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New Results on Partial Fraction Expansion Based Frequency Weighted Balanced Truncation

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Abstract— **In this paper, we present some new results on frequency weighted balanced truncation technique based on well-known partial-fraction-expansion idea. The reduced order models which are guaranteed to be stable in case of double-sided weighting are obtained by direct truncation. Two sets of simple, elegant and easily calculatable *a priori* error bounds are also derived. A numerical example and comparison with other well-known techniques show the effectiveness of the proposed method.**

I. INTRODUCTION

Enns [3] has presented a scheme for reducing a stable high order model with frequency weighting, based on a modification of balanced truncation [9]. The method, known as frequency weighted balanced truncation, may use input weighting, output weighting, or both. With only one weighting present, stability of the reduced order model is guaranteed. With both weightings present, the method may yield unstable models. A slight modification to the Enns' technique was presented in [13] which not only yields stable models in case of double-sided weightings, but also gives easily computable error bounds. Under certain conditions, this technique is equivalent to Enns' technique.

Another group of methods which is based on partial fraction expansion was originally proposed in [7]. Inspired by this, Al-Saggaf and Franklin [1] proposed a technique for frequency weighted model reduction. In their technique, the numerator of the reduced order model is calculated by forcing the reduction error to have zeros at the poles of the weighting function. The limitations of this method are (i) it can be used with single-sided weighting only, (ii) the output matrix of input weight or input matrix of output weight have to be square and non-singular and (iii) the original system and weighting function need to be strictly proper. Sreeram and Anderson [11] then generalized [1] to include double-sided weightings. However, the method can only handle strictly proper weighting functions. To overcome this, Ghafoor and Sreeram [4] proposed a parametrized method which combines the advantages of the unweighted balancing and Sreeram and Anderson's [11] method. This method can handle both proper and strictly proper weighting functions. Optimal Hankel norm approximation based on partial fraction expansion can be found in [14]. A detailed survey of all the well-known model order reduction and frequency weighted model reduction techniques can be found in $[2, 9]$

In this paper, we present some new results on frequency weighted balanced truncation based on partial-fraction-expansion idea. The method has the following advantages: (i) guaranteed stability of models in case of double-sided weighting, (ii) simple, elegant and easily calculatable error bounds, (iii) applicability to both continuous and discrete systems, and (iv) easily extendable to frequency weighted optimal Hankel norm approximation. Simulation results show that by properly chosing free parameters it is possible to obtain reduced order models with smaller approximation error than other well-known techniques from [3], [12], [13].

II. PRELIMINARIES

This section reviews some well-known frequency weighted balanced model reduction techniques.

A. Frequency Weighted Balanced Truncation Technique

Let the transfer function of the original stable system be given by $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $\{A, B, C, D\}$ is a minimal state-space realization. Let the transfer functions of the stable input and output weights be $V(s) = \left[\begin{array}{c|c} A_V & B_V \\ \hline C_V & D_V \end{array}\right]$ and $W(s) = \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right]$ where $\{A_V, B_V, C_V, D_V\}$ and $\{A_W, B_W, C_W, D_W\}$ are minimal realizations. The state-space realization of the augmented system $W(s)G(s)V(s)$ is given by

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{ccc|c} A_W & B_W C & B_W D C_V & B_W D D_V \\ 0 & A & B C_V & B D_V \\ 0 & 0 & A_V & B_V \\ \hline C_W & D_W C & D_W D C_V & D_W D D_V \end{array}\right] \quad (1)$$

The controllability and observability Gramians of the augmented realization $\left\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\right\}$ are given by

$$\tilde{P} = \begin{bmatrix} P_W & P_{12} & P_{13} \\ P_{12}^T & P & P_{23} \\ P_{13}^T & P_{23}^T & P_V \end{bmatrix} \text{ and } \tilde{Q} = \begin{bmatrix} Q_W & Q_{12} & Q_{13} \\ Q_{12}^T & Q & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_V \end{bmatrix} \quad (2)$$

where P and Q are the frequency weighted controllability and observability Gramians defined by Enns [3] which satisfies the following Lyapunov equations:

$$\tilde{A}\tilde{P} + \tilde{P}\tilde{A}^T + \tilde{B}\tilde{B}^T = 0 \quad (3)$$
$$\tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{C}^T\tilde{C} = 0 \quad (4)$$

Assuming that there are no pole-zero cancellations in $W(s)G(s)V(s)$, the Gramians, $\tilde{P}$ and $\tilde{Q}$ are positive definite.

B. Enns' Method

Expanding the (2,2) blocks of (3) and (4) yield the following equations:

$$AP + PA^T + P_E = 0 \quad (5)$$
$$A^TQ + QA + Q_E = 0 \quad (6)$$

where

$$P_E = BC_VP_{23}^T + P_{23}C_V^TB^T + BD_VB^T \quad (7)$$
$$Q_E = C^TB_W^TQ_{12} + Q_{12}^TB_WC + C^TD_W^TC \quad (8)$$

The Gramians P and Q are then diagonalized simultaneously

$$T_E^{-1}PT_E^{-T} = T_E^TQT_E = \Sigma = diag(\sigma_1, \sigma_2, .., \sigma_n)$$

where $\sigma_1 > \sigma_2 > ... > \sigma_n > 0$ are the frequency weighted Hankel singular values. Transforming and partitioning the original system will yield the following

$$\left[\begin{array}{c|c} T_E^{-1}AT_E & T_E^{-1}B \\ \hline CT_E & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \quad (9)$$

where A_{11} depends on the order of the required truncated model. Hence giving Enns' reduced order model $G_r(s)$

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \quad (10)$$

Essentially, Enns' method is based on diagonalizing simultaneously the solutions of Lyapunov equations as given in equations (5) and (6). However, Enns' method cannot guarantee the stability of reduced order models as P_E and Q_E may not be positive semidefinite. Several modifications to the Enns' technique are proposed in the literature to overcome the stability problem.

C. Sreeram and Anderson's Partial Fraction Expansion based Technique [11]

In Sreeram and Anderson's partial fraction expansion based technique [11], the system matrix in (1) is block diagonalized by $\tilde{T} = \begin{bmatrix} I & -Y & R \\ 0 & I & X \\ 0 & 0 & I \end{bmatrix}$. Note that although the technique [11] was proposed for strictly proper weights and the original system, the derivation presented here is generalized to include proper weights and the original system. Transforming the augmented system of (1) yields

$$\hat{G}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{c|c} \tilde{T}^{-1}\tilde{A}\tilde{T} & \tilde{T}^{-1}\tilde{B} \\ \hline \tilde{C}\tilde{T} & \tilde{D} \end{array}\right] = \left[\begin{array}{ccc|c} A_W & X_{12} & X_{13} & X_1 \\ 0 & A & X_{23} & X_2 \\ 0 & 0 & A_V & B_V \\ \hline C_W & Y_1 & Y_2 & D_WDD_V \end{array}\right]$$

where X, Y and R are obtained by solving the matrix equations:

$$X_{12} = YA - A_WY + B_WC = 0 \quad (11)$$
$$X_{23} = AX - XA_V + BC_V = 0 \quad (12)$$
$$X_{13} = A_WR - RA_V + B_WCX + YAX + B_WDC_V + YBC_V - YXA_V = 0 \quad (13)$$
$$X_1 = B_WDD_V + YBD_V - YXB_V \quad (14)$$
$$X_2 = BD_V - XB_V \quad (15)$$
$$Y_1 = D_WC - C_WY \quad (16)$$
$$Y_2 = D_WCX + D_WDC_V \quad (17)$$

Note that $\hat{D} = \tilde{D} = D_WDD_V$. Using the same similarity transformation $\tilde{T}$, the transformed Gramians of (2) are given by

$$\tilde{T}^{-T}\tilde{P}\tilde{T}^{-1} = \hat{P} = \begin{bmatrix} \hat{P}_W & \hat{P}_{12} & \hat{P}_{13} \\ \hat{P}_{12}^T & \hat{P}_{PF} & \hat{P}_{23} \\ \hat{P}_{13}^T & \hat{P}_{23}^T & \hat{P}_V \end{bmatrix}$$
$$\tilde{T}^T\ \tilde{Q}\ \tilde{T} = \hat{Q} = \begin{bmatrix} \hat{Q}_W & \hat{Q}_{12} & \hat{Q}_{13} \\ \hat{Q}_{12}^T & \hat{Q}_{PF} & \hat{Q}_{23} \\ \hat{Q}_{13}^T & \hat{Q}_{23}^T & \hat{Q}_V \end{bmatrix}$$

In [11], instead of diagonalizing P and Q as in [3], they simultaneously diagonalize $\hat{P}_{PF}$ and $\hat{Q}_{PF}$ as shown below

$$T_{PF}^{-1}\hat{P}_{PF}T_{PF}^{-T} = T_{PF}^T\hat{Q}_{PF}T_{PF} = \Sigma = diag(\sigma_1, \sigma_2, .., \sigma_n)$$

where $\sigma_1 > \sigma_2 > ... > \sigma_n > 0$ and

$$\hat{P}_{PF} = P - P_{23}X^T - XP_{23}^T + XP_VX^T \quad (18)$$
$$\hat{Q}_{PF} = Q - Q_{12}Y - Y^TQ_{12}^T + Y^TQ_WY \quad (19)$$

The diagonalized Gramians $\hat{P}_{PF}$ and $\hat{Q}_{PF}$ satisfy the following Lyapunov equations

$$A\hat{P}_{PF} + \hat{P}_{PF}A^T + X_2X_2^T = 0$$
$$A^T\hat{Q}_{PF} + \hat{Q}_{PF}A + Y_1^TY_1 = 0$$

Since the realization $\{A, X_2, Y_1\}$ is minimal and the diagonalized Gramians satisfy the Lyapunov equations, the partial fraction technique yields stable models in the case of double-sided weighting. Note that the frequency weighted error can be large with this method. However, the error can be reduced for strictly proper original system and the weights ($D = 0$, $D_V = 0$ and $D_W = 0$) if the reduction error is made to have zeros at the poles of the frequency weighting as shown in [11].

D. Ghafoor and Sreeram's Partial Fraction Expansion based Technique [4]

Sreeram and Anderson's [11] method was later generalized by Ghafoor and Sreeram to include proper weights. In this method, a new frequency weighted balanced reduction technique is proposed which is based on parametrized combination of the unweighted technique [9] and the partial fraction expansion technique [11].

Instead of simultaneously diagonalizing $\hat{P}_{PF}$ and $\hat{Q}_{PF}$, P_X and Q_Y

$$\begin{aligned} P_X &= P + \alpha_{PF}^2 \hat{P}_{PF} \\ Q_Y &= Q + \beta_{PF}^2 \hat{Q}_{PF} \end{aligned}$$

are simultaneously diagonalized. In the above equations α_{PF} and β_{PF} are real constants, while P and Q are the unweighted Gramians satisfying

$$\begin{aligned} AP + PA^T + BB^T &= 0 \\ A^TQ + QA + C^TC &= 0 \end{aligned}$$

The Gramians $\hat{P}_{PF}$ and $\hat{Q}_{PF}$ satisfy (18) and (19) respectively while Gramians P_X and Q_Y satisfy

$$\begin{aligned} AP_X + P_XA^T + B_XB_X^T &= 0 \\ A^TQ_Y + Q_YA + C_Y^TC_Y &= 0 \end{aligned}$$

where $B_X = \begin{bmatrix} B & \alpha_{PF}X_2 \end{bmatrix}$ and $C_Y = \begin{bmatrix} C \\ \beta_{PF}Y_1 \end{bmatrix}$ are fictitious input and output matrices.

Remark 1: Although, the method gives lower error, the method is adhoc with no theoretical justification, for simultaneously diagonalizing P_X and Q_Y.

III. Main Result

The proposed method can be explained conceptually. The augmented system $W(s)G(s)V(s)$ is first decomposed using partial fraction expansion to obtain $\hat{W}(s) + \hat{G}(s) + \hat{V}(s)$ where the system matrix is block diagonalized. The decomposed system $\left(\hat{W}(s) + \hat{G}(s) + \hat{V}(s)\right)$ is then expressed as a new augmented system $\bar{W}(s)\bar{G}(s)\bar{V}(s)$ such that the system matrix of $\bar{W}(s)\bar{G}(s)\bar{V}(s)$ is the same as the system matrix of $\hat{W}(s)+\hat{G}(s)+\hat{V}(s)$, i.e., block diagonal. Instead of reducing $\hat{G}(s)$ to $\hat{G}_r(s)$ as in the conventional partial fraction expansion based balanced truncation technique, $\bar{G}(s)$ is reduced to $\bar{G}_r(s)$ using the balanced truncation. Advantage of this apporach is finding the final reduced order model $G_r(s)$ from $\bar{G}_r(s)$ is straightforward unlike the conventional partial fraction approaches [1], [11]. Other advantages include: (i) guaranteed stability in case of double sided weighting, (ii) two sets of easily calculatable *a priori* error bound, (iii) application to both continuous as well as discrete systems, (iv) having a choice of free parameters to reduce the weighted errors and error bounds and (v) easily applicable to controller reduction problems. The only disadvantage of the newly proposed technique is the reduced-order models are variant under similarity transformation like [13].

Theorem 3.1: Given $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, and the input and output weights $V(s) = \left[\begin{array}{c|c} A_V & B_V \\ \hline C_V & D_V \end{array}\right]$, $W(s) = \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right]$, then the new original system and the new weights satisfy the following relationship:

$$W(s)G(s)V(s) = \hat{W}(s) + \hat{G}(s) + \hat{V}(s) = \bar{W}(s)\bar{G}(s)\bar{V}(s)$$

Proof:

$$\begin{aligned}
& W(s)G(s)V(s) \\
= & \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \left[\begin{array}{c|c} A_V & B_V \\ \hline C_V & D_V \end{array}\right] \\
= & \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] \\
= & \left[\begin{array}{ccc|c} A_W & B_WC & B_WDC_V & B_WDD_V \\ 0 & A & BC_V & BD_V \\ 0 & 0 & A_V & B_V \\ \hline C_W & D_WC & D_WDC_V & D_WDD_V \end{array}\right] \\
= & \left[\begin{array}{c|c} \tilde{T}^{-1}\tilde{A}\tilde{T} & \tilde{T}^{-1}\tilde{B} \\ \hline \tilde{C}\tilde{T} & \tilde{D} \end{array}\right] \\
= & \left[\begin{array}{ccc|c} A_W & X_{12} & X_{13} & X_1 \\ 0 & A & X_{23} & X_2 \\ 0 & 0 & A_V & B_V \\ \hline C_W & Y_1 & Y_2 & D_WDD_V \end{array}\right] \\
= & \left[\begin{array}{ccc|c} A_W & 0 & 0 & X_1 \\ 0 & A & 0 & X_2 \\ 0 & 0 & A_V & B_V \\ \hline C_W & Y_1 & Y_2 & D_WDD_V \end{array}\right] \\
= & \hat{W}(s) + \hat{G}(s) + \hat{V}(s) \\
= & \left[\begin{array}{ccc|c} A_W & \bar{B}_W\bar{C} & \bar{B}_W\bar{D}\bar{C}_V & \bar{B}_W\bar{D}\bar{D}_V \\ 0 & A & \bar{B}\bar{C}_V & \bar{B}\bar{D}_V \\ 0 & 0 & A_V & B_V \\ \hline C_W & \bar{D}_W\bar{C} & \bar{D}_W\bar{D}\bar{C}_V & \bar{D}_W\bar{D}\bar{D}_V \end{array}\right] \\
= & \left[\begin{array}{c|c} A_W & \bar{B}_W \\ \hline C_W & \bar{D}_W \end{array}\right] \left[\begin{array}{c|c} A & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] \left[\begin{array}{c|c} A_V & B_V \\ \hline \bar{C}_V & \bar{D}_V \end{array}\right] \\
= & \bar{W}(s)\bar{G}(s)\bar{V}(s)
\end{aligned}$$

In the above equations, X_{12}, X_{23}, X_2 and Y_1 (see equations (11), (12), (15) and (16)) are factorized to obtain:

$$\begin{aligned}
X_{12} &= \begin{bmatrix} B_W & A_W & I \end{bmatrix} \begin{bmatrix} C \\ -Y \\ YA \end{bmatrix} = \bar{B}_W\bar{C} \\
X_{23} &= \begin{bmatrix} B & -X & AX \end{bmatrix} \begin{bmatrix} C_V \\ A_V \\ I \end{bmatrix} = \bar{B}\bar{C}_V \\
X_2 &= \begin{bmatrix} B & -X & AX \end{bmatrix} \begin{bmatrix} D_V \\ B_V \\ 0 \end{bmatrix} = \bar{B}\bar{D}_V
\end{aligned}$$

$$Y_1 = \begin{bmatrix} D_W & C_W & 0 \end{bmatrix} \begin{bmatrix} C \\ -Y \\ YA \end{bmatrix} = \bar{D}_W \bar{C}$$

Similarly, X_{13}, X_1 and Y_2 are factorized to obtain:

$$\begin{aligned} X_{13} &= A_W R - RA_V + B_W CX + YAX + B_W DC_V \\ &\quad + YBC_V - YXA_V \\ &= \begin{bmatrix} B_W & A_W & I \end{bmatrix} \begin{bmatrix} D & 0 & CX \\ 0 & 0 & R \\ YB & -R\text{–}YX & YAX \end{bmatrix} \begin{bmatrix} C_V \\ A_V \\ I \end{bmatrix} \\ &= \bar{B}_W \bar{D} \bar{C}_V \qquad (20) \\ X_1 &= B_W DD_V + YBD_V - YXB_V \\ &= \begin{bmatrix} B_W & A_W & I \end{bmatrix} \begin{bmatrix} D & 0 & CX \\ 0 & 0 & R \\ YB & -R\text{–}YX & YAX \end{bmatrix} \begin{bmatrix} D_V \\ B_V \\ 0 \end{bmatrix} \\ &= \bar{B}_W \bar{D} \bar{D}_V \qquad (21) \\ Y_2 &= D_W CX + D_W DC_V \\ &= \begin{bmatrix} D_W & C_W & 0 \end{bmatrix} \begin{bmatrix} D & 0 & CX \\ 0 & 0 & R \\ YB & -R\text{–}YX & YAX \end{bmatrix} \begin{bmatrix} C_V \\ A_V \\ I \end{bmatrix} \\ &= \bar{D}_W \bar{D} \bar{C}_V \qquad (22) \end{aligned}$$

Theorem 3.2: If $\{A, B, C, D\}$ is stable and minimal, then the new realization $\{A, \bar{B}, \bar{C}, \bar{D}\}$ is also stable and minimal.
Proof: Follows from the stability and minimality of $\{A, B, C, D\}$.

Given the original system $\{A, B, C, D\}$ and the weights $\{A_V, B_V, C_V, D_V\}$ and $\{A_W, B_W, C_W, D_W\}$, the proposed technique is based on balancing the realization $\{A, \bar{B}, \bar{C}\}$. The reduced order models are then obtained by direct truncation.

A. The Generalized Algorithm

1) Given a stable, original minimal realization $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and minimal realizations of the weights $V(s) = \left[\begin{array}{c|c} A_V & B_V \\ \hline C_V & D_V \end{array}\right]$ and $W(s) = \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right]$, compute X and Y

$$\begin{aligned} AX - XA_V + BC_V &= 0 \\ YA - A_W Y + B_W C &= 0 \end{aligned}$$

2) Compute the fictitious input and output matrices $\bar{B}$ and $\bar{C}$

$$\bar{B} = \begin{bmatrix} B & -X & AX \end{bmatrix} \quad \bar{C} = \begin{bmatrix} C \\ -Y \\ YA \end{bmatrix}$$

3) Solve the Lyapunov equations for $\bar{P}$ and $\bar{Q}$

$$\begin{aligned} A\bar{P} + \bar{P}A^T + \bar{B}\bar{B}^T = 0 \\ A^T\bar{Q} + \bar{Q}A + \bar{C}^T\bar{C} = 0 \end{aligned}$$

4) Calculate the transformation matrix T which balances $\{A, \bar{B}, \bar{C}\}$

$$T^{-1}\bar{P}T^{-T} = T^T\bar{Q}T = diag\,(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)$$

where $\bar{\sigma}_i \geq \bar{\sigma}_{i+1}$, $i = 1, 2, \ldots, n-1$

5) Compute the frequency weighted balanced truncation

$$\check{G}(s) = \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right] = \left[\begin{array}{c|c} \check{A} & \check{B} \\ \hline \check{C} & \check{D} \end{array}\right]$$

6) Partition $\{\check{A}, \check{B}, \check{C}\}$ as follows

$$\check{G}(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

$A_{11} \in R^{r\times r}$, $B_1 \in R^{r\times p}$ and $C_1 \in R^{q\times r}$ $r < n$

7) The reduced order model is then given as follows:

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

Theorem 3.3: The reduced order models $\{A_{11}, B_1, C_1, D\}$ obtained using the proposed technique are stable.
Proof: Follows immediately from the unweighted balanced truncation.

IV. Error Bounds

In this section we derive the error bounds for the reduced order models obtained using the proposed technique. To derive the error bounds, we need to establish relationships between the input and output matrices (B and C) and the new fictitious input and output matrices ($\bar{B}$ and $\bar{C}$). Let

$$K^T = \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T \qquad L = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \qquad (23)$$

Then $B = \bar{B}K$ and $C = L\bar{C}$

Theorem 4.1: Let $G(s)$ be a proper, stable transfer function of order n and $W(s)$ and $V(s)$ be proper weighting functions. If $G_r(s)$ is a proper, stable reduced-order model obtained using the proposed technique then the following error bound holds:

$$\|W(s)[G(s) - G_r(s)]V(s)\|_\infty \leq \gamma \sum_{i=r+1}^{n} \sigma_i$$

where $\gamma = 2\|V(s)\|_\infty \|W(s)\|_\infty$
Proof: Partitioning

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} \text{ and } \bar{C} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$$

and substituting $B_1 = \bar{B}_1 K$ and $C_1 = L\bar{C}_1$ we have

$$\begin{aligned} &\|W(s)(G(s) - G_r(s))V(s)\|_\infty \\ = &\|W(s)(C(sI - A)^{-1}B + D \\ &\quad - C_1(sI - A_{11})^{-1}B_1 - D)V(s)\|_\infty \\ = &\|W(s)(L\bar{C}(sI - A)^{-1}\bar{B}K \\ &\quad - L\bar{C}_1(sI - A_{11})^{-1}\bar{B}_1 K)V(s)\|_\infty \\ \leq &\|W(s)L\|_\infty \|(\bar{C}(sI - A)^{-1}\bar{B} \\ &\quad - \bar{C}_1(sI - A_{11})^{-1}\bar{B}_1)\|_\infty \|KV(s)\|_\infty. \\ \leq &\|W(s)\|_\infty \|(\bar{C}(sI - A)^{-1}\bar{B} \\ &\quad - \bar{C}_1(sI - A_{11})^{-1}\bar{B}_1)\|_\infty \|V(s)\|_\infty. \end{aligned}$$

Since $\left[\begin{array}{c|c} A & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ is a balanced realization and $\left[\begin{array}{c|c} A_{11} & \bar{B}_1 \\ \hline \bar{C}_1 & \bar{D} \end{array}\right]$ is its reduced order model, we have from [3], [5]

$$\|\bar{C}(sI - A)^{-1}\bar{B} + D - \bar{C}_1(sI - A_{11})^{-1}\bar{B}_1 - D\|_\infty$$
$$= \|\bar{C}(sI - A)^{-1}\bar{B} - \bar{C}_1(sI - A_{11})^{-1}\bar{B}_1\|_\infty \leq 2\sum_{i=r+1}^{n}\sigma_i$$

Let $k = 2\|W(s)\|_\infty\|V(s)\|_\infty$, then

$$\|W(s)(G(s) - G_r(s))V(s)\|_\infty \leq k\sum_{i=r+1}^{n}\sigma_i.$$

Corollary 1: In the case of input weighting alone, the error bound is given by

$$\|(G(s) - G_r(s))V(s)\|_\infty \leq \gamma_v\sum_{i=r+1}^{n}\sigma_i$$

where $\gamma_v = 2\|V(s)\|_\infty$. Similarly, in the case of output weighting alone, we have

$$\|W(s)(G(s) - G_r(s))\|_\infty \leq \gamma_w\sum_{i=r+1}^{n}\sigma_i$$

where $\gamma_w = 2\|W(s)\|_\infty$.

Remark 2: If the reduced order model $G_r(s)$ is obtained without frequency weighting, then $V(s) = W(s) = I$. The following result of $[3, 5]$ can be obtained easily:

$$\|(G(s) - G_r(s))\|_\infty \quad\leq\quad 2\sum_{i=r+1}^{n}\sigma_i$$

A second set of error bound formulas given in the following theorem can be easily shown:

Theorem 4.2:

$$\|W(s)\left[G(s) - G_r(s)\right]V(s)\|_\infty$$
$$\leq 2\left\|\bar{V}(s)\right\|_\infty\left\|\bar{W}(s)\right\|_\infty\sum_{i=r+1}^{n}\sigma_i$$
$$\|(G(s) - G_r(s))\,V(s)\|_\infty \leq \left\|\bar{V}(s)\right\|_\infty\sum_{i=r+1}^{n}\sigma_i$$
$$\|W(s)\,(G(s) - G_r(s))\|_\infty \leq \left\|\bar{W}(s)\right\|_\infty\sum_{i=r+1}^{n}\sigma_i$$

Proof: The above theorems can be easily proved using the proof of Theorem 3.1 and Theorem 4.1.

A. Limitatons

Even though the frequency weighted errors obtained using the new method are generally lower than Enns' as well as other well known techniques, the technique is realization dependant. For different realization of input and output weights, different reduced order models and weighted approximation errors are obtained. Hence, to obtain the optimum weighted errors, simple transformation αI for the input weight and βI for the output weight are utilized. By varying the scalars α and β, one can easily reduce the weighted approximation errors.

V. Example

Consider the fourth-order system used in [8], [12], [13]

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 5 \\ 1/2 & -3/2 \\ 1 & -5 \\ -1/2 & 1/6 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4/15 & 1 & 0 & 1 \end{bmatrix}$$

with the following input and output weights:

$$V(s) = W(s) \quad = \quad \{-4.5I_2,\ 3I_2,\ 1.5I_2,\ I_2\}$$

where I_2 denotes an identity matrix of 2nd order. Error and error bounds using Example for Enns' method [3], Lin and Chiu's [8], Wang et al's [13] and the proposed method are shown in the table below. It is clear from the table that the proposed technique compares well with the well-known techniques.

VI. Conclusion

An improved frequency weighted balanced truncation based on partial fraction expansion idea is presented. The method has the following advantages : (i) guaranteed stability in case of double-sided weighting (ii) two sets of easily calculatable *a priori* error bounds. The only disadvantage of the technique is it is dependant on the realization of the weights. However, this property can be used to reduce the approximated error by varying the realization of the weights. Choosing a general transformation matrix for the weights to reduce the weighted error is a challenging open problem and is currently under investigation.

The proposed method not only holds good for both continuous and discrete systems but can easily be extended to optimal Hankel norm approximations.

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Table 1 : Error and error bound for double-sided case

	Enns'	Lin and Chiu's	Wang's		New Method			
Order	Error	Error	Error	E_b	α	β	Error	E_b
1	2.1291	2.5744	2.1213	7.2898	8.1	6.7	2.1158	13.922
					8.2	6.9	2.1161	13.488
					8.0	7.0	2.1201	13.580
2	0.2660	0.5607	0.2720	1.4895	180	110	0.3149	0.5290
					90	100	0.3219	0.5686
					147.5	32.5	0.3166	0.6996
3	0.1131	0.1645	0.1151	0.3228	103.9	35.8	0.1081	0.3104
					105.7	35.7	0.1082	0.3092
					105.3	35.3	0.1083	0.3120

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