, \ldots, \delta_{\ell-1}$, we define

$$\overline{\Phi}_j(\delta_1, \ldots, \delta_j) = e^{A_{\mathcal{C}}\delta_j} A_{\mathcal{I}} \cdots e^{A_{\mathcal{C}}\delta_1} A_{\mathcal{I}}$$

for $j = 1, \ldots, \ell - 1$ and

$$\overline{M}_\alpha(\delta_1, \ldots, \delta_{\ell-1}) = C_{\mathcal{I}}^T C_{\mathcal{I}} \\ + \sum_{j=1}^{\ell-1} \alpha^{2j} \overline{\Phi}_j^T(\delta_1, \ldots, \delta_j)\, C_{\mathcal{I}}^T C_{\mathcal{I}}\, \overline{\Phi}_j(\delta_1, \ldots, \delta_j) \tag{10}$$

on the compact domain $\mathcal{D} := [\underline{\delta}, \overline{\delta}] \times [\underline{\delta}, \overline{\delta}] \times \cdots \times [\underline{\delta}, \overline{\delta}] \subset \mathbb{R}^{\ell-1}$ from which

$$M_\alpha(\tau_{k-\ell}^-, \tau_{k-1}) = \overline{M}_\alpha(\delta_{k-\ell}, \ldots, \delta_{k-2})$$

The strong observability hypothesis implies that $\overline{M}_\alpha(\delta_1, \ldots, \delta_{\ell-1})$ is positive definite and bounded at each point in $\mathcal{D}$. Since the elements of $\overline{M}_\alpha(\delta_1,\ldots,\delta_{\ell-1})$, and therefore its pointwise-defined eigenvalues, are continuous functions on $\mathcal{D}$, it follows that the maximum eigenvalue of $\overline{M}_\alpha(\delta_1,\ldots,\delta_{\ell-1})$ attains its finite maximum value and the minimum eigenvalue of $\overline{M}_\alpha(\delta_1,\ldots,\delta_{\ell-1})$ attains its positive minimum value on $\mathcal{D}$. We refer to these extrema as λ_{max} and λ_{min}, respectively, and conclude that

$$\lambda_{\text{min}} I \leq \overline{M}_\alpha(\delta_1,\ldots,\delta_\ell) \leq \lambda_{\text{max}} I$$

on $\mathcal{D}$ and so these bounds hold for $M_\alpha(\tau_{k-\ell}^-,\tau_k)$ for all $k\in\mathbb{Z}$ and for all impulse time sets $\mathcal{T}$ satisfying Assumption 2.1.

Now, for any impulse time set $\mathcal{T}$ satisfying Assumption 2.1 and any interval $(t_0,t_f]$ containing ℓ impulse times denoted $\tau_{k-\ell},\ldots,\tau_{k-1}$ we can write

$$\begin{aligned} M_\alpha(t_0,t_f) &= M_\alpha(t_0,\tau_{k-1}) \\ &= e^{A_{\mathcal{C}}^T(\tau_{k-\ell}-t_0)} M_\alpha(\tau_{k-\ell}^-,\tau_{k-1}) e^{A_{\mathcal{C}}(\tau_{k-\ell}-t_0)} \end{aligned}$$

From this we conclude that

$$e^{-2\|A_{\mathcal{C}}\|\overline{\delta}}\lambda_{\text{min}} I \leq M_\alpha(t_0,t_f) \leq e^{2\|A_{\mathcal{C}}\|\overline{\delta}}\lambda_{\text{max}} I \tag{11}$$

which completes the proof.

III. Error Dynamics

We show that strong observability of the linear impulsive system (1) is sufficient for the existence of an impulsive observer (3) yielding uniformly exponentially stable error dynamics (4). Specifically, we use the weighted observability gramian (9) to construct an observer gain and to demonstrate uniform exponential stability via Theorem 2.2. We now formally state the main result of the paper.

Theorem 3.1: For a strongly observable linear impulsive system (1) with impulse time set $\mathcal{T}$ satisfying Assumption 2.1 and positive integer ℓ as in Definition 2.3, the observer gain defined by

$$\begin{aligned} &L_{\mathcal{I}}[k] = \\ &-\alpha^{2(\ell-1)}\Phi(\tau_k,\tau_{k-\ell})M_\alpha^{-1}(\tau_{k-\ell},\tau_k)\Phi^T(\tau_k^-,\tau_{k-\ell})C_{\mathcal{I}}^T \end{aligned} \tag{12}$$

specifies a linear impulsive observer (3) for which the associated impulsive error dynamics (4) are uniformly exponentially stable.

The remainder of this section is devoted to a proof of Theorem 3.1. We begin by writing the error dynamics (4) with observer gain (12) as

$$\begin{aligned} \dot{\tilde{x}}(t) &= A_{\mathcal{C}}\tilde{x}(t) & t\in\mathbb{R}\setminus\mathcal{T} \\ \tilde{x}(\tau_k) &= \tilde{A}_{\mathcal{I}}[k]\tilde{x}(\tau_k^-) & \tau_k\in\mathcal{T} \end{aligned} \tag{13}$$

in which, with $\Gamma[k] = \alpha^{(\ell-1)}C_{\mathcal{I}}\,\Phi(\tau_k^-,\tau_{k-\ell})$,

$$\begin{aligned} \tilde{A}_{\mathcal{I}}[k] &= A_{\mathcal{I}} + L_{\mathcal{I}}[k]C_{\mathcal{I}} \\ &= A_{\mathcal{I}} - \alpha^{(\ell-1)}\Phi(\tau_k,\tau_{k-\ell})M_\alpha^{-1}(\tau_{k-\ell},\tau_k)\Gamma^T[k]C_{\mathcal{I}} \end{aligned}$$

Uniform exponential stability of (13) will be established by proving uniform exponential stability of a related state equation defined as follows. First, we define, for a given impulse time set $\mathcal{T}$ and positive integer ℓ, the map

$$\rho_\ell\,:\,[\tau_{k-1},\tau_k)\longrightarrow[\tau_{k-\ell-1},\tau_{k-\ell}) \qquad k\in\mathbb{Z}$$

via

$$\rho_\ell(t) = \tau_{k-\ell-1} + \frac{\delta_{k-\ell-1}}{\delta_{k-1}}(t-\tau_{k-1})$$

The mapping $\rho_\ell(t)$ defines a past time instant that represents the same proportion of elapsed time with respect to the interval $[\tau_{k-\ell-1},\tau_{k-\ell})$ that t does with respect to the interval $[\tau_{k-1},\tau_k)$. That is,

$$\frac{\rho_\ell(t)-\tau_{k-\ell-1}}{\delta_{k-\ell-1}} = \frac{t-\tau_{k-1}}{\delta_{k-1}}$$

Note that this mapping satisfies $\rho_{i+j}(t) = \rho_i(\rho_j(t)) = \rho_j(\rho_i(t))$ for all $i,j\in\mathbb{Z}$. We then define the impulsive state equation

$$\begin{aligned} \dot{z}(t) &= \hat{A}_{\mathcal{C}}(t)z(t) & t\in\mathbb{R}\setminus\mathcal{T} \\ z(\tau_k) &= \hat{A}_{\mathcal{I}}[k]z(\tau_k^-) & \tau_k\in\mathcal{T} \end{aligned} \tag{14}$$

where

$$\hat{A}_{\mathcal{C}}(t) = \frac{\delta_{k-\ell-1}}{\delta_{k-1}}A_{\mathcal{C}} \qquad t\in(\tau_{k-1},\tau_k) \tag{15}$$

$$\hat{A}_{\mathcal{I}}[k] = A_{\mathcal{I}} - \alpha^{(\ell-1)}M_\alpha^{-1}(\tau_{k-\ell},\tau_k)\Gamma^T[k]C_{\mathcal{I}}\Phi(\tau_k^-,\tau_{k-\ell}^-)$$

It is not difficult to show that (13) and (14) are related via the open-loop state transition matrix as follows. If $\tilde{x}(t_0) = \Phi(t_0,\rho_\ell(t_0))z(t_0)$ then $\tilde{x}(t) = \Phi(t,\rho_\ell(t))z(t)$ for all $t\geq t_0$. Although this relationship is in general not invertible, it is still possible to establish the following connection between (13) and (14) with respect to uniform exponential stability.

Lemma 3.2: For an impulse time set $\mathcal{T}$ satisfying Assumption 2.1, the state equation (13) is uniformly exponentially stable if and only if the state equation (14) is uniformly exponentially stable.

This lemma is similar in spirit to Lemma 4.2 appearing in [6] for time-varying discrete-time linear systems with singular transition matrices, and a similar result used to address feedback stabilization of linear impulsive systems is available in [2], so a proof is omitted here. It remains to be shown that the impulsive system (14) is uniformly exponentially stable, which we accomplish by showing that the conditions of Theorem 2.2 hold for

$$P(t) = M_\alpha(\rho_\ell(t),t)$$

Assuming that (1) is strongly observable, it follows from Lemma 2.6 that the bounds in (11) also apply to $P(t)$ so the first condition in Theorem 2.2 is satisfied.

Next, we have for $t\in(\tau_{k-1},\tau_k)$

$$\begin{aligned} \dot{P}(t) &= -A_{\mathcal{C}}^T P(t)\dot{\rho}_\ell(t) - P(t)A_{\mathcal{C}}\dot{\rho}_\ell(t) \\ &= -\left(\frac{\delta_{k-\ell-1}}{\delta_{k-1}}A_{\mathcal{C}}^T\right)P(t) - P(t)\left(\frac{\delta_{k-\ell-1}}{\delta_{k-1}}A_{\mathcal{C}}\right) \\ &= -\hat{A}_{\mathcal{C}}^T(t)P(t) - P(t)\hat{A}_{\mathcal{C}}(t) \end{aligned}$$

from which the second condition in Theorem 2.2 is satisfied for $Q_{\mathcal{C}}(t)\equiv 0$.

For the final step, using

$$P(\tau_k) = M_\alpha(\tau_{k-\ell}, \tau_k) = \sum_{j=k-\ell+1}^{k} \alpha^{2(j-k+\ell-1)} \Phi^T(\tau_j^-, \tau_{k-\ell}) C_{\mathcal{I}}^T C_{\mathcal{I}} \Phi(\tau_j^-, \tau_{k-\ell})$$

along with

$$P(\tau_k^-) = M_\alpha(\tau_{k-\ell}^-, \tau_k^-) = \sum_{j=k-\ell}^{k} \alpha^{2(j-k+\ell)} \Phi^T(\tau_j^-, \tau_{k-\ell}^-)\, C_{\mathcal{I}}^T C_{\mathcal{I}}\, \Phi(\tau_j^-, \tau_{k-\ell}^-)$$

we compute

$$A_{\mathcal{I}}^T P(\tau_k) A_{\mathcal{I}} = \alpha^{-2} P(\tau_k^-) - \alpha^{-2} C_{\mathcal{I}}^T C_{\mathcal{I}} + \alpha^{2(\ell-1)} \Phi^T(\tau_k^-, \tau_{k-\ell}^-) C_{\mathcal{I}}^T C_{\mathcal{I}} \Phi(\tau_k^-, \tau_{k-\ell}^-)$$

This yields

$$\hat{A}_{\mathcal{I}}^T[k] P(\tau_k) \hat{A}_{\mathcal{I}}[k] = \alpha^{-2} P(\tau_k^-) - \alpha^{-2} C_{\mathcal{I}}^T C_{\mathcal{I}} - A_{\mathcal{I}}^T \Gamma^T[k] \Big(I - \Gamma[k] M_\alpha^{-1}(\tau_{k-\ell}, \tau_k) \Gamma^T[k] \Big) \Gamma[k] A_{\mathcal{I}}$$

From well-known results on Schur complements, positive semi-definiteness of

$$I - \Gamma[k] M_\alpha^{-1}(\tau_{k-\ell}, \tau_k) \Gamma^T[k]$$

is equivalent to positive semi-definiteness of

$$M_\alpha(\tau_{k-\ell}, \tau_k) - \Gamma^T[k]\Gamma[k] = \sum_{j=k-\ell+1}^{k-1} \alpha^{2(j-k+\ell-1)} \Phi^T(\tau_j^-, \tau_{k-\ell}) C_{\mathcal{I}}^T C_{\mathcal{I}} \Phi(\tau_j^-, \tau_{k-\ell}) \geq 0$$

This allows us to conclude that

$$A_{\mathcal{I}}^T P(\tau_k) A_{\mathcal{I}} - P(\tau_k^-) + (1 - \alpha^{-2}) P(\tau_k^-) \leq 0$$

Since $\alpha > 1$, the third condition of Theorem 2.2 is satisfied for bounded, uniformly positive definite

$$Q_{\mathcal{I}}[k] = (1 - \alpha^{-2}) P(\tau_k^-)$$

We therefore conclude that the state equation (14) is uniformly exponentially stable, implying, by Lemma 2.6, the same for the impulsive error dynamics (13). This proves our main result.

IV. Example

We consider the linear impulsive state equation (1) specified by

$$A_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$A_{\mathcal{I}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad E_{\mathcal{I}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C_{\mathcal{I}} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

This system is capable of generating twice-continuously-differentiable cubic splines by properly selecting the initial conditions, the impulse times, and the discrete-time input signal.

Given an impulse time set $\mathcal{T}$ and initial time $t_0 \in [\tau_{k_0-1}, \tau_{k_0})$, the initial state

$$x(t_0) = \begin{bmatrix} -\frac{1}{6}(\tau_{k_0} - t_0)^3 & \frac{1}{2}(\tau_{k_0} - t_0)^2 & -(\tau_{k_0} - t_0) & 1 \end{bmatrix}^T$$

yields a zero-input response for which

$$x(\tau_{k_0}^-) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

This gives $y[k_0] = 0$ along with $x(\tau_{k_0}) = 0 \in \mathbb{R}^4$. The latter yields $x(t) \equiv 0$ for all $t \geq \tau_{k_0}$ from which $y[k] = 0$ for all $k \geq k_0$. It follows that for any $t_f \geq \tau_{k_0}$, $\mathcal{Q}_{\text{fixed}}(t_0, t_f, \mathcal{T})$ is a nonzero subspace and, consequently, this impulsive system is not strongly observable. However, we will show that this system possesses a detectability property in that it is still possible to construct an observer gain using the techniques presented in this paper for which the associated error dynamics are uniformly exponentially stable.

To proceed, we consider observer gain vectors whose last component is zero

$$L_{\mathcal{I}}[k] = \left[\begin{array}{c} L_{\mathcal{I}}^{(1)}[k] \\ \hline 0 \end{array} \right]$$

and partition the state estimation error accordingly

$$\tilde{x}(t) = \left[\begin{array}{c} \tilde{x}^{(1)}(t) \\ \hline \tilde{x}_4(t) \end{array} \right]$$

along with the state equation coefficients

$$A_{\mathcal{C}} = \left[\begin{array}{c|c} A_{\mathcal{C}}^{(1,1)} & A_{\mathcal{C}}^{(1,2)} \\ \hline 0 & 0 \end{array} \right]$$
$$A_{\mathcal{I}} = \left[\begin{array}{c|c} A_{\mathcal{I}}^{(1,1)} & 0 \\ \hline 0 & 0 \end{array} \right]$$
$$C_{\mathcal{I}} = \left[\begin{array}{c|c} C_{\mathcal{I}}^{(1)} & 0 \end{array} \right]$$

This, in turn, leads to the following error dynamics with block triangular coupling

$$\left[\begin{array}{c} \dot{\tilde{x}}^{(1)}(t) \\ \hline \dot{\tilde{x}}_4(t) \end{array} \right] = \left[\begin{array}{c|c} A_{\mathcal{C}}^{(1,1)} & A_{\mathcal{C}}^{(1,2)} \\ \hline 0 & 0 \end{array} \right] \left[\begin{array}{c} \tilde{x}^{(1)}(t) \\ \hline \tilde{x}_4(t) \end{array} \right] \quad t \in \mathbb{R} \setminus \mathcal{T}$$
$$\left[\begin{array}{c} \tilde{x}^{(1)}(\tau_k) \\ \hline \tilde{x}_4(\tau_k) \end{array} \right] = \left[\begin{array}{c|c} A_{\mathcal{I}}^{(1,1)} + L_{\mathcal{I}}^{(1)}[k] C_{\mathcal{I}}^{(1)} & 0 \\ \hline 0 & 0 \end{array} \right] \left[\begin{array}{c} \tilde{x}^{(1)}(\tau_k^-) \\ \hline \tilde{x}_4(\tau_k^-) \end{array} \right]$$

It is clear that, again with $t_0 \in [\tau_{k_0-1}, \tau_{k_0})$, the fourth state variable exhibits the dead-beat response $\tilde{x}_4(t) = \tilde{x}_4(t_0)$ for $t \in [t_0, \tau_{k_0})$ and $\tilde{x}_4(t) = 0$ for all $t \geq \tau_{k_0}$.

We next focus our attention on the construction of a $3-$dimensional observer gain vector $L_{\mathcal{I}}^{(1)}[k]$ for which

$$\dot{\tilde{x}}^{(1)}(t) = A_{\mathcal{C}}^{(1,1)} \tilde{x}^{(1)}(t) \qquad t \in \mathbb{R} \setminus \mathcal{T}$$
$$\tilde{x}^{(1)}(\tau_k) = \Big(A_{\mathcal{I}}^{(1,1)} + L_{\mathcal{I}}^{(1)}[k] C_{\mathcal{I}}^{(1)} \Big) \tilde{x}^{(1)}(\tau_k^-) \quad \tau_k \in \mathcal{T}$$

is uniformly exponentially stable. For then, since $\tilde{x}(t)$ remains bounded on $[t_0, \tau_{k_0})$ for any initial error $\tilde{x}(t_0)$, uniform exponential stability of the entire error dynamics follows.

To demonstrate strong observability of the $\tilde{x}^{(1)}-$subsystem, we take $\ell = 3$ and for notational simplicity write $\mathcal{T} \cap (t_0, t_f] = \{\tau_1, \tau_2, \tau_3\}$. A direct computation facilitated by $A_{\mathcal{I}}^{(1,1)} = I$ gives

$$\mathcal{Q}^{(1)}_{\text{fixed}}(t_0, t_f, \mathcal{T}) = \text{Ker}\left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & (\tau_2 - \tau_1) & \frac{1}{2}(\tau_2 - \tau_1)^2 \\ 1 & (\tau_3 - \tau_1) & \frac{1}{2}(\tau_3 - \tau_1)^2 \end{pmatrix} e^{A_C^{(1,1)}(\tau_1 - t_0)}\right]$$

The left factor has nonzero determinant $\frac{1}{2}(\tau_3 - \tau_1)(\tau_2 - \tau_1)(\tau_3 - \tau_2) \geq \frac{1}{2}\delta^3$ indicating that $\mathcal{Q}^{(1)}_{\text{fixed}}(t_0, t_f, \mathcal{T}) = 0$ for any impulse time set $\mathcal{T}$ and any interval $(t_0, t_f]$ containing $\ell = 3$ impulse times. Hence the $\tilde{x}^{(1)}-$subsystem is strongly observable.

A direct application of (12) to the $\tilde{x}^{(1)}-$subsystem yields

$$L^{(1)}_{\mathcal{I}}[k] = -\left[\begin{array}{c} 1 \\ \frac{(\tau_k - \tau_{k-1}) + (\tau_k - \tau_{k-2})}{(\tau_k - \tau_{k-1})(\tau_k - \tau_{k-2})} \\ \frac{2}{(\tau_k - \tau_{k-1})(\tau_k - \tau_{k-2})} \end{array}\right]$$

which happens to be independent of α and is applicable for any impulse time set.

We simulate the impulsive system given by (16) and the associated impulsive observer for the spline shown in Fig. 1 that approximates a trapezoidal signal, also shown. The impulse times are taken to be the interior knots:

$$\begin{aligned}\mathcal{T} = \{\tau_k,\, k = 1, \ldots, 13\} = \{&0.1,\, 0.5,\, 0.9,\, 1.0,\, 1.1,\, 1.2, \\ &2.0,\, 2.8,\, 2.9,\, 3.0,\, 3.1,\, 3.5,\, 3.9\}\end{aligned}$$

The discrete-time input signal $w[k]$ is derived from the spline as follows. For the cubic polynomial segment on $[\tau_k, \tau_{k+1})$

$$p_k(t) = a_k(t - \tau_k)^3 + b_k(t - \tau_k)^2 + c_k(t - \tau_k) + d_k$$

we take $w[k] = 6a_k$ so that $x_4(\tau_k) = w[k] = \dddot{p}_k(\tau_k)$. With the aid of MATLAB, this produces

$$\begin{pmatrix} w[1] \\ w[2] \\ w[3] \\ w[4] \\ w[5] \\ w[6] \\ w[7] \\ w[8] \\ w[9] \\ w[10] \\ w[11] \\ w[12] \\ w[13] \end{pmatrix} = \begin{pmatrix} -58.0338 \\ 34.3681 \\ -400.6792 \\ 423.1681 \\ -91.9931 \\ 1.1499 \\ -1.1499 \\ 91.9931 \\ -423.1681 \\ 400.6792 \\ -34.3681 \\ 58.0338 \\ 58.0338 \end{pmatrix}$$

Finally, in order for the impulsive system specified by (16) to generate the spline in Fig. 1, the initial state must be set to

$$x(0) = \left[\begin{array}{cccc} 0 & -0.9836 & 21.6068 & -58.0338 \end{array}\right]^T$$

The impulsive observer is initialized with

$$\hat{x}(0) = \left[\begin{array}{cccc} 0 & 0 & 0 & 100 \end{array}\right]^T$$

where the value for $\hat{x}_4(0)$ was chosen so as to highlight the dead-beat response described above.

The state variable responses for the impulsive system and impulsive observer are shown in Figs. 2–5. We observe that not only does $\tilde{x}_4(t)$ exhibit a dead-beat response (after $\tau_1 = 0.1$s, as expected), so do $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, and $\tilde{x}_3(t)$ after $\tau_3 = 0.9$s.

V. CONCLUDING REMARKS

This paper has presented the construction of an impulsive observer for a class of linear impulsive systems featuring arbitrarily-spaced impulse times and possibly singular state transition matrices. As an illustration of the main ideas, an observer has been constructed for an impulsive system that produces general cubic spline signals for which the resulting observer gain is given explicitly in terms of the impulse times. Future work shall focus on output feedback stabilization using impulsive compensators featuring the impulsive observers derived herein together with the stabilizing state feedback laws developed in [3]. It is expected that these results for state feedback and state estimation shall also permit a more in depth treatment of the compensator synthesis framework initiated in [5].

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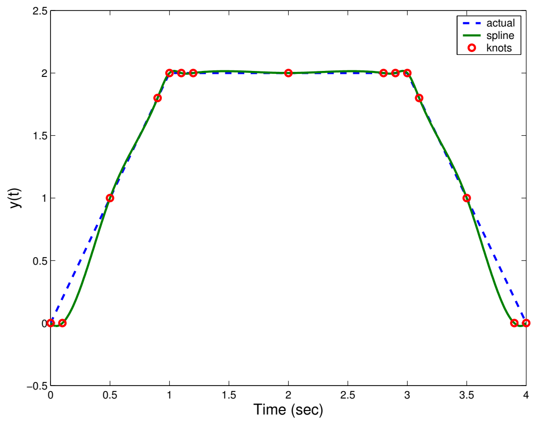

Fig. 1. Trapezoidal signal, spline approximation, and knots.

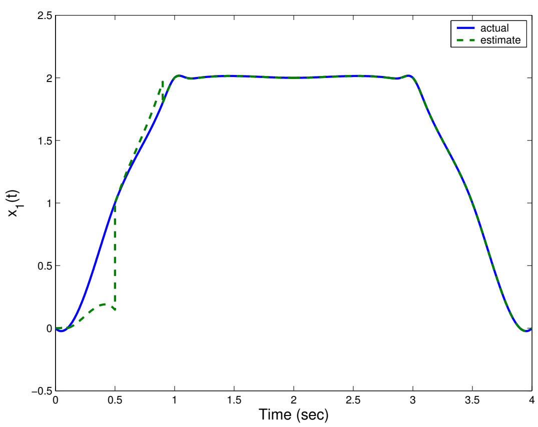

Fig. 2. $x_1(t)$ (solid) and $\hat{x}_1(t)$ (dashed) vs. t.

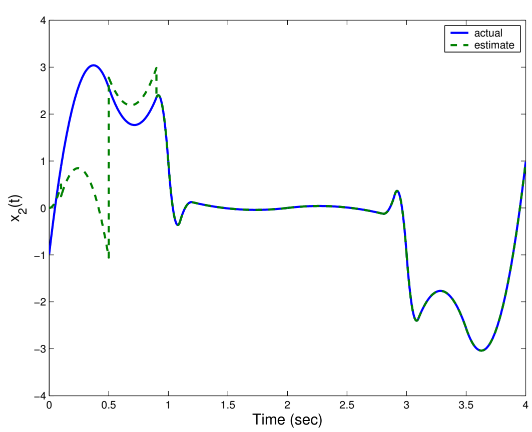

Fig. 3. $x_2(t)$ (solid) and $\hat{x}_2(t)$ (dashed) vs. t.

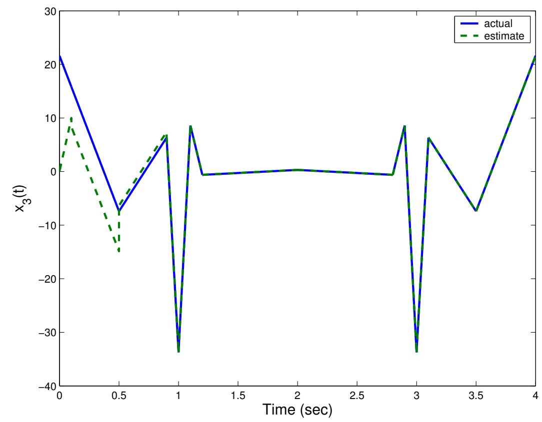

Fig. 4. $x_3(t)$ (solid) and $\hat{x}_3(t)$ (dashed) vs. t.

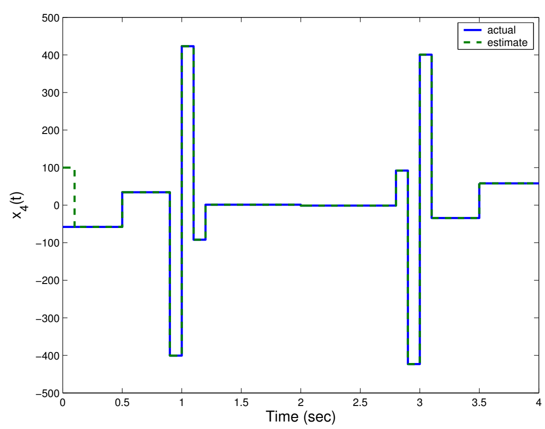

Fig. 5. $x_4(t)$ (solid) and $\hat{x}_4(t)$ (dashed) vs. t.