2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Low-complexity Model Predictive Control of Electromagnetic Actuators with a Stability Guarantee

R. M. Hermans, *Student Member* M. Lazar, *Member* S. Di Cairano, *Member* I.V. Kolmanovsky, *Fellow*

Abstract— **Electromagnetically driven mechanical systems are characterized by fast nonlinear dynamics that are subject to physical and control constraints, which makes controller design a challenging problem. This paper presents a novel model predictive control (MPC) scheme that can handle both the performance/physical constraints and the strict limits on computational complexity required in control of general electromagnetic (EM) actuators. The novel aspects of the MPC design are a one-step-ahead prediction horizon and an infinity-norm artificial Lyapunov function that is employed to drive the system to a desired reference. An additional optimization variable is introduced to relax the conditions on the Lyapunov function, which is not forced to decrease monotonically. In this way feasibility of the MPC algorithm is improved considerably. While the MPC scheme uses a full nonlinear model, which improves performance, we show that the resulting MPC problem can still be transformed into a low-complexity linear program that can be solved by modern microprocessors within tenths of milliseconds. Moreover, an even simpler piecewise affine explicit controller can be obtained via multiparametric programming. Simulation results are reported and compared with the results achieved by state-of-the-art explicit MPC based on a piecewise affine model.**

I. Introduction

Over the last few years, increasing operating demands for electromagnetic (EM) actuators in fields as diverse as precision, power, and automotive engineering [1] have intensified the need for fast and accurate stabilizing control strategies. Regardless of their application area, mechatronic systems are characterized by strict operating requirements (low power consumption, fast transition times, accurate reference tracking etc.), severe nonlinearities, and input and state constraints that need to be enforced. In addition, these operating requirements must be met robustly, considering component variability due to part-to-part differences and aging.

Due to these characteristics, the controller design task is challenging. Traditional methods such as proportional-integral-derivative (PID) or linear-quadratic regulator (LQR) control cannot explicitly enforce hard constraints. This is in fact one of the main reasons why model predictive control (MPC) has become successful [2]–[4]. In MPC, the actual control action is computed by solving a finite horizon open-loop optimization problem at each control sample instant, using the measured current state as starting condition, while satisfying input and state constraints.

Although until recently only "slow" systems found in chemical or process industry permitted implementation of MPC, the field of application of MPC is growing with advances in computing power. Recently, an explicit MPC approach was used to tackle the EM actuator control problem in [5], with promising results. However, some improvements can still be made to make it more suitable for practical implementation, both regarding the complexity of the control law and closed-loop stability. Existing MPC schemes with a stability guarantee [3] are either too complex or too conservative for real-time implementation. This paper proposes a new low-complexity nonlinear MPC strategy that is more suitable for real-time control and still provides a stability guarantee under certain assumptions. This is achieved by using a one-step-ahead prediction horizon in the MPC optimization problem and relaxed stabilization constraints formulated using infinity-norm based Lyapunov functions. This particular setup yields a single Linear Program (LP) to be solved on-line, although a full nonlinear model is still used for predictions. Compared to the MPC scheme presented in [5], which requires a piecewise affine (PWA) approximation of the nonlinear actuator model, this ultimately results in improved performance and lower complexity. Even tighter timing and control hardware requirements can be handled by making use of the multiparametric method [6] for obtaining an explicit version of the MPC controller.

II. Preliminaries

A. Basic notions and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ to denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, let $\operatorname{int}(\mathcal{S})$ represent the interior and $\operatorname{cl}(\mathcal{S})$ the closure of $\mathcal{S}$. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. For a vector $\xi \in \mathbb{R}^n$ let $\|\xi\|$ denote an arbitrary p-norm and let $[\xi]_i$, $i = 1, \ldots, n$ denote the i-th component of ξ. Let $\|\xi\|_\infty := \max_{i=1,\ldots,n} |[\xi]_i|$, where $|\cdot|$ denotes the absolute value. For a matrix $Z \in \mathbb{R}^{m\times n}$ let $\|Z\| := \sup_{\xi\neq 0} \frac{\|Z\xi\|}{\|\xi\|}$ denote its corresponding induced matrix norm. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ ($\varphi \in \mathcal{K}_\infty$) if $\varphi \in \mathcal{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$.

R. M. Hermans and M. Lazar are with the Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, E-mails: r.m.hermans@tue.nl, m.lazar@tue.nl.

S. Di Cairano and I.V. Kolmanovsky are with Powertrain Control R&A, Ford Motor Company, Dearborn, Michigan USA, E-mails: sdicaira@ford.com, ikolmano@ford.com.

978-1-4244-4524-0/09/$25.00 ©2009 AACC

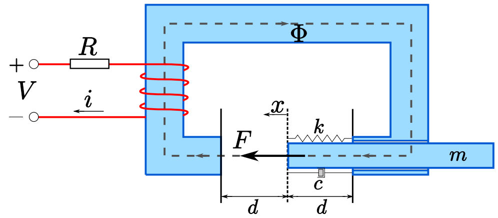

Fig. 1. A magnetically actuated mass-spring-damper system.

B. Lyapunov Stability

Consider the discrete-time, autonomous nonlinear system

$$\xi[k+1] \in \Phi\left(\xi[k]\right), \ k \in \mathbb{Z}_{+}, \tag{1}$$

with state $\xi[k] \in \mathbb{R}^n$ at discrete-time instant k. The mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an arbitrary set-valued function. We assume that the origin is an equilibrium of (1), i.e. $\Phi(0) = \{0\}$.

Definition II.1 A set $\mathcal{P} \subseteq \mathbb{R}^n$ is *Positively Invariant (PI)* for system (1) if $\forall \xi \in \mathcal{P}$ it holds that $\Phi(\xi) \subseteq \mathcal{P}$.

Definition II.2 *(i)* System (1) is *Lyapunov stable* if $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that for all state trajectories of (1) it holds that $\|\xi[0]\| \leq \delta(\varepsilon) \Rightarrow \|\xi[k]\| \leq \varepsilon$ for all $k \in \mathbb{Z}_+$. *(ii)* Let $\mathbb{X} \subseteq \mathbb{R}^n$ and $0 \in \text{int}(\mathbb{X})$. The origin of (1) is *attractive in* $\mathbb{X}$ if for any $\xi[0] \in \mathbb{X}$ it holds that all corresponding trajectories of (1) satisfy $\lim_{k\to\infty} \|\xi[k]\| = 0$. *(iii)* System (1) is *asymptotically stable in* $\mathbb{X}$ if it is Lyapunov stable and attractive in $\mathbb{X}$.

Theorem II.3 *Let $\mathbb{X}$ be a PI set for system* (1) *and let $0 \in \text{int}(\mathbb{X})$. Furthermore, let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that*

$$\alpha_1\left(\|\xi\|\right) \leq V\left(\xi\right) \leq \alpha_2\left(\|\xi\|\right), \tag{2a}$$

$$V(\xi^+) - V(\xi) \leq -\alpha_3\left(\|\xi\|\right), \tag{2b}$$

for all $\xi \in \mathbb{X}$ and all $\xi^+ \in \Phi(\xi)$. Then system (1) *is asymptotically stable in $\mathbb{X}$.*

The proof of the above theorem is similar in nature to the proof given in [7] by replacing the difference *equation* with the difference *inclusion* as in (1) and is omitted here for brevity. A function $V(\cdot)$ that satisfies the conditions of Theorem II.3 is called a *Lyapunov function*.

III. Physical Model and Constraints

The system to be controlled, shown in Fig. 1, is a typical magnetically actuated mass-spring-damper system (MSDS) that is common in mechatronic applications. The MSDS can be modeled by the second-order linear differential equation

$$\ddot{x} = \frac{F}{m} - \frac{c}{m}\dot{x} - \frac{k}{m}x, \tag{3}$$

with mass m [kg], position x [m], damping coefficient c [N·s·m^{-1}] and spring constant k [N·m^{-1}]. The mechanical subsystem is coupled to a nonlinear electromagnetic driving circuit via the magnetic force F [N], induced by applying an input voltage V [V] to the terminals of a coil with resistance R [Ω]. Controlling the nonlinear EM-actuator dynamics is complex and might therefore be inappropriate for implementation in standard microcontrollers. The complexity is decreased by adopting an inner-outer control loop strategy, in which the electromagnetic subsystem is driven by a feedback-linearizing control law [5]. In this way, the inner-loop closed-loop dynamics can be made much faster than the dynamics of the mechanical subsystem, so that the predictive outer-loop controller can be designed based on a time-discretization of the mechanical subsystem (3), with the magnetic force F as the control input.

Like most mechatronic actuator applications, the controlled system is subject to several constraints originating from physical limits or performance requirements. The mass position should not exceed the physical limits

$$-d \leq x \leq d. \tag{4}$$

For safety and performance reasons, a soft-landing constraint is imposed on the mass velocity with respect to its position,

$$-\varepsilon - \beta(d-x) \leq \dot{x} \leq \varepsilon + \beta(d-x), \tag{5}$$

where ε and β are chosen such that the constraint is essentially inactive for $x = 0$ mm (i.e. $\dot{x} \in [-10.2, 10.2]$), while the constraint is tight for $x = d$ (i.e. $\dot{x} \in [-0.2, 0.2]$). Furthermore, the magnetic force is only able to attract the mass, and the outer-loop controller needs to include a saturation constraint on the control input F that is a direct effect of the maximum coil current i_{max} [A] allowed:

$$0 \leq F \leq \frac{k_a i_{\text{max}}^2}{\left(d + k_b - x\right)^2}, \tag{6}$$

with constants k_a [N·m^2·A^{-2}], k_b [m] originating from the EM architecture. Note that this constraint is nonlinear and non-convex in the state variable x.

After defining the dynamics and the limitations of the actuator, the next sections will focus on controller design.

IV. Existing Outer-loop MPC Approach

Although the model (obtained by time-discretization of (3)) used by the outer-loop MPC controller is linear, the controller needs to find an input u from a non-convex feasible input set, defined by linear and nonlinear constraints (4), (5) and (6). The nonlinear constraint poses a computational problem: including the saturation bound in the MPC controller requires solving a nonlinear optimization problem each sampling instant. However, this is not feasible for the considered application due to the stringent limits on computational complexity.

In [5], an outer-loop MPC controller was proposed with a prediction horizon of 3 sampling instants and a quadratic cost function. The state and input restrictions were also enforced along a constraint horizon of 3 instants. The numerical complexity due to the nonlinear constraint was tackled by approximating the nonlinear saturation constraint (6) with a PWA function. This made it possible to formulate the MPC optimization problem as a mixed-integer quadratic program

(MIQP), which can be solved explicitly. However, it should be noted that in general the approximated PWA constraint can result in a deterioration of the closed-loop performance. Furthermore, the controller developed in [5] does not include stabilization as part of its design. Although stability can be checked a posteriori for the explicit form of the controller, if the check fails it is not clear how to modify the original MPC scheme such that stability is guaranteed.

As such, an MPC algorithm that can allow for the nonlinear model of the constraint and still offer a low computational complexity and an a priori stability guarantee is needed for controlling EM actuators. In the next section we propose a novel MPC scheme that attains these properties.

V. Low-complexity Nonlinear MPC

Before the new outer-loop MPC controller is described, we recall some preliminary notions on control Lyapunov functions, which will be instrumental in the MPC setup.

Consider the discrete-time constrained nonlinear system described by the difference equation

$$\xi[k+1] = \phi(\xi[k], u[k]), \quad \forall k \in \mathbb{Z}_+, \tag{7}$$

where $\xi[k] \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state and $u[k] \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at the discrete-time instant k. The function $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is arbitrary with $\phi(0,0) = 0$ and we assume that $\mathbb{X}$ and $\mathbb{U}$ are bounded sets with $0 \in \text{int}(\mathbb{X})$ and $0 \in \text{int}(\mathbb{U})$. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and let $\widetilde{\mathbb{X}}$ be a subset of $\mathbb{X}$ with the origin in its interior.

Definition V.1 A function $V : \mathbb{R}^n \to \mathbb{R}_+$ that satisfies (2a) for all $\xi \in \mathbb{R}^n$ and for which there exists a control law, possibly set-valued, $\pi : \mathbb{R}^n \rightrightarrows \mathbb{U}$ such that

$$V(\phi(\xi, u)) - V(\xi) \leq -\alpha_3 (\|\xi\|), \quad \forall \xi \in \widetilde{\mathbb{X}}, \forall u \in \pi(\xi)$$

is called a control Lyapunov function (CLF) in $\widetilde{\mathbb{X}}$ for the difference inclusion corresponding to system (7) in closed-loop with $u[k] \in \pi(\xi[k])$, $k \in \mathbb{Z}_+$.

Now we can define the new outer-loop MPC scheme, in which we use a one-step-ahead prediction scheme to reduce the overall controller complexity, whereas closed-loop stability is achieved in a non-conservative way by relaxing the conditions on a predetermined local CLF. By using stabilizing constraints not directly related to the MPC cost function, this approach decouples performance from stability and no longer requires the global optimum (with respect to the control input) to be attained at each sampling instant, as typically required in MPC for guaranteeing stability [3].

Let $J(\cdot)$ be a cost function that satisfies $J(\tau) \to \infty$ when $\tau \to \infty$ and $J(\tau) \to 0$ when $\tau \to 0$. Let $V(\cdot)$ be a CLF in $\widetilde{\mathbb{X}} \subseteq \mathbb{X}$ for system (7) and let $\alpha_3 \in \mathcal{K}_\infty$.

Problem V.2 At time $k \in \mathbb{Z}_+$, measure the state $\xi[k]$ and minimize the cost $J(\tau[k])$ over $u[k]$ and $\tau[k]$, subject to the constraints

$$u[k] \in \mathbb{U}, \ \phi(\xi[k], u[k]) \in \mathbb{X}, \ \tau[k] \geq 0, \tag{8a}$$

$$V(\phi(\xi[k], u[k])) - V(\xi[k]) + \alpha_3(\|\xi[k]\|) \leq \tau[k]. \tag{8b}$$

Let $\pi(\xi[k]) := \{u[k] \in \mathbb{R}^m \mid \exists \tau[k] \text{ s.t. (8) holds}\}$ and denote the set of corresponding closed-loop systems by $\phi_{\text{CL}}(\xi[k], \pi(\xi[k])) := \{\phi(\xi[k], u[k]) | u[k] \in \pi(\xi[k])\}$. Also, let $\mathcal{V}_\Gamma := \{\xi \in \mathbb{R}^n \mid V(\xi) \leq \Gamma\}$ for any $\Gamma \in \mathbb{R}_+$ and let $\tau^*[k]$ denote the optimum in Problem V.2 for all $k \in \mathbb{Z}_+$.

Theorem V.3 *Suppose that Problem V.2 is feasible for all ξ in $\mathbb{X}$ and assume that $\lim_{k\to\infty} \tau^*[k] = 0$. Then the closed-loop system*

$$\xi[k+1] \in \phi_{CL}(\xi[k], \pi(\xi[k])), \ k \in \mathbb{Z}_+, \tag{9}$$

is attractive in $\mathbb{X}$. Moreover, if $\exists \Gamma \in \mathbb{R}_{>0}$ such that $V(\cdot)$ is a CLF in $\mathcal{V}_\Gamma$ for (7), then (9) is asymptotically stable in $\mathbb{X}$.

The proof of Theorem V.3, which is omitted due to space limitations, follows more or less using standard arguments employed in proving input-to-state stability and stability, see for example, [8], [7]. Attractivity follows from the property $\lim_{k\to\infty} \tau^*[k] = 0$, which renders the difference inclusion (9) "converging-input converging-state", with $\tau^*[k]$ as input. This further implies that all closed-loop state trajectories reach the set $\mathcal{V}_\Gamma$ in finite time, where $V(\cdot)$ is a CLF. This in turn yields asymptotic stability. The interested reader is referred to the recent article [9] for more details on relaxation of the CLF concept for general discrete-time systems.

Next, we provide a non-conservative solution for guaranteeing that $\lim_{k\to\infty} \tau^*[k] = 0$, which is crucial for attaining asymptotic stability.

Lemma V.4 *Let $\rho \in \mathbb{R}_{[1,0)}$ and $N_\tau \in \mathbb{Z}_{\geq 1}$ be given. Assume that $\phi(\cdot,\cdot)$ and $V(\cdot)$ are bounded on bounded sets. If*

$$0 \leq \tau[k] \leq \rho \frac{1}{N_\tau} \sum_{i=1}^{N_\tau} \tau^*[k-i], \ \forall k \in \mathbb{Z}_{\geq N_\tau}, \tag{10}$$

then $\lim_{k\to\infty} \tau[k] = 0$.

Proof: First we establish an upper bound on $\tau^*[k]$ for all $k \in \mathbb{Z}_+$. Let

$$\bar{\tau} := \sup_{\xi \in \text{cl}(\mathbb{X}), u \in \text{cl}(\mathbb{U})} \{V(\phi(\xi, u)) - V(\xi) + \alpha_3(\|\xi\|)\}.$$

As $V(\cdot)$ is upper and lower bounded by a $\mathcal{K}_\infty$ function, the above supremum exists. Hence, due to constraint (8b) it holds that $\tau^*[k] \leq \bar{\tau}$ for all $k \in \mathbb{Z}_+$. From (10) we have that $\tau^*[k] \leq \rho \max_{i \in [1, N_\tau]} \tau^*[k-i]$, for $k \geq N_\tau$. Hence, recursive application of (10) gives

$$\tau[k] \leq \rho^{\lfloor \frac{k}{N_\tau} \rfloor} \max_{i \in \mathbb{Z}_{[1,N_\tau]}} \tau^*[N_\tau - i] \leq \rho^{\lfloor \frac{k}{N_\tau} \rfloor} \bar{\tau}.$$

The fact that $\lim_{k\to\infty} \rho^{\lfloor \frac{k}{N_\tau} \rfloor} = 0$ completes the proof. ■

As such, augmenting Problem V.2 with constraint (10) will guarantee that $\lim_{k\to\infty} \tau^*[k] = 0$. It is important to point out that this is achieved in a non-conservative way, in the sense that a non-monotonic evolution of $\tau^*[k]$ is allowed, within the asymptotically converging envelope generated by (10). Obviously, if $\tau^*[j] = 0$ for all $j \in \mathbb{Z}_{[k-N_\tau+1,k]}$, constraint (10) is equivalent to $\tau^*[k] = 0$, which is always feasible within $\mathcal{V}_\Gamma$. However, in real-life applications, where noise is present, or in case of reference tracking, the constraint (10) can become unfeasible. To maintain feasibility it suffices to discard (10) for the next N_τ discrete-time instants, as done initially. In Section VII we consider the case of reference tracking, which requires re-initialization of (10) whenever a change in the set-point occurs.

VI. Implementation as a Linear Program

In this section we present the ingredients that make it possible to implement Problem V.2 augmented with constraint (10) as a single linear program. The outer-loop controller uses a zero-order-hold time-discretized MSDS model, with sampling period $T_s = 0.5\,\text{ms}$. This gives the state-space representation

$$\xi[k+1] = A_d\xi[k] + B_d u[k], \tag{11}$$

with state $\xi[k] = \begin{bmatrix} x[k] \\ \dot{x}[k] \end{bmatrix}$ and input $u[k] = F[k]$, where $x[k]$ is the position and $\dot{x}[k]$ is the velocity. Additionally, let $u_r[k]$ and $\xi_r[k] = \begin{bmatrix} r_x[k] \\ r_{\dot{x}}[k] \end{bmatrix}$ be the corresponding input and state reference values respectively. The output is $y[k] = C\xi[k]$, where $y = \begin{bmatrix} x \\ \dot{x}+\beta x \\ \dot{x}-\beta x \end{bmatrix}$. Next, consider the following cost function to be minimized by the MPC controller:

$$\begin{aligned} &J_{\text{MPC}}(\xi, u, \tau) := \\ &\|Q_0(\xi - \xi_r)\|_\infty + \|Q_1(A_d\xi + B_d u - \xi_r)\|_\infty + \\ &\|R_u(u - u_r)\|_\infty + J(\tau), \end{aligned} \tag{12}$$

where we removed the dependency on time for all variables, for brevity. The cost that penalizes τ is defined as $J(\tau) := \|M\tau\|_\infty$, $M \in \mathbb{R}_{>0}$. Here, Q_0, Q_1, R_u are known full-column rank matrices of appropriate dimensions. Notice that the cost $J(\cdot)$ is chosen as required in Problem V.2.

A single sample prediction scheme is not only beneficial for decreasing the controller complexity, but also because all constraints depending nonlinearly on the measured state appear now linearly with respect to the variables in the optimization problem. The one-step-ahead saturation constraint (6) is linear in u for instance, as the right-hand-side is just a constant determined by the current position $x[k]$:

$$0 \le u[k] \le \frac{k_a i_{\max}^2}{(d + k_b - x[k])^2}. \tag{13}$$

The other performance or control constraints are linear in u and specified as

$$y_{\min} \le C(A_d\xi + B_d u) \le y_{\max}, \tag{14}$$

with $y_{\min} = \begin{bmatrix} -d \\ -\infty \\ -\varepsilon-\beta d \end{bmatrix}$ and $y_{\max} = \begin{bmatrix} d \\ \varepsilon+\beta d \\ \infty \end{bmatrix}$.

Now consider the following infinity-norm based CLF

$$V(\xi) = \|P\xi\|_\infty, \tag{15}$$

where $P \in \mathbb{R}^{p\times n}$ is a full column-rank matrix to be determined. This function satisfies (2a), with $\alpha_1(s) = \frac{\underline{\sigma}}{\sqrt{p}}s$, where $\underline{\sigma}$ is the smallest singular value of P, and with $\alpha_2(s) = \|P\|_\infty s$. Substituting (11) and (15) in (8b) yields

$$\begin{aligned} &\|P(A_d\xi + B_d u - \xi_r)\|_\infty \\ &\quad - \|P(\xi - \xi_r)\|_\infty + \alpha_3\|\xi - \xi_r\|_\infty \le \tau. \end{aligned} \tag{16}$$

Although (12) and (16) appear to be nonlinear in the optimization variables, the corresponding optimization problem can be recast as a linear program via a particular set of linear inequalities, without introducing conservatism, as follows. By definition of the infinity norm, for $\|\xi\|_\infty \le c$ to be satisfied, it is necessary and sufficient to require that $\pm[\xi]_j \le c$ for all $j \in \{1, 2, \ldots, n\}$. So, for (16) to be satisfied it is necessary and sufficient to require that

$$\begin{aligned} &\pm[P(A_d\xi + B_d u - \xi_r)]_j \\ &\quad - \|P(\xi - \xi_r)\|_\infty + \alpha_3\|\xi - \xi_r\|_\infty \le \tau \end{aligned} \tag{17}$$

for $j \in \{1, 2\}$. This yields a total of $2p$ linear inequalities in the optimization variables u and τ. Moreover, solving Problem V.2, which includes minimizing the cost (12), can be reformulated as:

$$\min \varepsilon_1 + \varepsilon_2 + J(\tau) \tag{18}$$

subject to (13), (14), (17), and

$$\begin{aligned} \pm[Q_1(A_d\xi + B_d u - \xi_r)]_j + \|Q_0(\xi - \xi_r)\|_\infty &\le \varepsilon_1 \\ \pm R_u(u - u_r) &\le \varepsilon_2, \end{aligned}$$

for $j = 1, 2$, which is a LP, as the cost $J(\tau)$ is linear in $\tau \ge 0$. The MPC algorithm now becomes:

Algorithm VI.1 *At each sampling instant k:*
Step 1: *Measure or estimate the current state $\xi_0 = \xi[k]$;*
Step 2: *Solve LP* (18) *and pick any feasible control action $\bar{u}_0$;*
Step 3: *Set $F[k] = \bar{u}_0$ as inner-control-loop reference.*

Notice that even with constraint (10) added, (18) is still a LP, as all $\tau^*[k-i]$, $i \in \mathbb{Z}_{[1,N_\tau]}$, are known at time $k \in \mathbb{Z}_{\ge N_\tau}$. In what follows, we will use the acronym IMPC-1 to denote the developed MPC scheme based on a CLF and cost function defined using the infinity norm.

A. Explicit form of the proposed MPC scheme

Although the optimization problem solved by the novel predictive controller is a simple linear program, implementation might still be hampered if the time required to find a feasible input exceeds the sampling period. However, [6] shows that the solution to an LP can be obtained as a function of parameters θ that appear linearly in the program, via multiparametric linear programming (mp-LP). The optimal explicit solution wass calculated off-line using the multiparametric toolbox (MPT) [10] in the form of a piecewise affine parametric feedback law $u(\theta)$.

The parameter vector was defined as

$$\theta(\xi_0, \xi_r) := \begin{bmatrix} \xi_0 \\ \xi_r \\ -\alpha_3\|\xi_0 - \xi_r\|_\infty + \|P(\xi_0 - \xi_r)\|_\infty \\ k_a \frac{i_{\max}^2}{(d+k_b-x)^2} \end{bmatrix}.$$

The parameter vector given above is suited for the explicit implementation of the IMPC-1 controller without constraint (10). Inclusion of (10) simply requires the augmentation of θ with $\rho \frac{1}{N_\tau} \sum_{i=1}^{N_\tau} \tau^*[k-i]$ for $k \geq N_\tau$.

It should be stressed that the optimal solution obtained by solving MPC problem on-line and its explicit counterpart return identical results, but there is a significant difference in the computational complexity of their implementation. This difference is consistent with the solution of an on-line optimization problem versus the evaluation of a set of linear equalities and the calculation of an affine state feedback term.

The explicit MPC algorithm is summarized as follows:

Algorithm VI.2 *At each sampling instant k:*
Step 1: *Measure or estimate the current state $\xi_0 = \xi[k]$;*
Step 2: *Detect in which region of the explicit-control parameter space the corresponding θ lies and calculate the optimal input u_0^* using the corresponding affine control law;*
Step 3: *Set $F[k] = u_0^*$ as inner-control-loop reference.*

In short, the explicit MPC controller acts as a simple PWA state feedback law, without introducing conservatism, whereas it preserves the beneficial properties of the on-line optimization based MPC scheme.

VII. Simulation Results

In this section we present the results attained by the one-step-ahead predictive controller developed in this article and we compare them with the ones obtained by the MPC scheme presented in [5]. All simulation results are obtained using the predictive controller in closed-loop with the Simulink implementation of the MSDS (3) that was also used in [5]. This mechanical subsystem is shaped to resemble a second-order under-damped system with damped frequency peak at $\omega_r = 950\,[\text{rad}\cdot\text{s}^{-1}]$ and 3 dB-bandwidth $BW_3 = 3 \cdot 10^3\,[\text{rad}\cdot\text{s}^{-1}]$, which is in accordance with real-life specifications for this device [5].

Figure 2 shows the closed-loop state trajectories obtained with the IMPC-1 controller, including constraint (10), when tracking a certain reference profile. The weight matrices of the cost (12) used by the IMPC-1 setup are $Q_1 = Q_0 = \begin{bmatrix} 10^4 & 0 \\ 0 & 7 \end{bmatrix}$, $R_u = 0$ and $M = 10^3$; the sampling period T_s was chosen equal to 0.5 ms. The technique of [11] was used to compute off-line the weight $P \in \mathbb{R}^{2\times 2}$ of the local CLF $V(\xi) = \|P\xi\|_\infty$ for $\alpha_3(s) = 10^{-4}s$ and the linear model of the MSDS in closed-loop with $u[k] := K\xi[k]$, $K \in \mathbb{R}^{1\times 2}$, yielding

$$P = \begin{bmatrix} 3.9650 & 0.0010 \\ 2.3012 & 0.0089 \end{bmatrix}, \quad K = \begin{bmatrix} 1.3475 & -0.0030 \end{bmatrix} \times 10^5.$$

Note that the control law $u[k] = K\xi[k]$ was only employed off-line, to calculate the weight matrix P of the local CLF $V(\cdot)$, and it was never used for controlling the system. The region of validity of $V(\xi) = \|P\xi\|_\infty$, i.e. the set $\mathcal{V}_\Gamma$, is obtained as the largest sublevel set contained in $\{\xi \in \mathbb{X} \mid K\xi \in \mathbb{U}\}$, which is much smaller than $\mathbb{X}$. This justifies the need of the relaxation variable $\tau[k]$.

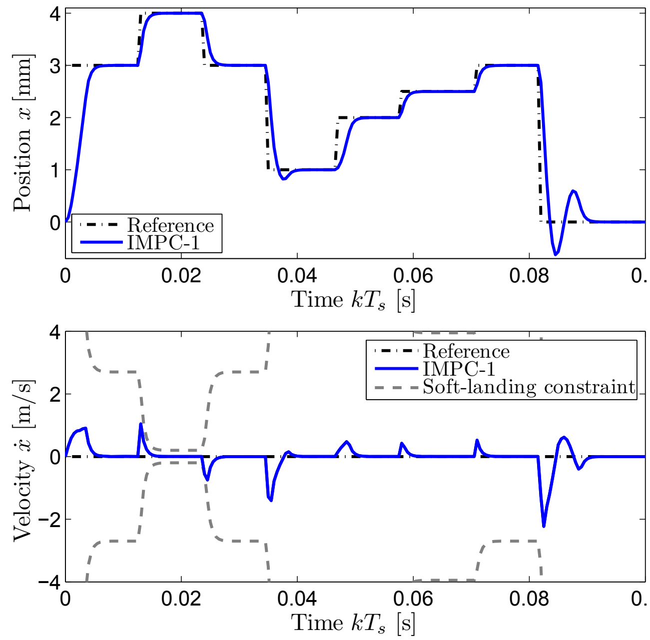

Fig. 2. Comparison of controller performance (top: Position trajectory tracking, bottom: Velocity trajectory tracking and soft-landing bounds).

TABLE I
Comparison of Reference Tracking Error and Control Effort

MPC controller	Cumulated position error [mm^2]	Input effort [kN^2]
QMPC-3	65.6008	27.0984
IMPC-1	59.8274	26.8519

Table I lists the cumulated squared position errors $\sum_k (x[k] - r_x[k])^2$ and the input effort $\sum_k u[k]^2$ for the quadratic-cost, 3 sample prediction controller (QMPC-3 for short) of [5] and the IMPC-1 scheme developed in this paper for the specific trajectory shown in Figure 2. Table I demonstrates the capabilities of the IMPC-1 scheme in terms of tracking performance. Furthermore, the effectiveness of the IMPC-1 scheme in terms of enforcing the nonlinear saturation constraint (13) is shown in Figure 3. The MPC controller developed in this paper is able to exploit the full feasible input range, whereas the performance of the QMPC-3 scheme is restricted by a slightly conservative PWA approximation of the saturation characteristic. Moreover, it is crucial to point out that the IMPC-1 setup never leads to violation of the soft-landing constraint, represented by the dashed lines in the bottom plot of Figure 2.

Finally, the evolution of the CLF relaxation variable $\tau^*[k]$ and the corresponding upper bound defined by (10) for $\rho = 0.9$ and $N_\tau = 7$, is shown in Figure 4, for this particular simulation and $k \in \mathbb{Z}_{[175,190]}$. It can be observed that $\tau^*[k]$ may be small or 0 for some time after which it is allowed to increase again, as long as this does not violate the upper bound. As $k \to \infty$, $\tau^*[k]$ is forced to converge to 0 however, which in turn implies asymptotic convergence of ξ to ξ_r and Lyapunov stability, as guaranteed by Theorem V.3.

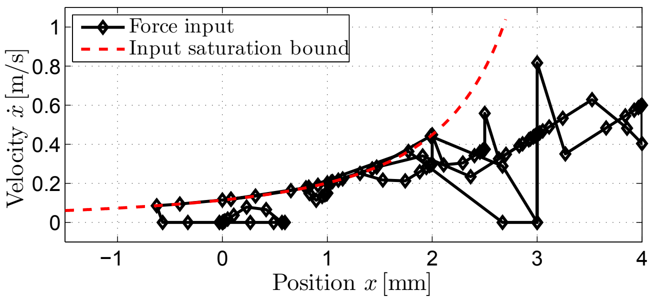

Fig. 3. Input-position trajectory and force-saturation bound.

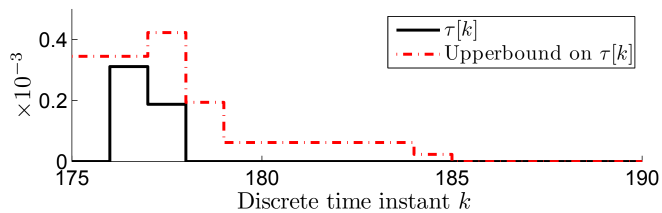

Fig. 4. Evolution of $\tau^*[k]$ and the upperbound on τ for $\rho = 0.9$ and $N_\tau = 7$.

A. Explicit Controller

As stated before, the explicit controller and its on-line optimizing counterpart achieve identical performance, although their computational complexities differ significantly. Therefore, we only discuss the differences in complexity and computational speed between the on-line and the explicit version of the QMPC-3 and IMPC-1 setups, respectively, without reporting the closed-loop state trajectories for the explicit controller.

Whereas the explicit QMPC-3 controller, based on a PWA model and a mp-MIQP problem, has a polyhedral complexity of 671 regions, the explicit IMPC-1 parametric space only consists of 42 polyhedrons, which is a significant complexity reduction. The number of regions can be further decreased if a desired reference signal is chosen in advance; this is a reasonable assumption if only on-off operation of the EM actuator is required. According to [5], where a Simulink platform running on a 1.0 GB RAM, 2.0 GHz Pentium-M PC with Cplex 9.1 and Matlab 7 was used, the worst case computational time for the explicit QMPC-3 controller was 0.3ms. Obviously, the explicit IMPC-1 controller would require a much smaller CPU time on the same platform, as its number of regions is much smaller compared to the explicit QMPC-3 controller. Optimized C-code implementations of the proposed controller running on dedicated computing hardware, for instance an FPGA device as proposed in [12], [13], are expected to reduce the required on-line computation time even further.

VIII. Conclusions

This article proposed a novel, low-complexity nonlinear model predictive control scheme for controlling electromagnetic actuators, which are used in many automotive components. The MPC controller optimizes the behavior of the mass-spring-damper system, decoupled from the electromagnetic subsystem, subject to hard performance and control constraints, while taking into account the nonlinear constraint arising from the magnetic driving circuit. Previous MPC approaches used PWA approximations of this constraint, see e.g. [5]. By adopting a one-step-ahead prediction strategy and an infinity-norm based optimization objective, the MPC optimization problem reduces to a single linear program, which makes the developed MPC scheme particularly attractive for systems with fast dynamics that require control at sampling periods below one millisecond. Even tighter chronometric requirements can be handled by using standard explicit MPC techniques.

IX. Acknowledgements

This research is supported by the Veni grant “Flexible Lyapunov Functions for Real-time Control”, grant number 10230, awarded by STW (Dutch Science Foundation) and NWO (The Netherlands Organization for Scientific Research).

References

[1] D. Hrovat, J. Asgari, and M. Fodor, *Automotive mechatronic systems*, ser. Mechatronic Systems, Techniques and Applications. New York, USA: Gordon and Breach Science Publishers, 2000, vol. 2, pp. 1–98.

[2] J. B. Rawlings, “Tutorial overview of model predictive control,” *IEEE Control Systems Magazine*, vol. 20, no. 3, pp. 38–52, 2000.

[3] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, “Constrained model predictive control: Stability and optimality,” *Automatica*, vol. 36, no. 6, pp. 789–814, 2000.

[4] S. J. Qin and T. A. Badgwell, “A survey of industrial model predictive control technology,” *Control Engineering Practice*, vol. 93, no. 316, pp. 733–764, 2003.

[5] S. Di Cairano, A. Bemporad, I. Kolmanovsky, and D. Hrovat, “Model predictive control of magnetically actuated mass spring dampers for automotive applications,” *International Journal of Control*, vol. 80, no. 11, pp. 1701–1716, 2007.

[6] A. Bemporad, F. Borrelli, and M. Morari, “Model predictive control based on linear programming—the explicit solution,” *IEEE Transactions on Automatic Control*, vol. 47, no. 12, pp. 1974–1985, 2002.

[7] M. Lazar, “Model predictive control of hybrid systems: Stability and robustness,” Ph.D. dissertation, Eindhoven University of Technology, Eindhoven, the Netherlands, 2006.

[8] Z.-P. Jiang and Y. Wang, “Input-to-state stability for discrete-time nonlinear systems,” *Automatica*, vol. 37, pp. 857–869, 2001.

[9] M. Lazar, “Flexible control Lyapunov functions,” *28th American Control Conference*, 2009.

[10] M. Kvasnica, P. Grieder, M. Baotić, and F. J. Christophersen, *Multi-parametric Toolbox (Reference Guide)*, Swiss Federal Institute of Technology Zürich, Zürich, Switzerland, 2006.

[11] M. Lazar, W. P. M. H. Heemels, S. Weiland, and A. Bemporad, “Stabilizing model predictive control of hybrid systems,” *IEEE Transactions on Automatic Control*, vol. 51, no. 11, pp. 1813–1818, 2006.

[12] L. G. Bleris, P. D. Vouzis, M. G. Arnold, and M. V. Kothare, “A co-processor FPGA platform for the implementation of real-time model predictive control,” in *25th American Control Conference*, Minneapolis, USA, 2006, pp. 1912–1917.

[13] K. V. Ling, S. P. Yue, and J. M. Maciejowski, “A FPGA implementation of model predictive control,” in *25th American Control Conference*, Minneapolis, USA, 2006, pp. 1930–1935.