2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Further Results on Lyapunov-Krasovskii Functionals via Nonlinear Small-Gain Conditions for Interconnected Retarded iISS Systems

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***Abstract*— This paper presents further results on the problem of establishing stability of retarded nonlinear interconnected systems comprising integral input-to-state stable subsystems. It is shown that the stability of the interconnected systems with respect to external signals can be verified by constructing Lyapunov-Krasovskii functionals explicitly whenever small-gain type conditions are satisfied. The primary result [12] is generalized in two aspects. One is to introduce a new flexibility in constructing Lyapunov-Krasovskii functionals to deal with distributed delays more effectively. The other is to cover systems involving time-varying delays in interconnecting channels.**

I. INTRODUCTION

Time-delay usually has a great influence on systems stability and performance. Recently, the popularity of research on teleoperation, networked control systems and consensus in cooperative control among multiple agents has regained attention to the importance of coping with time-delays in interconnected systems. There are various sources of delays in such networks and spatially distributed systems. One of typical delays is the transmission delay in communication between subsystems or agents, which is often time-varying.

There is a large body of literature on stability criteria for retarded systems, and some of them deal with time-varying delays (e.g. [7], [17], [5], [24] to name a few). Stability of interconnected systems and networks has been also investigated for many decades. Small-gain theorem states that an interconnected system is stable if the loop-gain is less than one. Although the small gain theorem has found numerous applications in control theory[3], the practical limitation of the $\mathcal{L}_p$-gain framework had been also pointed out. The emergence of the small-gain theorem for input-to-stable (ISS) systems has provided a way to remove the limitation[13], [23], and the ISS small-gain theorem is now widely used in many areas of nonlinear systems and control design. An ISS small-gain condition has been also developed for general ISS dynamical systems with weak semigroup properties in [14]. The philosophy of the ISS small-gain theorem has been generalized to a larger class of systems covering integral input-to-state stable (iISS) systems[10].

This paper is supported in part by Grant-in-Aid for Scientific Research of JSPS under grant 19560446, by the Italian MIUR Project PRIN 2005, and by NSF grants ECS-0093176 and DMS-0504462.

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Recently, in [12], the iISS small-gain theorem has been generalized to systems involving discrete delays and distributed delays which can appear everywhere in the systems. Interconnections of iISS retarded nonlinear systems are targeted, and the iISS property of the overall system is established by constructing iISS Lyapunov-Krasovskii functionals explicitly. The purpose of this paper is to improve the result of [12] further in the following two points:

(G1) to equalize the treatment of discrete delays and distributed delays, and construct more flexible iISS Lyapunov-Krasovskii functionals explicitly;

(G2) to allow time-varying delays in the communication between subsystems.

The results in this paper include [12] as a special case.

Notations: The symbol $|\cdot|$ stands for the Euclidean norm. The interval $[0,\infty)$ is denoted by $\mathbb{R}_+$. For a measurable and essentially bounded function $u:\mathbb{R}_+\to\mathbb{R}^m$, $\|u\|_\infty = ess\sup_{t\geq 0}|u(t)|$. We indicate with $u_{[T_1,T_2)}:\mathbb{R}_+\to\mathbb{R}^m$ the function given by $u_{[T_1,T_2)}(t)=u(t)$ for all $t\in[T_1,T_2)$ and $=0$ elsewhere. A function $\omega:\mathbb{R}_+\to\mathbb{R}_+$ is denoted by $\omega\in\mathcal{P}_0$ if it is continuous and satisfies $\omega(0)=0$. A function $\omega\in\mathcal{P}_0$ is said to be positive definite if $\omega(s)>0$ holds for all $s>0$, and written as $\omega\in\mathcal{P}$. A function is of class $\mathcal{K}$ if it belongs to $\mathcal{P}$ and is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A function $\beta:\mathbb{R}_+^2\to\mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed t the function $s\to\beta(s,t)$ is of class $\mathcal{K}$ and for each fixed s the function $t\to\beta(s,t)$ is non-increasing and goes to zero as $t\to+\infty$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively.

II. SYSTEM DESCRIPTION

Consider an interconnected system Σ described by the following functional differential equations

$$\Sigma\left\{\begin{array}{ll}\Sigma_1: & \dot{x}_1(t)=f_1(t,x_{1,t},x_{2,t},r_1(t)),\\ \Sigma_2: & \dot{x}_2(t)=f_2(t,x_{2,t},x_{1,t},r_2(t)),\end{array}\right. \tag{1}$$
$$x_{1,t_0}=\xi_{1,0},\quad x_{2,t_0}=\xi_{2,0},\quad t_0\in\mathbb{R}_+$$

where, for $i=1,2$, $x_i(t)\in\mathbb{R}^{n_i}$; $r_i(t)\in\mathbb{R}^{m_i}$ is an external input (measurable, locally essentially bounded); they are functions of time $t\in\mathbb{R}_+$; n_i and m_i are positive integers. For $t\in\mathbb{R}_+$, $x_{i,t}:[-\Delta,0]\to\mathbb{R}^{n_i}$ denotes the function $x_{i,t}(\tau)=x_i(t+\tau)$, where $\Delta>0$ is the maximum involved delay. Let $\mathcal{C}_i$ denote the space of continuous functions mapping the interval $[-\Delta,0]$ into $\mathbb{R}^{n_i}$. For $\phi_i\in\mathcal{C}_i$, we use $\|\phi_i\|_\infty=\sup_{-\Delta\leq\theta\leq 0}|\phi_i(\theta)|$. Suppose that $\xi_{i,0}\in\mathcal{C}_i$ and that $f_i:\mathbb{R}_+\times\mathcal{C}_i\times\mathcal{C}_{3-i}\times\mathbb{R}^{m_i}\to\mathbb{R}^{n_i}$ is a functional which is Lipschitz on any bounded set in $\mathcal{C}_i\times\mathcal{C}_{3-i}\times\mathbb{R}^{m_i}$ uniformly in $t\in\mathbb{R}_+$. We combine vectors as $x(t)=[x_1(t)^T,x_2(t)^T]^T\in\mathbb{R}^n$, $n=n_1+n_2$, $r(t)=[r_1(t)^T,r_2(t)^T]^T\in\mathbb{R}^m$, $m=m_1+$

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m_2, $\xi_0 = [\xi_{1,0}^T, \xi_{2,0}^T]^T \in \mathcal{C} := \mathcal{C}_1 \times \mathcal{C}_2$, $f() = [f_1()^T, f_2()^T]^T$, $\phi = [\phi_1^T, \phi_2^T]^T \in \mathcal{C}$. We define x_t and $\|\phi\|_\infty$ as done for their i-th components. If f_1 and f_2 are independent of t, let $t_0 = 0$ without loss of generality. It is assumed that $f_i(t, 0, 0, 0) = 0$, $i = 1, 2$, $\forall t \in \mathbb{R}_+$, thus ensuring that $x(t) = 0$ is the solution corresponding to zero input and zero initial conditions (i.e. the trivial solution). Note that the formulation (1) accepts non-commensurate discrete as well as distributed time-delays not only in the interconnecting channels, but also in the individual subsystems Σ_i.

In this paper, we let $M_{a,i} : \mathcal{C}_i \to \mathbb{R}_+$ and $M_a : \mathcal{C} \to \mathbb{R}_+$, $i = 1, 2$, be continuous functionals such that there exist $\underline{\gamma}_{a,i}$, $\overline{\gamma}_{a,i}$, $\underline{\gamma}_a$, $\overline{\gamma}_a \in \mathcal{K}_\infty$ such that

$$\underline{\gamma}_{a,i}(|\phi_i(0)|) \le M_{a,i}(\phi_i) \le \overline{\gamma}_{a,i}(\|\phi_i\|_\infty), \ \forall \phi_i \in C_i \quad (2)$$

$$\underline{\gamma}_a(|\phi(0)|) \le M_a(\phi) \le \overline{\gamma}_a(\|\phi\|_\infty), \qquad \forall \phi \in \mathcal{C} \ . \quad (3)$$

We borrow the definitions of ISS and iISS properties of the system (1) from the references[21], [22], [1], [20].

Definition 1: If the solution $x(t)$ of the interconnected system (1) exists for all $t \ge t_0$ and satisfies

$$\chi(|x(t)|) \le \beta(\|\xi_0\|_\infty, t - t_0) + \int_{t_0}^{t} \gamma_r(|r(\tau)|) d\tau \qquad (4)$$

for all $t \ge t_0$, with $\beta \in \mathcal{KL}$, $\chi \in \mathcal{K}_\infty$, $\gamma_r \in \mathcal{K}$, the system (1) is said to be iISS with respect to input r and state x.

Definition 2: If the solution $x(t)$ of the interconnected system (1) exists for all $t \ge t_0$ and satisfies

$$|x(t)| \le \beta(\|\xi_0\|_\infty, t - t_0) + \gamma_r(\|r_{[t_0,t)}\|_\infty) \qquad (5)$$

for all $t \ge t_0$, with $\beta \in \mathcal{KL}$, $\gamma_r \in \mathcal{K}$, the system (1) is said to be ISS with respect to input r and state x.

The property (4) (and (5)) implies the global asymptotic stability of $x = 0$ when $r(t) \equiv 0$. For short, it is called 0-GAS in this paper.

For a locally Lipschitz functional $V_{cl} : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}_+$, $D^+V_{cl}(\phi, r, t)$, which plays the central role in the Lyapunov-Krasovskii methodology[2], [18], [19], is defined as follows:

$$D^+V_{cl}(\phi, r, t) = \limsup_{h \to 0^+} \frac{V_{cl}(t + h, \phi^h) - V_{cl}(t, \phi)}{h},$$

$$\phi^h(s) = \begin{cases} \phi(s + h), & s \in [-\Delta, -h) \\ \phi(0) + (s + h) f(t, \phi, r), & s \in [-h, 0] \end{cases} \quad (6)$$

where $\phi \in \mathcal{C}$, $r \in \mathbb{R}^m$, $t \in \mathbb{R}_+$.

This paper addresses the problem of constructing Lyapunov-Krasovskii functionals V_{cl} to establish 0-GAS, iISS and ISS of the interconnected system (1) with respect to input r and state x under the following assumption imposed on each subsystem Σ_i.

Assumption 1: For each $i = 1, 2$, there exists a locally Lipschitz functional $V_i : \mathcal{C}_i \to \mathbb{R}_+$ such that

$$\underline{\alpha}_i(M_{a,i}(\phi_i)) \le V_i(\phi_i) \le \overline{\alpha}_i(M_{a,i}(\phi_i)), \qquad (7)$$

$$D^+V_i(\phi_i, \phi_{3-i}, r_i, t) \le \rho_i(t, \phi_i, \phi_{3-i}, r_i), \qquad (8)$$

$$\forall \ \phi_j \in \mathcal{C}_j, j = 1, 2, \forall \ r_i \in \mathbb{R}^{m_i}$$

hold, where $\underline{\alpha}_i, \overline{\alpha}_i$ are $\mathcal{K}_\infty$ functions, $\rho_i : \mathbb{R}_+ \times \mathcal{C}_i \times \mathcal{C}_{3-i} \times \mathbb{R}^{m_i} \to \mathbb{R}$ is a continuous functional satisfying $\rho_i(t, 0, 0, 0) = 0$ for all $t \in \mathbb{R}_+$, and

$$D^+V_i(\phi_i, \phi_{3-i}, r_i, t) = \limsup_{h \to 0^+} \frac{V_i(\phi_i^h) - V_i(\phi_i)}{h} \qquad (9)$$

$$\phi_i^h(s) = \begin{cases} \phi_i(s + h), & s \in [-\Delta, -h), \\ \phi_i(0) + (s+h) f_i(t, \phi_i, \phi_{3-i}, r_i), & s \in [-h, 0]; \end{cases}$$

Remark 1: In this paper, we make use of time-invariant Lyapunov functionals V_i for subsystems Σ_i, $i = 1, 2$ although the time variable t appears on the right hand side of (1). In general, a time-varying functional V_i is more useful unless subsystems posses a uniform characteristic in time t or time variable only appears in their input delays. In this sense, this paper mainly focuses on internally time-invariant subsystems Σ_i, $i = 1, 2$. In Section III, the uniform property in t is also implicitly required in interconnection channels, i.e., the feedback inputs since time-invariant functionals $V_{cl}(x)$ are sought for the overall system Σ. In contrast, Section IV constructs time-varying functionals $V_{cl}(t, x)$ for Σ, which aims mainly to deal with time-varying delays in interconnecting channels effectively.

III. A Generalization of Supply Rates and Lyapunov-Krasovskii Functionals

A. Preliminaries

This section uses the notion of iISS (ISS) Lyapunov-Krasovskii functional which is defined as follows:

Definition 3: A locally Lipschitz functional $V_{cl} : \mathcal{C} \to \mathbb{R}_+$ satisfying

$$\underline{\alpha}_{cl}(|\phi(0)|) \le V_{cl}(\phi) \le \overline{\alpha}_{cl}(M_a(\phi)), \forall \phi \in \mathcal{C}, \qquad (10)$$

$$D^+V_{cl}(\phi, r, t) \le -\alpha_{cl}(M_a(\phi)) + \sigma_r(|r|), \forall \phi \in \mathcal{C}, r \in \mathbb{R}^m \quad (11)$$

for some $\underline{\alpha}_{cl}$, $\overline{\alpha}_{cl}$, $\alpha_{cl} \in \mathcal{K}_\infty$, $\sigma_r \in \mathcal{P}_0$ is said to be an ISS Lyapunov-Krasovskii functional for the system (1).

Definition 4: A locally Lipschitz functional $V_{cl} : \mathcal{C} \to \mathbb{R}_+$ satisfying

$$\underline{\alpha}_{cl}(M_a(\phi)) \le V_{cl}(\phi) \le \overline{\alpha}_{cl}(M_a(\phi)), \forall \phi \in \mathcal{C}, \qquad (12)$$

$$D^+V_{cl}(\phi, r, t) \le -\alpha_{cl}(M_a(\phi)) + \sigma_r(|r|), \forall \phi \in \mathcal{C}, r \in \mathbb{R}^m \quad (13)$$

for some $\underline{\alpha}_{cl}$, $\overline{\alpha}_{cl} \in \mathcal{K}_\infty$, $\alpha_{cl} \in \mathcal{P}$, $\sigma_r \in \mathcal{P}_0$ is said to be an iISS Lyapunov-Krasovskii functional for the system (1).

The existence of iISS (ISS) Lyapunov-Krasovskii functionals is a sufficient condition for the iISS (ISS, respectively) of the system Σ (See Appendix). When we require $\alpha_{cl}(|\phi(0)|)$ instead of $\alpha_{cl}(|M_a(\phi)|)$ in (11) for $r(t) \equiv 0$, the functional V_{cl} reduces to the well-known Lyapunov-Krasovskii functional for 0-GAS [8], [16], [2].

B. Motivation

In [12], the supply rate ρ_i in Assumption 1 is chosen as

$$\rho_i = -\alpha_i(M_{a,i}(\phi_i)) + S_{i,0}\sigma_{i,0}(M_{a,3-i}(\phi_{3-i})) + \sum_{j=1}^{h} S_{i,j}\sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right) + \sigma_{r,i}(|r_i|) \quad (14)$$

where Δ_j's are time-invariant time-delays, and $S_{i,j} \in \{0, 1\}$, $\alpha_i, \sigma_{i,j} \in \mathcal{K}$ and $\sigma_{r,i} \in \mathcal{P}_0$. This supply rate explicitly includes discrete delays in the interconnection channels, i.e., the third term in (14), while distributed delays in the interconnection channels do not appear explicitly in (14). The distributed delays are required to be incorporated in $M_{a,3-i}$ in ρ_i. This requirement together with (7) implies that we need to choose V_{3-i} for Σ_{3-i} taking into account the distributed delays at the input of Σ_i. In [12], this idea is employed only for the distributed delays, and discrete delays in the interconnecting channels are handled by the composite functional V_{cl} instead. In fact, the iISS Lyapunov-Krasovskii functional constructed for the overall system Σ is

$$V_{cl}(\phi) = \int_0^{V_1(\phi_1)} \lambda_1(s)ds + \int_0^{V_2(\phi_2)} \lambda_2(s)ds + \sum_{j=1}^{h} S_{1,j} \int_{-\Delta_j}^{0} F_{1,j}(\tau)\tilde{\sigma}_{1,j}\left(\underline{\gamma}_{a,2}(|\phi_2(\tau)|)\right) d\tau + \sum_{j=1}^{h} S_{2,j} \int_{-\Delta_j}^{0} F_{2,j}(\tau)\tilde{\sigma}_{2,j}\left(\underline{\gamma}_{a,1}(|\phi_1(\tau)|)\right) d\tau \quad (15)$$

The last two terms in (15) cope with the discrete delays. The definition of λ_i, $\tilde{\sigma}_{i,j}$ and $F_{i,j}$ will be shown later. It can be expected here that we can use the same idea for distributed delays. To deal with delays in the interconnecting channels, we should be able to take either of the following ways:

- preprocessing the input delays in choosing the pair of dissipative inequalities of the individual subsystems;
- leaving the input delays explicitly in the supply rates of individual subsystems, and dealing with the input delays in the process of constructing a composite Lyapunov-Krasovskii functional for the overall system.

These two ways should be applicable to discrete delays and distributed delays equally. This section formulates a stability criterion providing the above two options explicitly.

C. A Generalized Small-Gain Theorem

The following is the main result in this section.

Theorem 1: Suppose that supply rate functionals ρ_i, $i = 1, 2$, are as follows:

$$\begin{aligned}\rho_i(\phi_i, \phi_{3-i}, r_i) =& \\ &-\alpha_i(M_{a,i}(\phi_i)) + S_{i,0}\sigma_{i,0}(M_{a,3-i}(\phi_{3-i})) \\ &+\sum_{j=1}^{h} S_{i,j}\sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right) \\ &+\sum_{j=h+1}^{h+h_d} S_{i,j}\int_{-\Delta_j}^{0} \sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau \\ &+\sigma_{r,i}(|r_i|),\end{aligned} \quad (16)$$

where h and h_d are non-negative integers and, for $i = 1, 2$, α_i are functions of class $\mathcal{K}$, $\sigma_{r,i} \in \mathcal{P}_0$, and for $j = 0, 1, \dots, h+h_d$, $S_{i,j}$ belong to $\{0, 1\}$, $\sigma_{i,j}$ are functions of class $\mathcal{K}$, and for $j = 1, 2, \dots, h + h_d$, $\Delta_j \in (0, \Delta]$. Assume that one of the following three conditions holds:

(H1) $\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_1(s) = \infty,$

(H2) $\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty,$

(H3) $\lim_{s\to\infty} \sigma_1(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty,$

where σ_i, $i = 1, 2$, are defined as

$$\sigma_i(s) = \left(\sum_{k=0}^{h+h_d} S_{i,k}\right) \max\left\{\max_{j=0,1,\dots,h} S_{i,j}\sigma_{i,j}(s), \max_{j=h+1,\dots,h+h_d} S_{i,j}\Delta_j\sigma_{i,j}(s)\right\} \quad (17)$$

Suppose that there exist $c_i > 1$, $i = 1, 2$, such that

$$c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \quad (18)$$

Then, the interconnected system (1) is iISS with respect to input r and state x. In addition, it is ISS with respect to input r and state x in the case of *(H1)*. Furthermore, an iISS (ISS in the *(H1)* case) Lyapunov-Krasovskii functional for (1) is

$$\begin{aligned}V_{cl}(\phi) =& \int_0^{V_1(\phi_1)} \lambda_1(s)ds + \int_0^{V_2(\phi_2)} \lambda_2(s)ds \\ &+\sum_{j=1}^{h} S_{1,j} \int_{-\Delta_j}^{0} F_{1,j}(\tau)\tilde{\sigma}_{1,j}\left(\underline{\gamma}_{a,2}(|\phi_2(\tau)|)\right) d\tau \\ &+\sum_{j=h+1}^{h+h_d} S_{1,j}\int_{-\Delta_j}^{0} F_{1,j}(\tau)\int_{\tau}^{0} \tilde{\sigma}_{1,j}\left(\underline{\gamma}_{a,2}(|\phi_2(\theta)|)\right) d\theta d\tau \\ &+\sum_{j=1}^{h} S_{2,j} \int_{-\Delta_j}^{0} F_{2,j}(\tau)\tilde{\sigma}_{2,j}\left(\underline{\gamma}_{a,1}(|\phi_1(\tau)|)\right) d\tau \\ &+\sum_{j=h+1}^{h+h_d} S_{2,j}\int_{-\Delta_j}^{0} F_{2,j}(\tau)\int_{\tau}^{0} \tilde{\sigma}_{2,j}\left(\underline{\gamma}_{a,1}(|\phi_1(\theta)|)\right) d\theta d\tau\end{aligned} \quad (19)$$

where λ_1, λ_2 and $\tilde{\sigma}_{i,j}$ are given in (54), (55) and (46) with

$$\varpi_{i,j} = 1, \qquad j = 0, 1, ..., h+h_d, \quad (20)$$

and $F_{i,j} : [-\Delta_j, 0] \to \mathbb{R}$ is defined for $0 < \epsilon_i < c_i - 1$ as

$$F_{i,j}(\tau) = \frac{-\tau}{\Delta_j} + (1 + \epsilon_i)\frac{\tau + \Delta_j}{\Delta_j} \quad (21)$$

Proof: Using $F_{i,j}(\tau) \leq 1 + \epsilon_i$ for $\tau \in [-\Delta_j, 0]$, we can verify that V_{cl} given by (19) satisfies (12) for some $\underline{\alpha}_{cl}$, $\overline{\alpha}_{cl} \in \mathcal{K}_\infty$ and some $M_a : \mathcal{C} \to \mathbb{R}_+$ fulfilling (3) with $\underline{\gamma}_a$, $\overline{\gamma}_a \in \mathcal{K}_\infty$. The following inequality also holds:

$$\begin{aligned}D^+V_{cl}(\phi, r, t) \leq& \sum_{i=1}^{2} \lambda_i(V_i(\phi_i))D^+V_i(\phi_i, \phi_{3-i}, r_i, t) \\ &+\sum_{j=1}^{h}(1 + \epsilon_i)S_{i,j}\tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \\ &\quad -S_{i,j}\tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right) \\ &\quad -S_{i,j}\frac{\epsilon_i}{\Delta_j}\int_{-\Delta_j}^{0} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau \\ &+\sum_{j=h+1}^{h+h_d}\left(1 + \frac{\epsilon_i}{2}\right) S_{i,j}\Delta_j\tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \\ &\quad -S_{i,j}\int_{-\Delta_j}^{0} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau \\ &\quad -S_{i,j}\frac{\epsilon_i}{\Delta_j}\int_{-\Delta_j}^{0}\int_{\tau}^{0} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\theta)|)\right) d\theta d\tau\end{aligned} \quad (22)$$

Combining (22) with Lemma 1, we obtain the following for suitable functions $\tilde{\alpha}_i$, $\tilde{\sigma}_{i,j}, \in \mathcal{K}$ and $\tilde{\sigma}_{r,i} \in \mathcal{P}_0$.

$$\begin{aligned}
D^+V_{cl} \le & \sum_{i=1}^{2} -\tilde{\alpha}_i(M_{a,i}(\phi_i)) + S_{i,0}\tilde{\sigma}_{i,0}(M_{a,3-i}(\phi_{3-i})) \\
& + \Bigg\{ \sum_{j=1}^{h} (1+\epsilon_i) S_{i,j}\tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \\
& \quad - S_{i,j}\frac{\epsilon_i}{\Delta_j}\int_{-\Delta_j}^{0} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau \Bigg\} \\
& + \Bigg\{ \sum_{j=h+1}^{h+h_d} (1+\epsilon_i) S_{i,j}\Delta_j\tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \\
& \quad - S_{i,j}\frac{\epsilon_i}{\Delta_j}\int_{-\Delta_j}^{0}\int_{\tau}^{0} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\theta)|)\right) d\theta d\tau \Bigg\} \\
& + \tilde{\sigma}_{r,i}(|r_i|)
\end{aligned} \tag{23}$$

By Lemma 2, taking into account (18) and (2), it follows that there exist class $\mathcal{K}$ functions α_{cl} and σ_{cl} such that (13) is satisfied. In the case of *(H1)*, We obtain $\tilde{\alpha}_i, \alpha_{cl} \in \mathcal{K}_\infty$. Finally, Theorem 3 completes the proof. ∎

By $h = 0$ ($h_d = 0$), we mean that the first (second, respectively) sum in (16) vanishes. Note that (18) with c_1, $c_2 > 1$ requires

$$\lim_{s\to\infty} \alpha_2(s) = \infty \quad \vee \quad \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s)$$

When we take $h_d = 0$ in (16), Theorem 1 reduces to the result of [12]. The summation from $j = h+1$ to $h+h_d$ in (16) is not utilized in [12] for dealing with distributed delays at the feedback input of the each subsystem Σ_i. Thus, Theorem 1 includes [12] as a special case and add a degree of flexibility. In order to deal with the new term in the supply rate (16), this paper introduces the double integrals in the Lyapunov-Krasovskii functional V_{cl}. It is worth mentioning that such terms are often utilized in the Lyapunov-Krasovskii framework, e.g., [7], [17], [4]. This paper shows that such a technique can be also utilized to deal with interconnected iISS systems by selecting $\tilde{\sigma}_{i,j}$'s judiciously (See Appendix).

Remark 2: In the supply rates (16), $i = 1, 2$, the functions

$$M_{a,3-i}(\phi_{3-i}), \quad \int_{-\Delta_j}^{0} S_{i,j}\sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau \tag{24}$$

provide some mutual redundancy. For example, a distributed delay at the input of the system Σ_i can be incorporated in either $M_{a,3-i}$ or $\int S_{i,j}\sigma_{i,j}$ in the supply rate ρ_i of the system Σ_i. Here, the catch is that the choice of $M_{a,3-i}$ restricts the selection of the functional V_{3-i} of the system Σ_{3-i} and its supply rate. In some cases, the relation between $M_{a,3-i}$ and Σ_{3-i} prevents $M_{a,3-i}$ from representing distributed delays at the input of Σ_i. The use of non-zero $\int S_{i,j}\sigma_{i,j}$'s to represent distributed delays at the input of Σ_i would be sometimes easier since the functional V_{3-i} and the supply rate ρ_{3-i} of the system Σ_{3-i} can be chosen independently of the system Σ_i. However, the inequality (18) might result in a conservative stability condition in some cases. We can utilize this flexibility of (24) to reduce the conservativeness accordingly to the situation. For systematic utilization, the development of optimization techniques is needed. The redundancy among

$$M_{a,3-i}(\phi_{3-i}), \quad S_{i,j}\sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right) \tag{25}$$

also has a degree of flexibility for discrete delays.

Remark 3: Assumption 1 with the new supply rate (16) implies that each subsystem Σ_i is iISS with respect to input ($x_{3-i,t}$, r_i) and state x_i. Indeed, since

$$\rho_i = -\alpha_i(M_{a,i}(\phi_i)) + \sigma_i(\overline{\gamma}_{a,3-i}(\|\phi_{3-i}\|_\infty)) + \sigma_{r,i}(|r_i|) \tag{26}$$

is also a supply rate for Σ_i, we can verify

$$\begin{aligned}
\chi_i(|x_i(t)|) \le & \beta_i(\|\xi_{i,0}\|_\infty, t-t_0) + \\
& \textstyle\int_{t_0}^{t} \gamma_i(\|x_{3-i,\tau}\|_\infty) d\tau + \int_{t_0}^{t} \gamma_{r,i}(|r_i(\tau)|) d\tau
\end{aligned}$$

for all $t \ge t_0$ with $\beta_i \in \mathcal{KL}$, $\chi_i \in \mathcal{K}_\infty$, and γ_i, $\gamma_{r,i} \in \mathcal{K}$. If *(H1)* holds, we obtain

$$\begin{aligned}
|x_i(t)| \le & \beta_i(\|\xi_{i,0}\|_\infty, t-t_0) + \\
& \gamma_i\left(\sup\nolimits_{\tau\in[t_0,t)} \|x_{3-i,\tau}\|_\infty\right) + \gamma_{r,i}(\|(r_i)_{[t_0,t)}\|_\infty)
\end{aligned}$$

for all $t \ge t_0$ with $\beta_i \in \mathcal{KL}$ and γ_i, $\gamma_{r,i} \in \mathcal{K}$, which shows Σ_i to be ISS with respect to input ($x_{3-i,t}$, r_i) and state x_i.

Remark 4: The selection of the individual functionals V_i, $i = 1, 2$ has an influence on the result of stability analysis based on the inequality (18). To reduce the associated conservativeness arising in each case, we can make use of a large literature on detailed techniques of delay systems for the effective choices of V_i of the individual Σ_i.

Remark 5: The small-gain condition (18) can be relaxed into a non-uniform small-gain condition[11] by using a little complex λ_i's.

IV. Time-Varying Delays

This section considers the following time-varying delays:

$$0 \le \Delta_j(t) \le \bar{\Delta}_j, \quad \frac{d\Delta_j(t)}{dt} \le b_j < 1, \quad \begin{matrix} j = 1, 2, ..., h+h_d, \\ \forall t \in \mathbb{R}_+ \end{matrix} \tag{27}$$

Note that (27) implies $b_j \ge 0$. In the case where these time-varying delays appear in (1), we can establish 0-GAS of Σ in the following way.

Theorem 2: Suppose that supply rate functionals ρ_i, $i = 1, 2$, satisfy

$$\begin{aligned}
& \rho_i(t, \phi_i, \phi_{3-i}, r_i) = \\
& \quad -\alpha_i(M_{a,i}(\phi_i)) + S_{i,0}\sigma_{i,0}(M_{a,3-i}(\phi_{3-i})) \\
& \quad + \sum_{j=1}^{h} S_{i,j}\sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j(t))|)\right) \\
& \quad + \sum_{j=h+1}^{h+h_d} S_{i,j}\int_{-\Delta_j(t)}^{0} \sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau \\
& \quad + \sigma_{r,i}(|r_i|)
\end{aligned} \tag{28}$$

for $r_i(t) = 0$, where h and h_d are non-negative integers and, for $i = 1, 2$, $\alpha_i \in \mathcal{K}$, $\sigma_{r,i} \in \mathcal{P}_0$, and for $j = 0, 1, \ldots, h+h_d$, $S_{i,j}$ belong to $\{0, 1\}$, $\sigma_{i,j}$ are of class $\mathcal{K}$. Assume that

(J1) $\lim_{s\to\infty} \alpha_1(s) = \infty \quad \vee \lim_{s\to\infty} \sigma_2(s) < \infty$,

holds, where σ_i, $i = 1, 2$, are defined as

$$\begin{aligned}
\sigma_i(s) = & \left(\sum_{k=0}^{h+h_d} S_{i,k}\right) \max\Bigg\{ S_{i,0}\sigma_{i,0}(s), \\
& \max_{j=1,\ldots,h} \frac{S_{i,j}}{1-b_j}\sigma_{i,j}(s), \max_{j=h+1,\ldots,h+h_d} \frac{S_{i,j}\bar{\Delta}_j}{1-b_j}\sigma_{i,j}(s) \Bigg\}
\end{aligned} \tag{29}$$

Suppose that there exist $c_i > 1$, $i = 1, 2$, such that (18) holds. Then, the interconnected system (1) is 0-GAS. Furthermore, a Lyapunov-Krasovskii functional for (1) is

$$
\begin{aligned}
V_{cl}(t,\phi) = &\int_0^{V_1(\phi_1)} \lambda_1(s)ds + \int_0^{V_2(\phi_2)} \lambda_2(s)ds \\
&+ \sum_{j=1}^{h} \frac{S_{1,j}}{1-b_j} \int_{-\Delta_j(t)}^{0} \tilde{\sigma}_{1,j}\left(\underline{\gamma}_{a,2}(|\phi_2(\tau)|)\right) d\tau \\
&+ \sum_{j=h+1}^{h+h_d} \frac{S_{1,j}}{1-b_j} \int_{-\Delta_j(t)}^{0} \int_{\tau}^{0} \tilde{\sigma}_{1,j}\left(\underline{\gamma}_{a,2}(|\phi_2(\theta)|)\right) d\theta d\tau \\
&+ \sum_{j=1}^{h} \frac{S_{2,j}}{1-b_j} \int_{-\Delta_j(t)}^{0} \tilde{\sigma}_{2,j}\left(\underline{\gamma}_{a,1}(|\phi_1(\tau)|)\right) d\tau \\
&+ \sum_{j=h+1}^{h+h_d} \frac{S_{2,j}}{1-b_j} \int_{-\Delta_j(t)}^{0} \int_{\tau}^{0} \tilde{\sigma}_{2,j}\left(\underline{\gamma}_{a,1}(|\phi_1(\theta)|)\right) d\theta d\tau \quad (30)
\end{aligned}
$$

where λ_1, λ_2 and $\tilde{\sigma}_{i,j}$ are given in (54), (55) and (46) with

$$
\varpi_{i,0} = 1, \quad \varpi_{i,j} = \frac{1}{1-b_j}, \qquad j = 1, 2, ..., h+h_d. \quad (31)
$$

Proof: The following inequality holds for (30) and (1):

$$
\begin{aligned}
D^+V_{cl}(t,\phi) \le &\sum_{i=1}^{2} \lambda_i(V_i(\phi_i)) D^+V_i(\phi_i, \phi_{3-i}, t) \\
&+ \sum_{j=1}^{h} \frac{S_{i,j}}{1-b_j} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \\
&- \frac{S_{i,j}}{1-b_j}\left(1 - \frac{d\Delta_j(t)}{dt}\right) \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right) \\
&+ \sum_{j=h+1}^{h+h_d} \frac{S_{i,j}\Delta_j(t)}{1-b_j} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \\
&- \frac{S_{i,j}}{1-b_j}\left(1 - \frac{d\Delta_j(t)}{dt}\right) \int_{-\Delta_j}^{0} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(\tau)|)\right) d\tau \quad (32)
\end{aligned}
$$

Combining (32) with Lemma 1 and using (27), we obtain the following for suitable functions $\tilde{\alpha}_i$ and $\tilde{\sigma}_{i,j} \in \mathcal{K}$.

$$
\begin{aligned}
D^+V_{cl} \le &\sum_{i=1}^{2} -\tilde{\alpha}_i(M_{a,i}(\phi_i)) + S_{i,0}\tilde{\sigma}_{i,0}(M_{a,3-i}(\phi_{3-i})) \\
&+ \sum_{j=1}^{h} \frac{S_{i,j}}{1-b_j} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \\
&+ \sum_{j=h+1}^{h+h_d} \frac{S_{i,j}\bar{\Delta}_j}{1-b_j} \tilde{\sigma}_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(0)|)\right) \quad (33)
\end{aligned}
$$

From Lemma 3, (18) and (2), it follows that there exist $\alpha_{cl} \in \mathcal{K}$ and a functional M_a such that

$$
\underline{\alpha}_{cl}(|\phi(0)|) \le V(t,\phi) \le \overline{\alpha}_{cl}(M_a(\phi)), \quad \forall \phi \in \mathcal{C} \quad (34)
$$

$$
D^+V_{cl}(t,\phi) \le -\alpha_{cl}(|\phi(0)|), \quad \forall \phi \in \mathcal{C} \quad (35)
$$

and (3) hold for some $\underline{\alpha}_{cl}, \overline{\alpha}_{cl} \in \mathcal{K}_\infty$. Hence, the Lyapunov-Krasovskii Theorem proves the claim. ■

There are a large number of studies on stability of systems with time-varying delays using Lyapunov-Krasovskii functionals(See, e.g., [5], [24], [9] and references therein). This paper shows how to extend the fundamentals of such ideas to cover essentially nonlinear interconnected systems consisting of iISS subsystems. The novelty of the Lyapunov-Krasovskii functional V_{cl} given by (30) lies in $\tilde{\sigma}_{i,j}$'s which lead us to the small-gain condition (18) (See Appendix).

In the proof of Theorem 2, the function α_{cl} is obtained as a function of $|\phi(0)|$. Since α_{cl} is not guaranteed to be a function of $M_a(\phi)$, Theorem 2 proves 0-GAS based on the Lyapunov-Krasovskii Theorem instead of iISS and ISS.

Remark 6: The functional V_{cl} in (30) is time-dependent due to the presence of $\Delta_j(t)$. On the other hand, Definitions 3 and 4 require properties uniform in time t since V_{cl} and M_a are time-invariant. In order to succeed in verifying the iISS property for a larger class of time-varying systems, we need to develop a methodology of iISS Lyapunov-Krasovskii functionals which are allowed to be non-uniform in t.

Remark 7: Theorem 2, in general, requires that state delays in individual subsystem Σ_i are time-invariant. In fact, the inequality (8) with (28) implies that the property of Σ_i with respect to ϕ_i is uniform in time t since V_i and $M_{a,i}$ are time-independent functional of ϕ_i. Nevertheless, it is stressed that time-varying delays in the interconnection channels can be incorporated into the supply rates (28) of Σ_i, $i = 1, 2$, effectively, so that the selection of V_i's and $M_{a,i}$'s can be independent of the interconnection delays.

Remark 8: The assumption *(J1)* in Theorem 2 can be relaxed by using a non-uniform small-gain condition as in the delay-free case[11].

V. An Example

Consider the interconnected system described by

$$
\begin{aligned}
\dot{x}_1(t) &= -\frac{x_1(t)^3}{1+x_1(t)^2} + x_1(t)\int_{t-\Delta_1}^{t} x_2(\tau)^2 d\tau + x_1(t)r_1(t) \\
\dot{x}_2(t) &= -\gamma x_2(t) + x_2(t-\Delta_2) + \frac{x_1(t-\Delta_3(t))}{1+x_1(t-\Delta_3(t))^2}
\end{aligned} \quad (36)
$$

where $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, $r(t) = r_1(t) \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. The time-delays $\Delta_1, \Delta_2 > 0$ are time-invariant, while the time-varying delay $\Delta_3(t)$ is supposed to satisfy

$$
0 \le \Delta_3(t) \le \bar{\Delta}_3, \quad \frac{d\Delta_3(t)}{dt} \le b_3 < 1, \quad \forall t \in \mathbb{R}_+
$$

For $\phi_1 \in \mathcal{C}_1, \phi_2 \in \mathcal{C}_2$, let

$$
\begin{aligned}
V_1(\phi_1) &= \log(1+\phi_1(0)^2) \\
V_2(\phi_2) &= \phi_2^2(0) + \int_{-\Delta_2}^{0}\left(\frac{-\tau}{\Delta_2} + (1+\epsilon_2)\frac{\tau+\Delta_2}{\Delta_2}\right)\phi_2(\tau)^2 d\tau
\end{aligned}
$$

where $\epsilon_2 > 0$ has yet to be determined. Then, we obtain

$$
\begin{aligned}
D^+V_1 \le &-(1-\delta)\frac{\phi_1(0)^4}{(1+\phi_1(0)^2)^2} + \Delta_1\int_{-\Delta_1}^{0}\phi_2(\tau)^4 d\tau + \frac{1}{\delta}r_1(t)^2 \\
D^+V_2 \le &-(2\gamma - 3 - \epsilon_2)\phi_2(0)^2 + \\
&\frac{\phi_1(-\Delta_3(t))^2}{1+\phi_1(-\Delta_3(t))^2} - \frac{\epsilon_2}{\Delta_2}\int_{-\Delta_2}^{0}\phi_2(\tau)^2 d\tau
\end{aligned}
$$

Here, Young inequality and Cauchy-Schwarz inequality are used. We can choose the following functions and parameters:

$$M_{a,1}(\phi_1)=\phi_1(0)^2,\ M_{a,2}(\phi_2)=\phi_2(0)^2+\int_{-\Delta_2}^{0}\phi_2(\tau)^2d\tau$$
$$\underline{\alpha}_1(s)=\overline{\alpha}_1(s)=\log(1+s),\quad \underline{\gamma}_{a,1}(s)=\overline{\gamma}_{a,1}(s)=s^2$$
$$\underline{\alpha}_2(s)=s,\ \overline{\alpha}_2(s)=(1+\epsilon_2)s,\ \overline{\gamma}_{a,2}(s)=(1+\Delta_2)s^2$$
$$\underline{\gamma}_{a,2}(s)=s^2,\ S_{1,0}=S_{1,1}=S_{2,0}=S_{2,2}=0,\ h=h_d=1$$
$$S_{1,2}=S_{2,1}=1,\ \alpha_1(s)=\tfrac{(1-\delta)s^2}{(1+s)^2},\ \sigma_{1,2}(s)=\Delta_1 s^2$$
$$\alpha_2(s)=\min\left\{2\gamma-3-\epsilon_2,\tfrac{\epsilon_2}{\Delta_2}\right\}s,\ \sigma_{2,1}(s)=\tfrac{s}{1+s}$$
$$\sigma_1(s)=\Delta_1^2s^2,\ \sigma_2(s)=\tfrac{s}{(1-b_3)(1+s)}$$

Thus, the dissipation inequality (8) with ρ_i in the form of (28) implies that the x_2 subsystem is ISS with respect to input x_1 and state x_2, while the x_1 subsystem is only iISS with respect to input (x_2, r_1) and state x_1.

First, we suppose $b_3 = 0$, i.e., $\Delta_3(t)$ is a constant. For the supply rates in the form of (16), there exist $c_1, c_2 > 1$ such that the small-gain condition (18) is satisfied if

$$\gamma > \frac{3+\Delta_1-2\Delta_1\Delta_2}{2(1-\Delta_1\Delta_2)} \quad\wedge\quad \Delta_1\Delta_2<1 \tag{37}$$

holds, which is independent of Δ_3. Thus, the interconnected system (36) is iISS with respect to input r and state x if (37) is satisfied. Theorem 1 provides an iISS Lyapunov-Krasovskii functional as (19)

Next, in order to illustrate Theorem 2 in the case of time-varying $\Delta_3(t)$ with $b_3 > 0$, we consider the interconnected system (36) for $r_1(t) \equiv 0$. Taking $\delta = 0$, we have the supply rates in the form of (28). There exist $c_1, c_2 > 1$ satisfying the small-gain condition (18) if

$$\gamma > \frac{3(1-b_3)+\Delta_1-2\Delta_1\Delta_2}{2(1-b_3-\Delta_1\Delta_2)} \ \wedge\ \Delta_1\Delta_2<1-b_3 \tag{38}$$

holds. Theorem 2 asserts that the system (36) is 0-GAS if (38) is satisfied. A Lyapunov-Krasovskii functional is (30).

VI. Conclusions

This paper has investigated the problem of establishing stability of interconnected nonlinear systems involving discrete as well as distributed time-delays. It has been shown that Lyapunov-Krasovskii functionals establishing stability of the interconnections can be constructed when small-gain conditions are satisfied. This paper has improved the previous result [12] of the authors by generalizing supply rates of individual subsystems to deal with discrete delays and distributed delays in a unified manner. Lyapunov-Krasovskii functionals are tailored in order to handle the generalized supply rates. This paper has also covered time-varying delays in communication channels.

This paper has employed time-invariant functionals V_i for individual subsystems although time-varying functionals V_i have more potential to cope with fully time-varying networks. Indeed, Section IV aims mainly at time-varying delays only in the channels connecting time-invariant subsystems. When we deal with internally time-varying subsystems, time-invariant V_i's can fulfill (8) with (28) only when properties of the subsystems are uniform in t. Recently, Lyapunov-Krasovskii type characterizations of non-uniform in time ISS of nonlinear retarded systems have been derived in [15]. Therefore, our further research includes extending the result in this paper to more general stability which is allowed to be non-uniform in t.

Appendix

Theorem 3: If there exists an ISS (iISS) Lyapunov-Krasovskii functional V_{cl} for (1), then the system (1) is ISS (iISS, respectively) with respect to input r and state x.

This theorem was originally proved for time-invariant systems in [20]. The arguments given there remain true since Definition 3 and Definition 4 use time-invariant functional V_{cl} and require uniform properties in time t. The following lemmas are extension of lemmas proved in [12] to the cases of distributed and time-varying delays.

Lemma 1: For $i=1,2$, consider

$$\alpha_i,\sigma_{i,j}\in\mathcal{K},\quad \sigma_{r,i}\in\mathcal{P}_0,,\quad j=0,1,...,h+h_d \tag{39}$$
$$S_{i,j}\in\{0,1\},\ \varpi_{i,j}\in[1,\infty),\quad j=0,1,...,h+h_d \tag{40}$$
$$0\le\Delta_j(t)\le\bar{\Delta}_j\le\Delta,\ \forall t\in\mathbb{R}_+,\ j=1,2,...,h+h_d \tag{41}$$

with non-negative integers h, h_d, and non-decreasing continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$. Assume that

$$\lim_{s\to\infty}\alpha_i(s)<\infty\ \Rightarrow\ \lim_{s\to\infty}\lambda_i(s)<\infty \tag{42}$$

holds. If functionals $V_i, M_{a,i} : \mathcal{C}_i \to \mathbb{R}_+$ satisfy

$$\underline{\alpha}_i(M_{a,i}(\phi_i))\le V_i(\phi_i)\le\overline{\alpha}_i(M_{a,i}(\phi_i)),\quad \forall\phi_i\in\mathcal{C}_i, \tag{43}$$

for some $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, then it holds that

$$\begin{aligned}
&\lambda_i(V_i(\phi_i))\Bigg\{-\alpha_i(M_{a,i}(\phi_i))+\sum_{j=0}^{h}S_{i,j}\sigma_{i,j}(w_{i,j})\\
&\quad+\sum_{j=h+1}^{h+h_d}S_{i,j}\int_{-\Delta_j(t)}^{0}\sigma_{i,j}(v_{i,j}(s))ds+\sigma_{r,i}(z_i)\Bigg\}\\
&\quad\le-\tilde{\alpha}_i(M_{a,i}(\phi_i))+\sum_{j=0}^{h}S_{i,j}\tilde{\sigma}_{i,j}(w_{i,j})\\
&\quad\quad+\sum_{j=h+1}^{h+h_d}S_{i,j}\int_{-\Delta_j(t)}^{0}\tilde{\sigma}_{i,j}(v_{i,j}(s))ds+\tilde{\sigma}_{r,i}(z_i)\\
&\quad\forall\phi_i\in\mathcal{C}_i,\ \forall w_{i,j},z_i\in\mathbb{R}_+,\ \forall v_{i,j}\in\mathbf{C}^0([-\Delta,0];\mathbb{R}_+)
\end{aligned}\tag{44}$$

where $\tilde{\alpha}_i, \tilde{\sigma}_i \in \mathcal{K}$ and $\tilde{\sigma}_{ri} \in \mathcal{P}_0$ are

$$\tilde{\alpha}_i(s)=\delta\left(1-\frac{1}{\tau_i}\right)\lambda_i(\underline{\alpha}_i(s))\alpha_i(s) \tag{45}$$

$$\tilde{\sigma}_{i,j}(s)=\begin{cases}\lambda_i(\theta_{i,j}(s))\sigma_{i,j}(s)\\ \qquad\text{if }\lim_{v\to\infty}\alpha_i(v)\ge N_i\tau_i\beta_{i,j}(s)\\ \lim_{v\to\infty}\lambda_i(v)\sigma_{i,j}(s)\quad\text{otherwise}\end{cases} \tag{46}$$

$$\tilde{\sigma}_{ri}(s)=\begin{cases}\lambda_i(\theta_{ri}(s))\sigma_{ri}(s)\\ \qquad\text{if }\lim_{v\to\infty}\alpha_i(v)\ge\tau_{ri}\sigma_{ri}(s)\\ \lim_{v\to\infty}\lambda_i(v)\sigma_{ri}(s)\quad\text{otherwise}\end{cases} \tag{47}$$

$$\begin{aligned}
\theta_{i,j}(s)&=\overline{\alpha}_i\circ\alpha_i^{-1}\circ N_i\tau_i\beta_{i,j}(s),\ j=0,1,...,h+h_d\\
\beta_{i,j}(s)&=\varpi_{i,j}\sigma_{i,j}(s),\qquad j=0,1,...,h\\
\beta_{i,j}(s)&=\varpi_{i,j}\bar{\Delta}_j\sigma_{i,j}(s),\ j=h+1,h+2,...,h+h_d
\end{aligned}\tag{48}$$

$$\theta_{ri}(s)=\overline{\alpha}_i\circ\alpha_i^{-1}\circ\tau_{ri}\sigma_{ri}(s),\quad N_i=\textstyle\sum_{k=0}^{h+h_d}S_{i,k} \tag{49}$$

for any $\delta \in (0,1)$ and $\tau_i, \tau_{ri} \in (1,\infty)$ satisfying $1-\frac{1}{\tau_i}-\frac{1}{\tau_{ri}} \geq \delta\left(1-\frac{1}{\tau_i}\right)$.

Lemma 2: Given (39), (40), (41) and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ satisfying $\underline{\alpha}_i(s) \leq \overline{\alpha}_i(s)$, $\forall s \in \mathbb{R}_+$, for $i=1,2$, suppose that $\tilde{\alpha}_i, \tilde{\sigma}_{i,j}, \beta_{i,j} \in \mathcal{K}$ and $\tilde{\sigma}_{ri} \in \mathcal{P}_0$ are given by (45), (46), (48) and (47), and assume that *(H1)* $\vee$ *(H2)* $\vee$ *(H3)* holds, where

$$\sigma_i(s) = \textstyle\sum_{k=0}^{h+h_d} S_{i,k} \max_{j=0,1,\ldots,h+h_d}\{S_{i,j}\beta_{i,j}(s)\} \quad (50)$$

Let $M_{a,i}, M_{\sigma,3-i,j} : \mathcal{C}_i \to \mathbb{R}_+$ be functionals fulfilling

$$M_{a,1}(\phi_1) \geq M_{\sigma,2,j}(\phi_1),\ \forall\phi_1 \in \mathcal{C}_1,\ j=0,1,...,h+h_d \quad (51)$$
$$M_{a,2}(\phi_2) \geq M_{\sigma,1,j}(\phi_2),\ \forall\phi_2 \in \mathcal{C}_2,\ j=0,1,...,h+h_d \quad (52)$$

for $i=1,2$. If there exist $c_i > 1$, $i=1,2$ such that (18) is satisfied, there exist $\alpha_{cl,i} \in \mathcal{K}$, $i=1,2$, such that

$$\begin{aligned}\sum_{i=1}^{2}\Big\{&-\tilde{\alpha}_i(M_{a,i}(\phi_i))+(1+\epsilon_i)\sum_{j=0}^{h}\varpi_{i,j}S_{i,j}\tilde{\sigma}_{i,j}(M_{\sigma,i,j}(\phi_{3-i}))\\ &+(1+\epsilon_i)\sum_{j=h+1}^{h+h_d}\varpi_{i,j}S_{i,j}\bar{\Delta}_j\tilde{\sigma}_{i,j}(M_{\sigma,i,j}(\phi_{3-i}))+\tilde{\sigma}_{r,i}(|r_i|)\Big\}\\ &\leq \sum_{i=1}^{2}\{-\tilde{\alpha}_{cl,i}(M_{a,i}(\phi_i))+\tilde{\sigma}_{r,i}(|r_i|)\},\\ &\forall\phi_1 \in \mathcal{C}_1,\ r_1 \in \mathbb{R}^{m_1},\ \forall\phi_2 \in \mathcal{C}_2,\ r_2 \in \mathbb{R}^{m_2} \end{aligned} \quad (53)$$

holds for each $\epsilon_i \in [0, c_i-1)$ with

$$\lambda_1(s) = \left[\alpha_2 \circ \hat{\mu}_1^{-1} \circ \frac{1}{\tau_1}\hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right]\left[\frac{1}{\tau_1}\hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s)\right]^{\psi} \quad (54)$$

$$\lambda_2(s) = \frac{k_2}{(k_2-1)}\sqrt{\frac{k_1}{\tau_1}}\left[\hat{\mu}_1 \circ \underline{\alpha}_2^{-1}(s)\right]^{\psi+1} \quad (55)$$

for any k_1, k_2, τ_1, δ, $\psi \in \mathbb{R}$ and $\hat{\mu}_1, \hat{\alpha}_1 \in \mathcal{K}$ satisfying

$$\mu_i(s) = (1+\epsilon_i)\sigma_i(s), \quad k_i > 1, \quad i=1,2 \quad (56)$$
$$0 < \sqrt{\tau_1/k_1} < \delta < 1 \quad (57)$$
$$0 \leq \psi,\ 1 < \tau_1,\ (\tau_1/k_1)^{\psi} \leq (\tau_1-1)(k_2-1) \quad (58)$$
$$\hat{\mu}_1(s)\begin{cases} = \mu_1(s), & \forall s \in \mathbb{R}_+ \quad \text{if } \lim_{s\to\infty}\alpha_2(s) < \infty \\ \geq \mu_1(s), & \forall s \in \mathbb{R}_+ \quad \text{otherwise}\end{cases} \quad (59)$$
$$\hat{\alpha}_1(s)\begin{cases} \leq \alpha_1(s), & \forall s \in \mathbb{R}_+ \quad \text{if } \lim_{s\to\infty}\alpha_2(s) < \infty \\ = \alpha_1(s), & \forall s \in \mathbb{R}_+ \quad \text{otherwise}\end{cases} \quad (60)$$
$$\lim_{s\to\infty}\hat{\alpha}_1(s) \leq \lim_{s\to\infty}\hat{\mu}_1(s) \quad (61)$$
$$k_1\hat{\mu}_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ k_2\mu_2(s) \leq \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s),\ \forall s \in \mathbb{R}_+ \quad (62)$$

Furthermore, $\tilde{\alpha}_{cl,1}, \tilde{\alpha}_{cl,2} \in \mathcal{K}_\infty$ holds if $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

The existence of k_1, k_2, τ_1, δ, $\psi \in \mathbb{R}$ and $\hat{\mu}_1, \hat{\alpha}_1 \in \mathcal{K}$ fulfilling (56)-(62) is guaranteed when (18) and *(H1)* $\vee$ *(H2)* $\vee$ *(H3)* hold. Note that (42) is fulfilled by (54) and (55) with the help of (60)-(61) when *(H1)* $\vee$ *(H2)* $\vee$ *(H3)* holds.

Lemma 3: In the case of $\sigma_{ri}(s) \equiv 0$, $i=1,2$, the assumption *(H1)* $\vee$ *(H2)* $\vee$ *(H3)* in Lemma 2 can be replaced by *(J1)*.

Lemma 4: Given a locally Lipschitz functional $V_i : \mathcal{C}_i \to \mathbb{R}_+$ and a continuous function $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, let $W_i : \mathcal{C}_i \to \mathbb{R}_+$ be a continuous functional defined as $W_i(\phi_i) = \int_0^{V_i(\phi_i)} \lambda_i(s)ds$. Then,

$$D^+W_i(\phi_i, \phi_{3-i}, r_i, t) \leq \lambda_i(V_i(\phi_i))D^+V_i(\phi_i, \phi_{3-i}, r_i, t), \quad (63)$$

where

$$D^+W_i(\phi_i, \phi_{3-i}, r_i, t) = \limsup_{h\to 0^+}\frac{W_i(\phi_i^h) - W_i(\phi_i)}{h}, \quad (64)$$
$$D^+V_i(\phi_i, \phi_{3-i}, r_i, t) = \limsup_{h\to 0^+}\frac{V_i(\phi_i^h) - V_i(\phi_i)}{h}. \quad (65)$$

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