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Stability and l^2 Gain Analysis for the Particle Swarm Optimization Algorithm

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Abstract— **Stability of the particle swarm optimization algorithm is analyzed without any simplifying assumptions made in the previous works. To evaluate the convergence speed of the algorithm, the decay rate is introduced, and a method for finding the largest lower bound of the decay rate is presented. Moreover, it is pointed out that the l^2 gain of the algorithm can be used to measure exploration ability of the algorithm, and a method for finding of the smallest upper bound of the l^2 gain is provided. The above methods are based on linear matrix inequality techniques and therefore are carried out efficiently by using convex optimization tools. Numerical examples are given to show that the analysis methods are reasonable and effective to select the parameters in the algorithm.**

I. INTRODUCTION

Particle swarm optimization (PSO) is a kind of stochastic optimization, which is based on swarm intelligence such as bird flocking and fish schooling [4]. Recently, the PSO has been applied to various nonconvex optimization problems [5], [8] because it is not only effective to nondifferentiable problems but also easy to implement.

The PSO algorithm is described as a simple dynamical system with stochastic variables, and therefore, we can analyze the behavior of the PSO algorithm by control theoretic approaches [6], [3], [2]. Trelea showed a convergence condition of the PSO algorithm when the stochastic variables are assumed to be constant, which leads to the limitations of the results [6]. Kadirkamanathan et al. showed a more precise convergence condition based on a model with the stochastic variables [3]. However, additive time-varying noise involved in the state variable of the model is assumed to be time-invariant, which is still different from the exact behavior of the PSO algorithm.

To resolve the above problems, we propose a method for the stability analysis of the PSO algorithm without any simplifying assumptions on the stochastic variables. At first, we model the PSO dynamics as a system with multiplicative noise. Then we provide a stability condition by applying a linear matrix inequality (LMI) technique [1]. Also, to evaluate the convergence speed of the algorithm, we introduce the decay rate of the PSO dynamics, and present a method for finding the largest lower bound of the decay rate. Moreover, we point out that the l^2 gain of the algorithm can be used to measure exploration ability of the algorithm, and provide a method for finding of the smallest upper bound of the l^2 gain. The proposed methods are based on LMI techniques, and therefore, the computation for the analysis can be carried out efficiently by convex optimization tools.

This paper is organized as follows. In Section II, we briefly describe the results of stability and l^2 gain analysis for systems with multiplicative noise via LMI techniques. Next we address the basic PSO algorithm, and show the two previous works on the stability analysis of the PSO algorithm in Section III. In Sections IV and V, we present the main results on stability and l^2 gain analysis, respectively. Then, we provide numerical examples to show the effectiveness of the presented analysis methods in Section VI. Finally, Section VII concludes the paper.

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II. PRELIMINARIES

In this section, we briefly provide the results on stability and l^2 gain analysis of systems with multiplicative noise [1]. The results are applied to analysis of the PSO algorithm in Sections IV and V.

A. Stability analysis of systems with multiplicative noise

We consider the discrete-time stochastic system

$$\xi_{k+1} = \left(A + \sum_{i=1}^{L} A_i \theta_{i,k} \right) \xi_k, \tag{1}$$

where $\xi_k \in \Re^{n_\xi}$ is the state, $A, A_1, \ldots, A_L \in \Re^{n_\xi \times n_\xi}$ are the coefficient matrices, and $\theta_k := [\theta_{1,k}, \ldots, \theta_{L,k}]^T$ is a random variable. Especially, denoting the expectation by $\mathbf{E}$, we assume that $\theta_0, \theta_1, \ldots$ are independent identically distributed random variables with

$$\begin{aligned} \mathbf{E}\,\theta_k &= 0 \\ \mathbf{E}\,\theta_k \theta_k^T &= \mathbf{diag}(\sigma_1^2, \ldots, \sigma_L^2), \end{aligned}$$

where, for $i = 1, \ldots, L$, σ_i^2 denotes the variance of $\theta_{i,k}$. We also assume that ξ_0 is independent of the process θ_k.

For the state ξ_k, we define the state correlation matrix as

$$M_k := \mathbf{E}\,\xi_k \xi_k^T.$$

From (1), we see that M_k satisfies the linear recursion

$$\begin{aligned} M_{k+1} &= A M_k A^T + \sum_{i=1}^{L} \sigma_i^2 A_i M_k A_i^T \qquad (2) \\ M_0 &= \mathbf{E}\,\xi_0 \xi_0^T. \end{aligned}$$

If this linear recursion is stable, i.e., regardless of ξ_0, $\lim_{k\to\infty} M_k = 0$, we say the system is *mean-square stable*. Mean-square stability implies, for example, that $\xi_k \to 0$ almost surely. The following theorem holds for system (1) [1].

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Theorem 1: System (1) is mean-square stable if and only if there exists a matrix $P > 0$ satisfying the LMI

$$A^T PA - P + \sum_{i=1}^{L} \sigma_i^2 A_i^T PA_i < 0.$$

Theorem 1 is derived by applying the Lyapunov stability theorem to system (2) as shown in [1].

B. l^2 gain analysis of systems with multiplicative noise

Next we show the result of the l^2 gain analysis. We consider the following system with $\xi_0 = 0$ and with the same assumptions on θ_k as in the previous subsection.

$$\begin{aligned} \xi_{k+1} &= A\xi_k + Bw_k + \sum_{i=1}^{L}(A_i\xi_k + B_iw_k)\theta_{i,k} \\ z_k &= C\xi_k + Dw_k + \sum_{i=1}^{L}(C_i\xi_k + D_iw_k)\theta_{i,k}, \end{aligned} \tag{3}$$

where $w_k \in \Re^{n_w}$ is the exogenous input, $z_k \in \Re^{n_z}$ is the output, and $B, B_1, \dots, B_L \in \Re^{n_\xi \times n_w}$, $C, C_1, \dots, C_L \in \Re^{n_z \times n_\xi}$, $D, D_1, \dots, D_L \in \Re^{n_z \times n_w}$ are the coefficient matrices. We assume that w_k is deterministic.

We define the l^2 gain η of this system as

$$\eta^2 := \sup \left\{ \mathbf{E} \sum_{k=0}^{\infty} z_k^T z_k \,\middle|\, \sum_{k=0}^{\infty} w_k^T w_k \le 1 \right\}.$$

Suppose that $V(\xi) = \xi^T P\xi$, with $P > 0$, satisfies

$$\mathbf{E}\, V(\xi_{k+1}) - \mathbf{E}\, V(\xi_k) \le \gamma^2 w_k^T w_k - \mathbf{E}\, z_k^T z_k. \tag{4}$$

Then $\gamma \ge \eta$. The condition (4) can be shown to be equivalent to the LMI

$$\begin{aligned} &\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix} \\ &+ \sum_{i=1}^{L} \sigma_i^2 \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \le 0. \end{aligned} \tag{5}$$

Therefore, we can obtain the following theorem shown in [1].

Theorem 2: Minimizing γ^2 subject to $P > 0$ and LMI (5) yields an upper bound on the l^2 gain η of system (3).

III. PARTICLE SWARM OPTIMIZATION AND THE PREVIOUS RESULTS

In the PSO algorithm, each particle position is a potential solution to an optimization problem in n-dimensional space, and its previous best position and the best position among all particles are stored. Since each dimension of a particle position vector is updated independently of the others, the analysis of particle behavior can be carried out on one dimension without loss of generality as in [6], [3]. The basic PSO algorithm in one dimension is given by

$$\begin{aligned} v_{k+1} &= \alpha v_k + \beta_k^{(p)}(x_k^{(p)} - x_k) + \beta_k^{(g)}(x_k^{(g)} - x_k) \\ x_{k+1} &= x_k + v_{k+1}, \end{aligned} \tag{6}$$

where v_k is the particle velocity at the kth iteration, x_k is the particle position at the kth iteration, $x_k^{(p)}$ is the personal best position, i.e., the best position of an individual particle achieved up to the kth iteration, $x_k^{(g)}$ is the global best position among all particles, α is the inertia factor, and $\beta_k^{(p)} \sim \mathcal{U}[0, c^{(p)}]$ and $\beta_k^{(g)} \sim \mathcal{U}[0, c^{(g)}]$ are random parameters according to uniform distributions with constants $c^{(p)}$ and $c^{(g)}$ known as acceleration coefficients.

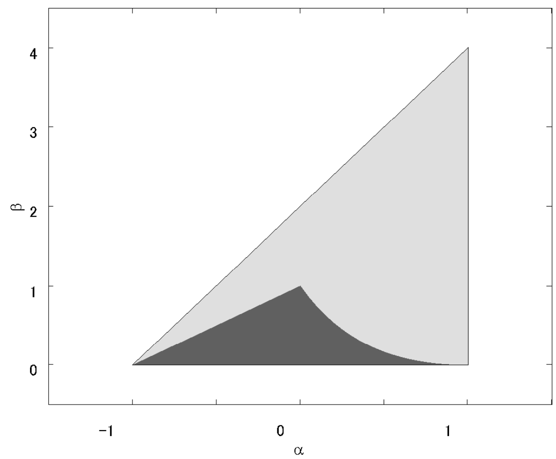

Fig. 1. Stability regions by the conventional stability analysis; union of the light and deep gray regions: stability region by Trelea [6]; deep gray region: stability region by Kadirkamanathan et al. [3].

To simplify the PSO dynamics (6), we use the following notation:

$$\begin{aligned} \beta_k &:= \beta_k^{(p)} + \beta_k^{(g)} \\ q_k &:= \frac{\beta_k^{(p)}}{\beta_k^{(p)} + \beta_k^{(g)}} x^{(p)} + \frac{\beta_k^{(g)}}{\beta_k^{(p)} + \beta_k^{(g)}} x^{(g)}. \end{aligned}$$

Then, (6) can be represented as the system in state-space form

$$\begin{bmatrix} x_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} 1 - \beta_k & \alpha \\ -\beta_k & \alpha \end{bmatrix} \begin{bmatrix} x_k \\ v_k \end{bmatrix} + \begin{bmatrix} \beta_k \\ \beta_k \end{bmatrix} q_k. \tag{7}$$

In the previous stability analysis [6], β_k is assumed to be constant as shown by $\beta_k = \beta, \forall k$. Namely, $\beta = (c^{(p)} + c^{(g)})/2$ is assumed by setting the expected values of $\beta_k^{(p)}$ and $\beta_k^{(g)}$. System (7) is a simple linear time-invariant second-order dynamic model. Therefore, the convergence condition derived in [6] in our notation is given by

$$\begin{aligned} &\alpha < 1 \\ &\beta > 0 \\ &2\alpha - \beta + 2 > 0. \end{aligned} \tag{8}$$

The parameter region for (8) is triangular, and is shown with the union of the light and deep gray regions in Fig. 1.

In another previous stability analysis [3], the PSO algorithm (6) is represented by

$$\begin{aligned} \begin{bmatrix} x_{k+1} \\ v_{k+1} \end{bmatrix} &= \begin{bmatrix} 1 & \alpha \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x_k \\ v_k \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k \\ y_k &= [1 \quad 0] \begin{bmatrix} x_k \\ v_k \end{bmatrix} \\ u_k &= -\beta_k (y_k - q_k), \end{aligned}$$

where u_k and y_k are interpreted as the control input and output, respectively. By assuming that q_k is constant, i.e., $q_k = q, \forall k$, and by introducing the state vector

$$\zeta_k := \begin{bmatrix} x_k - q \\ v_k \end{bmatrix},$$

the resulting state-space representation is

$$\begin{aligned} \zeta_{k+1} &= \begin{bmatrix} 1 & \alpha \\ 0 & \alpha \end{bmatrix} \zeta_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k \\ y_k &= [1 \quad 0] \zeta_k \\ u_k &= -\beta_k y_k. \end{aligned} \qquad (9)$$

In [3], the sufficient condition for stability of system (9) is derived by using the concept of passive systems and Lyapunov stability. As a result, the sufficient condition for convergence of the PSO algorithm is given by

$$\begin{aligned} & |\alpha| < 1 \\ & \alpha \neq 0 \\ & \beta < \frac{1 - 2|\alpha| + \alpha^2}{1 + \alpha}. \end{aligned} \qquad (10)$$

The parameter region for (10) is shown with the deep gray region in Fig. 1.

Note that the two methods for stability analysis mentioned above make assumptions that the random variables are constant, and therefore, the results are approximate and may be far from a real convergence condition.

IV. STABILITY ANALYSIS

We are now ready to analyze the PSO algorithm via the LMI techniques described in Section II. We first introduce the following 2-dimensional vectors:

$$\xi_{k+1} := \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}, \qquad w_k := \begin{bmatrix} x_k^{(p)} \\ x_k^{(g)} \end{bmatrix}.$$

Using these vectors, the PSO algorithm can be expressed as follows:

$$\begin{aligned} \xi_{k+1} &= \begin{bmatrix} 0 & 1 \\ -\alpha & 1 + \alpha - \beta_k^{(p)} - \beta_k^{(g)} \end{bmatrix} \xi_k \\ &+ \begin{bmatrix} 0 & 0 \\ \beta_k^{(p)} & \beta_k^{(g)} \end{bmatrix} w_k. \end{aligned} \qquad (11)$$

System (11) is a system whose coefficient matrices contain random variables. To analyze system (11) more easily, we also express the random variables $\beta_k^{(p)}$, $\beta_k^{(g)}$ as

$$\begin{aligned} \beta_k^{(p)} &= c^{(p)} \theta_{1,k} + \frac{c^{(p)}}{2} \\ \beta_k^{(g)} &= c^{(g)} \theta_{2,k} + \frac{c^{(g)}}{2}, \end{aligned}$$

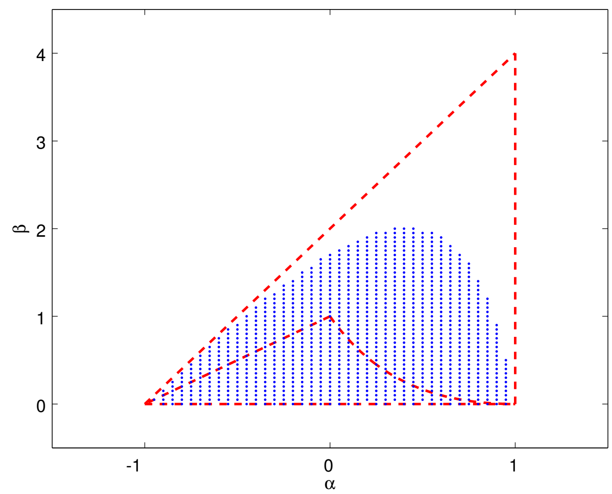

Fig. 2. Stability region (dotted region) by the proposed stability analysis.

where $\theta_{1,k}, \theta_{2,k} \sim \mathcal{U}[-1/2, 1/2]$ are both random variables with uniform distributions. Here system (11) is represented by

$$\xi_{k+1} = A\xi_k + Bw_k + \sum_{i=1}^{2} (A_i \xi_k + B_i w_k)\theta_{i,k}, \qquad (12)$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -\alpha & 1 + \alpha - \frac{c^{(p)}}{2} - \frac{c^{(g)}}{2} \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 \\ \frac{c^{(p)}}{2} & \frac{c^{(g)}}{2} \end{bmatrix} \\ A_1 &= \begin{bmatrix} 0 & 0 \\ 0 & -c^{(p)} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -c^{(g)} \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 & 0 \\ c^{(p)} & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & c^{(g)} \end{bmatrix}. \end{aligned}$$

Since system (12) is a system with multiplicative noise, we can analyze the stability of the PSO algorithm via the LMI technique.

Although the exogenous input w_k in the PSO algorithm depends on the state ξ_k, we can assume this dependence is neglected in the case where $x_k^{(p)}$ and $x_k^{(g)}$ are not updated so frequently. Also, we may regard $\theta_k := [\theta_{1,k}, \theta_{2,k}]^T$ as an independent, identically distributed random variable with

$$\mathbf{E}\,\theta_k = 0, \qquad \mathbf{E}\,\theta_k\theta_k^T = \mathbf{diag}(1/12, 1/12).$$

Then we obtain the following theorem from Theorem 1.

Theorem 3: The PSO algorithm (12) is asymptotically stable in the mean-square sense, if and only if there exists a matrix $P > 0$ satisfying the LMI

$$A^T P A - P + \sum_{i=1}^{2} \frac{1}{12} A_i^T P A_i < 0. \qquad (13)$$

Note that the condition (13) is derived without any simplifying assumptions on the random variables of the PSO algorithm, while such stochastic variables are assumed to be constant in [6], [3].

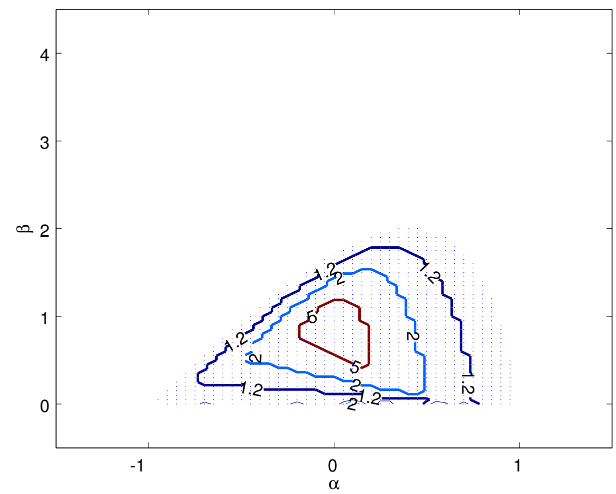

Fig. 3. Contours of the parameters giving the largest lower bound of the decay rate of the PSO algorithm.

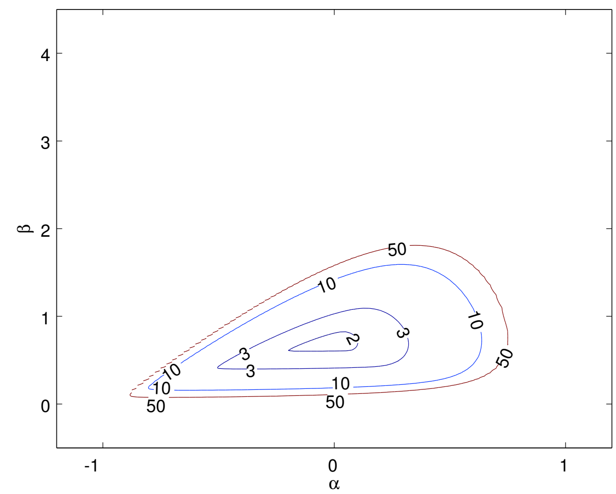

Fig. 4. Contours of the parameters giving the smallest upper bound of the l^2 gain of the PSO algorithm.

Fig. 2 shows the stability region by the proposed stability analysis. In this analysis, the constants $c^{(p)}$ and $c^{(g)}$ are set as $c^{(p)} = c^{(g)} = \beta$. We see from the figure that the region by the conventional analysis in [6] includes that by the proposed analysis. This fact means that the proposed method is more strict than the conventional one in [6], which is confirmed by a numerical example in Section VI.

As an extension of this analysis, we consider a *decay rate* ν [1], [7] for system (12) that can be used as a measure for the convergence speed of the PSO algorithm. The decay rate is defined as the largest $\nu > 1$ such that $\lim_{k\to\infty} \nu^k(\mathbf{E}\,\xi_k\xi_k^T) = 0$. After the same discussion as in [1], [7], we can obtain the fact that $\tilde{\nu} = 1/\mu$ is a lower bound of the decay rate for system (12) if and only if there exists a matrix $P > 0$ satisfying the LMI

$$A^T PA - \mu P + \sum_{i=1}^{2} \frac{1}{12} A_i^T PA_i < 0. \tag{14}$$

We can compute the largest lower bound of the decay rate by applying a bisection algorithm for μ and checking the feasibility of LMI (14).

Note that the decay rate corresponds to the convergence speed of the PSO algorithm. Fig. 3 shows the contours for the largest lower bound of the decay rate. As expected, the parameters α and β around the center of the stability region give fast convergence, while those close to the boundary give slow convergence.

V. l^2 GAIN ANALYSIS

To analyze the l^2 gain of the PSO algorithm, we use the state equation (12) and simply set the output equation by

$$z_k = \xi_k. \tag{15}$$

Thus, the PSO algorithm corresponds to $C = I$, $D = C_i = D_i = 0\ \forall i$ in system (3). For this system, the l^2 gain η is defined as

$$\eta^2 = \sup \left\{ \mathbf{E} \sum_{k=0}^{\infty} (x_k^2 + x_{k+1}^2) \;\middle|\; \sum_{k=0}^{\infty} \left\{ \left(x_k^{(p)}\right)^2 + \left(x_k^{(g)}\right)^2 \right\} \le 1 \right\}. \tag{16}$$

The performance index (16) can be used to estimate how large the search region is against the trajectories of the personal best position and the global best position in the worst case. In general, to improve search ability of global optimization algorithms, it is important to take a balance between exploration and exploitation. The l^2 gain can be interpreted as the exploration ability of the PSO algorithm, while the decay rate can be interpreted as the exploitation ability.

Then we obtain the following theorem from Theorem 2.

Theorem 4: Minimizing γ^2 subject to $P > 0$ and the following LMI yields an upper bound of the l^2 gain η of system (12), (15).

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} P\,[A\ B] + \begin{bmatrix} I-P & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \sum_{i=1}^{2} \frac{1}{12} \begin{bmatrix} A_i^T \\ B_i^T \end{bmatrix} P\,[A_i\ B_i] \le 0. \tag{17}$$

Fig. 4 shows the contours for the smallest upper bound of the l^2 gain. In [6], it is pointed out that the domain around $\alpha = 0.75$ and $\beta = 1.6$ provides good results for some examples. We can see from Fig. 4 that this fact is reasonable because the l^2 gain is large for the parameter couple.

VI. NUMERICAL EXAMPLES

A. Example for the stability analysis

We first test stability region for four benchmark functions used in [6]. In particular, the parameter sets (A) $\alpha = 0.9$, $\beta = 3.5$ ($c^{(p)} = c^{(g)} = 3.5$) and (B) $\alpha = 0.75$, $\beta = 1.6$ ($c^{(p)} = c^{(g)} = 1.6$) are examined. The parameter set (A)

TABLE I

PSO ALGORITHM PERFORMANCE WITH THE PARAMETER SET (A).

Function	Average	Minimum	Maximum
Sphere	0.8090	0.3797	1.6074
Rosenbrock	48.8574	16.0215	125.5004
Rastrigin	22.0112	10.9469	28.2953
Griewank	0.1168	0.0351	0.1725

TABLE II

PSO ALGORITHM PERFORMANCE WITH THE PARAMETER SET (B).

Function	Average	Minimum	Maximum
Sphere	0	0	0
Rosenbrock	0.5058	0.0339	3.9967
Rastrigin	1.3929	0	2.9849
Griewank	0	0	0

is in the stability region by the conventional method [6], while it is not in that by the proposed method. Also, the parameter set (B) is in the stability region by the propose method, while it is not in that by the conventional method [3]. The test functions are as follows.

$$\begin{aligned} f_{\text{Sphere}} &= \sum_{i=1}^{n} x_i^2 \\ f_{\text{Rosenbrock}} &= \sum_{i=1}^{n-1} \left(100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2\right) \\ f_{\text{Rastrigin}} &= \sum_{i=1}^{n} \left(x_i^2 - 10\cos(2\pi x_i) + 10\right) \\ f_{\text{Griewank}} &= \frac{1}{4000}\sum_{i=1}^{n} x_i^2 - \prod_{i=1}^{n} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1. \end{aligned}$$

We consider a minimization problem for each function in 5 dimensional space. The optimal values of the four functions are all 0. Let the number of particles be 10, and the number of iterations be 1000. Initial particle positions are randomly given from $\mathcal{U}[-1, 1]$. Under the above settings, we performed the PSO algorithm 10 times for each function. The average, minimum and maximum values obtained from the PSO algorithms are shown in Tables 1 and 2.

From Table 1, we see that the PSO algorithm with the parameter set (A) does not converge in some cases, while the parameter set (A) is in the stability region by [6]. Also, we see from Table 2 that the PSO algorithm with the parameter set (B) converges, while the parameter set (B) is not in the stability region by [3]. Therefore, the conventional analysis methods are not reasonable. In contrast, the proposed analysis method gives a more reasonable stability region than the conventional ones.

B. Example for the l^2 gain analysis

To show that the l^2 gain of the PSO algorithm can be used for a measure of the exploration ability of the algorithm, the two parameter sets (B) and (C) $\alpha = 0.5$, $\beta = 1.0$ ($c^{(p)} = c^{(g)} = 1.0$) are examined. The l^2 gain for the parameter sets (B) and (C) are 331.43 and 2.74, respectively.

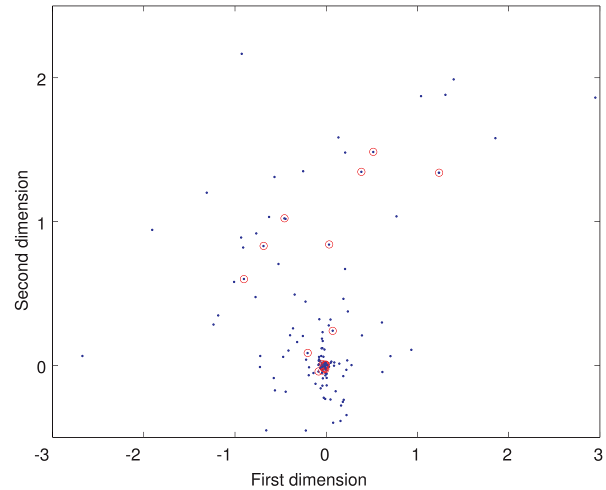

Fig. 5. Particles ($\cdot$) and the global best particles ($\circ$) for the parameter set (B).

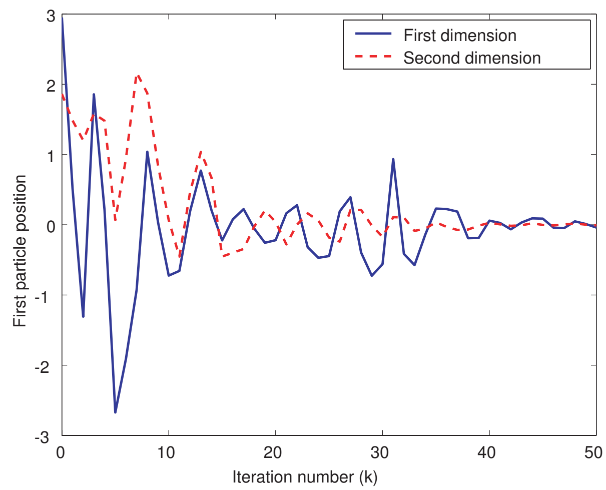

Fig. 6. Trajectory of the first particle for the parameter set (B).

We consider a simple minimization problem of f_{Sphere} in 2 dimensional space. Let the number of particles be 3, and the number of iterations be 50. We set initial particle positions randomly from $\mathcal{U}[1, 3]$.

Figs. 5 and 6 show the distribution of the particles and the global best particles by the PSO algorithm with the parameter set (B), and the trajectories of the first particle, respectively. Fig. 7 shows the best function values corresponding to the global best particles for the parameter set (B). Figs. 8–10 show the results for the parameter set (C).

The particles of the PSO algorithm with the parameter set (B) are widely spread in comparison with those with the parameter set (C). Consequently, the global best particle for the parameter set (B) can reach the global optimal point, i.e., the origin. This observation shows that the exploration ability of the PSO algorithm is high when its l^2 gain is large.

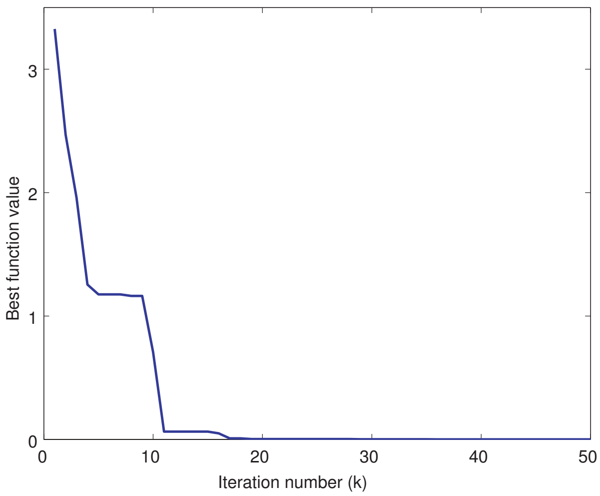

Fig. 7. Best function value for the parameter set (B).

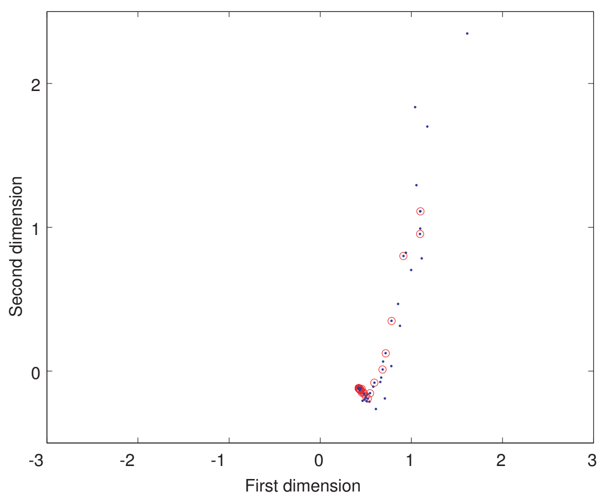

Fig. 8. Particles ($\cdot$) and the global best particles ($\circ$) for the parameter set (C).

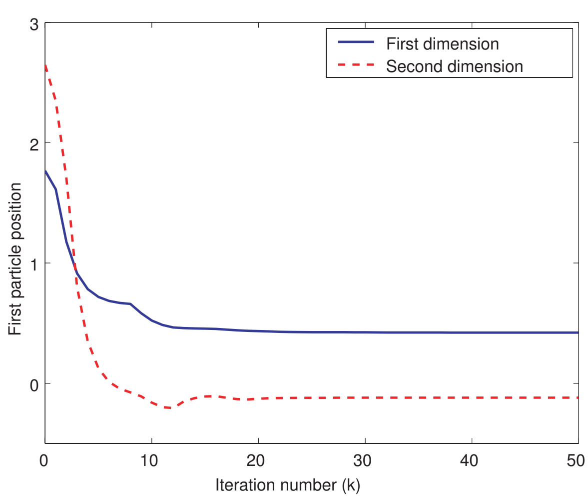

Fig. 9. Trajectory of the first particle for the parameter set (C).

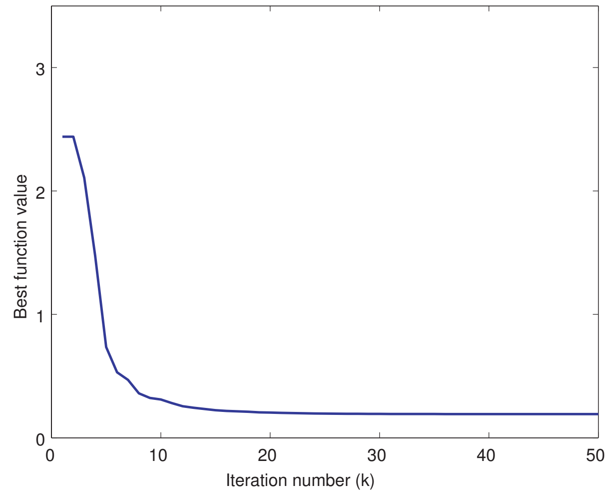

Fig. 10. Best function value for the parameter set (C).

VII. Conclusion

In this paper, we have proposed a method for the stability analysis of the PSO algorithm without any simplifying assumptions on the stochastic variables. Also, to evaluate the convergence speed of the algorithm, we have introduced the decay rate of the PSO dynamics, and have presented a method for finding the largest lower bound of the decay rate. Moreover, we have shown that the l^2 gain of the algorithm can be used to measure exploration ability of the algorithm, and provided a method for finding of the smallest upper bound of the l^2 gain.

We have considered the case for $c^{(p)} = c^{(g)} = \beta$ in this paper. However, the contours of the l^2 gain of the PSO algorithm with $c^{(p)} \neq c^{(g)}$ can be different from those with $c^{(p)} = c^{(g)}$. Further research on this property will be needed.

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