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Parameter-Dependent Slack Variable Approach for Positivity Check of Polynomials over Hyper-Rectangle

Masayuki Sato

Abstract— **This paper addresses the positivity check of polynomials in which the region of indeterminates is given as a hyper-rectangle. A new tractable quadratically parameter-dependent condition is proposed using Parameter-Dependent Slack Variables (PDSVs) which are up to second-order with respect to the indeterminates. It is proved that our derived condition always holds if the condition via Sum-Of-Squares (SOS) approach holds. In addition, as the polynomials defining the region of the indeterminates always satisfy constraint qualification, our proposed method is proved to be a sufficient condition which asymptotically becomes a necessary and sufficient condition for our addressed problem with increase of the sizes of PDSVs.**

I. INTRODUCTION

Identifying given matrix-valued polynomials as being positive definite is a very important problem for control theory, because it is relevant to recently developed control theory, such as, robust stability/performance analysis, robust controller design, gain-scheduled controller design, etc. some of which are reported in [1]. One of the most effective approaches for this problem is Sum-Of-Squares (SOS) approach in which SOS decomposition technique is applied for given polynomials to identify the non-negativity. Following this approach, Shor has proposed a method for obtaining the global minimum of scalar polynomials [2]. After his proposal, Chesi *et al.* [3], Lasserre [4], and Parrilo [5] independently have shown that SOS decomposition can be cast to Semi-Definite Programming (SDP). Unfortunately, the set of SOS polynomials generally has a gap from the set of non-negative polynomials apart from very simple cases; single indeterminate case, quadratic form with arbitrary many indeterminates case, and quartic form in two indeterminates case [2], [6]. However, Lasserre has proved that all non-SOS polynomials, which are non-negative, can be approximated by other polynomials [4], and they can be approximated by some SOS polynomials with small SOS polynomial perturbations [7], [8], [9]. Scherer and Hol have shown the matrix version of Putinar's SOS representation [10]. After these papers, SOS approach has been known as a very powerful tool for control theory and been applied to a wide variety of control problems, such as nonlinear system analysis, hybrid system analysis, etc., some of which are surveyed in [11], [1]. Recently, some interesting results on the robust stability analysis for polytopic systems via SOS approach have been reported [12], [13], some of which have been proved their necessity for their problems [14], [15].

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M. Sato is with Institute of Space Technology and Aeronautics, Japan Aerospace Exploration Agency, Mitaka, Tokyo 181-0015, Japan sato.masayuki@jaxa.jp

On the non-negativity check of matrix-valued polynomials, it has been proved that SOS approach is equivalent to Slack Variable (SV) approach when the indeterminate region defined by polynomials is unbounded [16]. Further, if the indeterminate region is bounded, then it has been proved that SV approach encompasses SOS approach; that is, SV approach is no more conservative than SOS approach [16]. When the indeterminate region is bounded and given as a convex set, the methods therein give a series of parametrically affine Linear Matrix Inequality (LMI) conditions which combine the coefficients of the polynomials defining the indeterminate region, and they only have to be checked at the vertices of the convex set. However, it is not clear how combining the coefficients of the polynomials which define the indeterminate region works to reduce conservatism compared to SV approach without combining the coefficients, which has been used for some numerical examples in [17], [18] and robust stability/performance analysis in [19], [20]. In other words, if the indeterminate region is bounded and given as a convex set, the relationship between SV approach without combining coefficients of the polynomials defining the indeterminate region and SOS approach combining coefficients of the polynomials has not yet been clarified.

In this paper, we propose a new formulation for the positivity check of scalar polynomials, in which the indeterminate region is given as a hyper-rectangle, via SV approach. Our formulation uses Parameter-Dependent SVs (PDSVs), which generally leads to polynomially parameter-dependent LMI conditions that are hard to check for all possible values of the indeterminates due to lack of convexity. To circumvent this difficulty, we use structured PDSVs which are up to second-order with respect to the indeterminates and derive LMI conditions only with constant and quadratically parameter-dependent terms. Due to this parameter-dependency, the derived LMIs are equivalently converted to parametrically affine LMIs which can be easily solved numerically with some software, such as [21]. In addition, it is proved that our derived LMIs always hold if the condition via SOS approach holds. Further, since the polynomials which define the indeterminate region satisfy the constraint qualification [10] (it is called as Putinar's condition in [22]), the condition via SOS approach always holds with sufficiently high-order monomials in SOS decompositions, which consequently means that our derived condition is a sufficient condition which asymptotically becomes a necessary and sufficient condition for our addressed problem with increase of the

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sizes of PDSVs.

Methods using PDSVs have already been proposed by several researchers, e.g. Löfberg [23], Oliveira *et al.* [24], [25], [26], [27], [28], and Oishi [29], [30], [31]. However, the methods proposed by Oliveira *et al.* and Oishi focus on the proposition of less conservative conditions compared to existing methods and the asymptotic exactness of their methods with increase of numerical complexity using Pólya's theorem [32] or region dividing approach. Thus, the relationship between their methods and other effective methods, such as SOS approach method, has not been clarified. Löfberg [23] has proposed a very similar method to our method; however, the relationship between the proposed method and SOS approach method has not been clarified, as mentioned in the report. Thus, although his method is very effective compared to Parameter-inDependent SV (PiDSV) approach (e.g. [19], [20] and references therein) as demonstrated in the report, the necessity of the proposed method for the problem is not proved. Considering these, the contribution of this paper is summarized as follows: First, a new tractable LMI condition which identifies the positivity of given polynomials is proposed; second, the method is proved to be no more conservative than SOS approach when using same expressions; third and last, the method is proved to be a necessary and sufficient condition for the problem with sufficiently large sized PDSVs. Consequently, this paper clarifies the relationship between SOS approach and SV approach when the indeterminate region is given as a hyper-rectangle; that is, combining the products of SOS polynomials and polynomials which define the indeterminate region in SOS decomposition framework corresponds to adding parameter-dependent terms in SV framework.

Hereafter, $\langle X\rangle$ is the shorthand notation of $X + X^T$, $0_{n,m}$, I_n and $\mathbf{0}$ respectively denote an $n\times m$-dimensional zero matrix, an n-dimensional identity matrix and an appropriately dimensional zero matrix, $\mathcal{R}^{n\times m}$, $\mathcal{S}^n$ and $\mathcal{Q}^n$ respectively denote sets of $n\times m$ dimensional real matrices, $n\times n$ dimensional symmetric real matrices and $n\times n$ dimensional skew-symmetric real matrices, $\mathcal{Z}_+$ denotes the set of non-negative integers, $\otimes$ denotes Kronecker product, and $X^\perp \in \mathcal{R}^{n\times(n-r)}$ denotes a matrix satisfying $XX^\perp = \mathbf{0}$ and ${X^\perp}^T X^\perp > 0$, where $X \in \mathcal{R}^{m\times n}$ and $\mathrm{rank}(X) = r$. For vectors $v_i,\ (i = q,\cdots,r),\ q,r \in \mathcal{Z}_+$, $v_{[q,r]}$ denotes $v_q \otimes\cdots\otimes v_r$ with $v_{[q+1,q]}$ being defined as 1.

II. PRELIMINARIES

In this section, first, our addressed problem is defined, then several PiDSV approachs, one of which is equivalent to SOS approach, for the problem are recalled.

A. Problem Definition

Let us consider a scalar polynomial $f(\theta)$ with k indeterminates $\theta = [\theta_1\ \cdots\ \theta_k]^T$. To define such $f(\theta)$, let us introduce the vector of the power series of θ_i $(i = 1,\cdots,k)$ ranging from 0-th to m_i-th as

$$\breve{\theta}_i = \left[\theta_i^0\ \cdots\ \theta_i^{m_i}\right]^T \in \mathcal{R}^{\sigma_i},$$

where $\sigma_i = m_i+1$ $(i = 1,\cdots,k)$. Without loss of generality, it is assumed that the lowest degrees of θ_i are zeros in this paper.

Define the vector of all monomials obtained as products of all θ_i^j elements with $i = 1,\cdots,k,\ j = 0,\cdots,m_i$:

$$\breve{\theta} = \breve{\theta}_{[1,k]} \in \mathcal{R}^{\sigma},$$

where $\sigma = \sigma(1,k)$, and $\sigma(q,r) = \Pi_{i=q}^{r}\sigma_i$. By definition, let $\sigma(1,0) = 1$ and $\sigma(k+1,k) = 1$.

Using these definitions, all polynomials with k indeterminates, which have monomials obtained as products of all θ_i^j elements with $i = 1,\cdots,k$ and $j = 0,1,\cdots,2m_i$, can be expressed as follows:

$$f(\theta) = \breve{\theta}^T\hat{f}\breve{\theta} \tag{1}$$

with an appropriately defined $\hat{f} \in \mathcal{S}^\sigma$.

Remark 1: The matrix $\hat{f}$ is to be symmetric; however, the expression of $\hat{f}$ for $f(\theta)$ is not unique. Similarly, the expression of $\breve{\theta}$ is neither unique. For example, we can use arbitrary positively high-order indeterminate vector $\breve{\theta}$ to express positively low-order $f(\theta)$, similarly in [16], [17], [18].

In this paper, the region of the indeterminates θ is assumed to be a hyper-rectangle; that is, the region is defined as follows:

$$\Omega = \left\{\theta \in \mathcal{R}^k : |\theta_i| \le \delta_i\ (i = 1,\cdots,k)\right\},$$

where $\delta_i(\ge 0)$ are given *a priori*. Alternatively, this set is expressed as follows:

$$\Omega = \left\{\theta \in \mathcal{R}^k : g_i(\theta) = \delta_i^2 - \theta_i^2 \ge 0\ (i = 1,\cdots,k)\right\}. \tag{2}$$

Let us define vector $\theta^{[2]}$ and set $\Omega^{[2]}$ as follows:

$$\theta^{[2]} = \left[\theta_1^2\ \cdots\ \theta_k^2\right]^T,$$
$$\Omega^{[2]} = \left\{\theta^{[2]} \in \mathcal{R}^k : 0 \le \theta_i^2 \le \delta_i^2\ (i = 1,\cdots,k)\right\},$$

that is, $\Omega^{[2]}$ denotes the region of indeterminate vector $\theta^{[2]}$.

Now we are ready to pose our problem.

Problem 1: For a given polynomial $f(\theta)$ in (1), identify whether or not the polynomial is positive over Ω.

B. PiDSV Approach

We recall PiDSV approach for Problem 1.

To describe the PiDSV approach for Problem 1, first, let us define several notations.

Define the following matrices.

$$\eta_i(\theta_i) = \theta_i\eta_i^{[\infty]} + \eta_i^{[0]}$$
$$\eta_i^{[\infty]} = \begin{bmatrix} I_{m_i} \\ 0_{1,m_i}\end{bmatrix} \in \mathcal{R}^{\sigma_i\times m_i}$$
$$\eta_i^{[0]} = -\begin{bmatrix} 0_{1,m_i} \\ I_{m_i}\end{bmatrix} \in \mathcal{R}^{\sigma_i\times m_i}$$
$$\Psi_i(\theta_i) = \theta_i\Psi_i^{[\infty]} + \Psi_i^{[0]}$$
$$\Psi_i^{[\infty]} = I_{\sigma(1,i-1)} \otimes \eta_i^{[\infty]} \otimes I_{\sigma(i+1,k)} \in \mathcal{R}^{\sigma\times\pi_i}$$
$$\Psi_i^{[0]} = I_{\sigma(1,i-1)} \otimes \eta_i^{[0]} \otimes I_{\sigma(i+1,k)} \in \mathcal{R}^{\sigma\times\pi_i}$$

Here, π_i denotes $\sigma m_i/\sigma_i$. Note that matrix $\eta_i(\theta)$ satisifies $(\eta_i(\theta_i)^T)^\perp = \breve{\theta}_i$ and $\breve{\theta}^T\Psi_i(\theta_i) = \mathbf{0}$.

PiDSV approach without combining the polynomials $g_i(\theta)$ for Problem 1 is described as follows.

Lemma 1: If there exist a positive number ε, matrices $N_i \in \mathcal{R}^{\pi_i \times \sigma}$ $(i = 1, \cdots, k)$ such that (3) holds for all the vertices of Ω, then $f(\theta)$ is positive over Ω.

$$\hat{f} + \left\langle \sum_{i=1}^{k} \Psi_i(\theta_i) N_i \right\rangle - \begin{bmatrix} \varepsilon\, \mathbf{0} \\ \mathbf{0}\, \mathbf{0} \end{bmatrix} \geq 0 \tag{3}$$

Note that this lemma is a particular case of Theorem 1 in [16] setting matrices $\hat{H}_q^{[p]}$ be zeros. Thus, the proof is omitted here. In the next section, this lemma will be extended for the use of PDSVs (i.e. $N_i(\theta)$) instead of PiDSVs (i.e. N_i).

At the last of this section, we show a PiDSV approach which has been proved to be equivalent to a method using SOS decomposition technique.

We first recall Theorem 2 in [16] for scalar polynomials. To do that, polynomial constraints $g_i(\theta)$ are assumed to be expressed as $\breve{\theta}^T \hat{g}_i \breve{\theta}$ with some matrices $\hat{g}_i \in \mathcal{S}^\sigma$. Then Theorem 2 in [16] is given as follows with a slight revision.

Lemma 2 (Theorem 2 in [16]): $f(\theta)$ is positive over Ω if there exist a positive number ε, k scalars h_i and k skew-symmetric matrices $\hat{N}_i \in \mathcal{Q}^{\pi_i}$ such that (4) and (5) hold.

$$\hat{f} + \left\langle \sum_{i=1}^{k} \Psi_i^{[0]} \hat{N}_i \Psi_i^{[\infty]^T} \right\rangle - \begin{bmatrix} \varepsilon\, \mathbf{0} \\ \mathbf{0}\, \mathbf{0} \end{bmatrix} \geq \sum_{i=1}^{k} \hat{g}_i \otimes h_i \tag{4}$$

$$h_i \geq 0, i = 1, \cdots, k \tag{5}$$

Remark 2: Note that $\left\langle \Psi_i^{[0]} \hat{N}_i \Psi_i^{[\infty]^T} \right\rangle$ is equivalent to $\left\langle \Psi_i(\theta) \hat{N}_i \Psi_i^{[\infty]^T} \right\rangle$ because of $\hat{N}_i \in \mathcal{Q}^{\pi_i}$.

Remark 3: Considering Lemma 1 in [16], the condition of this lemma is a necessary and sufficient condition for $f(\theta) - \varepsilon - \sum_{i=1}^{k} g_i(\theta) h_i$ to be SOS.

As suggested in [16], it is possible to extend h_i be SOS polynomials. To do this extension, we make some more preliminaries. Let us assume that $m_i \geq 2$. Although this assumption generally increases the size of $\hat{f}$ as well as $\hat{N}_i$, this can be done without loss of generality. If polynomial constraints $g_i(\theta)$ is given as $g_i(\theta) = \breve{\theta}^T \hat{g}_i$, where $\hat{g}_i$ is defined as

$$e_{[1,i-1]} \otimes \begin{bmatrix} \delta_i^2 \\ 0 \\ -1 \\ 0_{m_i-2,1} \end{bmatrix} \otimes e_{[i+1,k]},\ e_i = \begin{bmatrix} 1 \\ 0_{m_i,1} \end{bmatrix},$$

and SOS polynomials $h_i(\theta)$ are set as $h_i(\theta) = \hat{h}_i^T \breve{\theta}$ with some vectors $\hat{h}_i \in \mathcal{R}^\sigma$, then Lemma 2 is revised as follows.

Lemma 3: $f(\theta)$ is positive over Ω if there exist a positive number ε, k vectors $\hat{h}_i \in \mathcal{R}^\sigma$, k skew-symmetric matrices $\hat{N}_i \in \mathcal{Q}^{\pi_i}$, and k^2 skew-symmetric matrics $\bar{N}_{j_i} \in \mathcal{Q}^{\pi_i}$ such that (6) and (7) hold.

$$\hat{f} + \left\langle \sum_{i=1}^{k} \Psi_i^{[0]} \hat{N}_i \Psi_i^{[\infty]^T} \right\rangle - \begin{bmatrix} \varepsilon\, \mathbf{0} \\ \mathbf{0}\, \mathbf{0} \end{bmatrix} \geq \left\langle \sum_{i=1}^{k} \hat{g}_i \hat{h}_i^T \right\rangle \tag{6}$$

$$\left\langle e \hat{h}_j^T \right\rangle + \left\langle \sum_{i=1}^{k} \Psi_i^{[0]} \bar{N}_{j_i} \Psi_i^{[\infty]^T} \right\rangle \geq 0,\ j = 1, \cdots, k \tag{7}$$

where e is defined as $e_{[1,k]}$.

Remark 4: Considering Lemma 1 in [16], Lemma 3 is equivalent to the following.

$$f(\theta) - \varepsilon - 2\sum_{i=1}^{k} g_i(\theta) h_i(\theta) = \text{SOS}$$
$$2h_i(\theta) = \text{SOS}$$

Remark 4 implies that Lemma 3 is an extention of Theorem 2 in [16] for the case in which SOS polynomials $h_i(\theta)$ are used instead of constants h_i.

Lemma 3 will be used in the next section to demonstrate that our new condition for Problem 1 is no more conservative than SOS approach (i.e. Lemma 3).

III. MAIN RESULTS

In this section, we first show a new tractable sufficient condition for Problem 1 using PDSVs, whose entries are up to second-order with respect to the indeterminates. Next, it is shown that the new condition is a necessary condition for Lemma 3. Finally, we show that the new condition is a sufficient condition which becomes a necessary and sufficient condition for Problem 1 with increase of the sizes of PDSVs.

A. PDSV Approach

A new condition for Problem 1 using PDSVs is given as follows.

Theorem 1: If there exist a positive number ε, k skew-symmetric matrices $\hat{M}_i \in \mathcal{Q}^{\pi_i}$, k^2 skew-symmetric matrices $\bar{M}_{i_j} \in \mathcal{Q}^{\pi_i}$ and k vectors $\hat{l}_i \in \mathcal{R}^\sigma$ such that (8) holds for all the vertices of $\Omega^{[2]}$, then $f(\theta)$ is positive over Ω.

$$\hat{f} + \left\langle \sum_{i=1}^{k} \Psi_i(\theta_i) M_i(\theta) \right\rangle - \begin{bmatrix} \varepsilon\, \mathbf{0} \\ \mathbf{0}\, \mathbf{0} \end{bmatrix} \geq 0, \tag{8}$$

where $M_i(\theta)$ $(i = 1, \cdots, k)$ are given as

$$\begin{aligned} M_i(\theta) = {} & \hat{M}_i \Psi_i^{[\infty]^T} + \left(\textstyle\sum_{j=1}^{k} \left(\delta_j^2 - \theta_j^2\right) \bar{M}_{i_j} \right) \Psi_i^{[\infty]^T} \\ & + \left(e_{[1,i-1]} \otimes \begin{bmatrix} \theta_i \\ 1 \\ 0_{m_i-2} \end{bmatrix} \otimes e_{[i+1,k]} \right) \hat{l}_i^T. \end{aligned} \tag{9}$$

Proof: We first show that (8) has only constant and quadratically parameter-dependent terms. Considering that $\hat{M}_i, \bar{M}_{i_j} \in \mathcal{Q}^{\pi_i}$, the followings hold (see Remark 2).

$$\left\langle \Psi_i(\theta_i) \hat{M}_i \Psi_i^{[\infty]^T} \right\rangle = \left\langle \Psi^{[0]} \hat{M}_i \Psi_i^{[\infty]^T} \right\rangle$$

$$\left\langle \Psi_i(\theta_i) \left(\sum_{j=1}^{k} (\delta_j^2 - \theta_j^2) \bar{M}_{i_j} \right) \Psi_i^{[\infty]^T} \right\rangle = \sum_{j=1}^{k} \left((\delta_j^2 - \theta_j^2) \left\langle \Psi_i^{[0]} \bar{M}_{i_j} \Psi_i^{[\infty]^T} \right\rangle \right)$$

That is, multiplications of $\Psi_i(\theta_i)$ and $\hat{M}_i \Psi_i^{[\infty]^T}$, $\sum_{j=1}^k (\delta_j^2 - \theta_j^2) \bar{M}_{i_j} \Psi_i^{[\infty]^T}$ produce only constant and quadratically parameter-dependent terms. Next, consider the remaining term, i.e. $\left\langle \Psi_i(\theta_i) \left(e_{[1,i-1]} \otimes \begin{bmatrix} \theta_i \\ 1 \\ \mathbf{0} \end{bmatrix} \otimes e_{[i+1,k]} \right) \hat{l}_i^T \right\rangle$. Some direct algebraic calculations give the following.

$$\left\langle \left(e_{[1,i-1]} \otimes \begin{bmatrix} \theta_i^2 \\ 0 \\ -1 \\ 0_{m_i-2} \end{bmatrix} \otimes e_{[i+1,k]} \right) \hat{l}_i^T \right\rangle$$

Thus, it is proved that inequality (8) has only constant and quadratically parameter-dependent terms.

Next, we show that if (8) holds at all the vertices of $\Omega^{[2]}$ then $f(\theta)$ is positive over Ω. After θ_i^2 is set as a new indeterminate $\zeta_i \in [0, \delta_i^2]$, (8) becomes affine with respect to new indeterminates ζ_i $(i = 1, \cdots, k)$. Therefore, if (8) holds at all the vertices of $\Omega^{[2]}$, i.e. the vertices of the existence region of $\zeta = [\zeta_1 \ \cdots \ \zeta_k]^T$, then (8) holds for all $\theta \in \Omega$. After multiplying $\breve{\theta}^T$ and its transpose to (8) from the left and the right respectively leads to $\breve{\theta}^T \hat{f} \breve{\theta} - \varepsilon \geq 0$; that is, $f(\theta)$ is identified to be positive over Ω. This completes the proof. ∎

Remark 5: As pointed out in the proof of Theorem 1, the proposed quadratically parameter-dependent condition (8) is equivalently converted to a parametrically affine LMI.

B. Connection to Lemma 3

In this subsection, we show that the condition of Theorem 1 is a necessary condition for that of Lemma 3. That is, we claim the following.

Theorem 2: If there exist a positive number ε, k matrices $\hat{N}_i \in \mathcal{Q}^{\pi_i}$, k^2 matrices $\bar{N}_{j_i} \in \mathcal{Q}^{\pi_i}$, and k vectors $\hat{h}_i$ such that (6) and (7) hold, then there always exist k matrices $\hat{M}_i \in \mathcal{Q}^{\pi_i}$, k^2 matrices $\bar{M}_{i_j} \in \mathcal{Q}^{\pi_i}$, and k vectors $\hat{l}_i$ such that (8) holds for all the vertices of $\Omega^{[2]}$ with the same ε.

Proof: From the assumption, suppose that there exist a positive number ε, k skew-symmetric matrices $\hat{N}_i$, k^2 skew-symmetric matrices $\bar{N}_{j_i}$, and k vectors $\hat{h}_i$ such that (6) and (7) hold.

As $\delta_j^2 - \theta_j^2 \geq 0$ $(j = 1, \cdots, k)$ hold for all the vertices of $\Omega^{[2]}$ and (7) holds, the following inequalities hold for all the vertices of $\Omega^{[2]}$.

$$\left(\delta_j^2 - \theta_j^2\right) \left(\left\langle e h_j^T \right\rangle + \left\langle \sum_{i=1}^{k} \Psi_i^{[0]} \bar{N}_{j_i} \Psi_i^{[\infty]^T} \right\rangle \right) \geq 0 \quad (10)$$

After summing up (6) and all (10), the following inequality holds for all the vertices of $\Omega^{[2]}$.

$$\hat{f} + \left\langle \sum_{i=1}^k \Psi_i^{[0]} \hat{N}_i \Psi_i^{[\infty]^T} \right\rangle - \begin{bmatrix} \varepsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \sum_{j=1}^k \left\langle \left(\hat{g}_j - \left(\delta_j^2 - \theta_j^2\right) e \right) \hat{h}_j^T \right\rangle + \sum_{j=1}^k \left(\left(\delta_j^2 - \theta_j^2\right) \left\langle \sum_{i=1}^k \Psi_i^{[0]} \bar{N}_{j_i} \Psi_i^{[\infty]^T} \right\rangle \right) \geq 0 \quad (11)$$

Considering that e is defined as $e_{[1,j-1]} \otimes e_j \otimes e_{[j+1,k]}$, the fourth term can be expressed as follows:

$$\left\langle \left(\hat{g}_j - \left(\delta_j^2 - \theta_j^2\right) e \right) \hat{h}_j^T \right\rangle = \left\langle \left(e_{1,[j-1]} \otimes \begin{bmatrix} \theta_j^2 \\ 0 \\ -1 \\ 0_{m_j-2,1} \end{bmatrix} \otimes e_{[j+1,k]} \right) \hat{h}_j^T \right\rangle.$$

Thus, the following inequality holds for all the vertices of $\Omega^{[2]}$.

$$\hat{f} + \left\langle \sum_{i=1}^k \Psi_i^{[0]} \hat{N}_i \Psi_i^{[\infty]^T} \right\rangle - \begin{bmatrix} \varepsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \sum_{j=1}^k \left\langle \left(e_{1,[j-1]} \otimes \begin{bmatrix} \theta_j^2 \\ 0 \\ -1 \\ 0_{m_j-2,1} \end{bmatrix} \otimes e_{[j+1,k]} \right) \hat{h}_j^T \right\rangle + \sum_{j=1}^k \left(\left(\delta_j^2 - \theta_j^2\right) \left\langle \sum_{i=1}^k \Psi_i^{[0]} \bar{N}_{j_i} \Psi_i^{[\infty]^T} \right\rangle \right) \geq 0 \quad (12)$$

The condition (12) is the same as (8) after setting $\hat{h}_j$, $\hat{N}_i$, and $\bar{N}_{j_i}$ as $\hat{l}_j$, $\hat{M}_i$, and $\bar{M}_{i_j}$ respectively. Thus, if inequalities (12) holds, then the condition of Theorem 1 always holds. This completes the proof. ∎

Theorem 2 shows that if the condition of Lemma 3, which is a necessary and sufficient condition for $f(\theta) - 2\sum_{i=1}^k g_i(\theta) h_i(\theta)$ and $h_i(\theta)$ be SOS polynomials, then the condition of Theorem 1 always holds. In other words, if the expressions for $f(\theta)$ and $g_i(\theta)$ are respectively given as in (1) and $\breve{\theta}^T \hat{g}_i$, and the formulation for SOS decomposition is set as in Remark 4, then PDSV approach (i.e. Theorem 1) is no more conservative than SOS approach (i.e. Lemma 3).

From Theorem 2, we obtain the following interpretation for combining $g_i(\theta)$ and the associated SOS polynomials $h_i(\theta)$ in SOS approach: Combining products of these polynomials in SOS approach corresponds to adding parameter-dependent terms in SV framework. Although this is neither new contribution to SOS approach nor to SV approach, it has not been clarified before; that is, Theorem 2 clarifies the relationship of those two approaches.

C. Necessity of Theorem 1

Finally, we give the last contribution of this paper. That is, we claim the following.

Theorem 3: If the polynomial $f(\theta)$ in (1) is positive over Ω, then for sufficiently large m_i $(i = 1, \cdots, k)$ there exist a positive number ε, k skew-symmetric matrices $\hat{M}_i \in \mathcal{Q}^{\pi_i}$, k^2 skew-symmetric matrices $\bar{M}_{i_j} \in \mathcal{Q}^{\pi_i}$ and k vectors $\hat{l}_i$ such that (8) holds for all the vertices of $\Omega^{[2]}$.

To prove this theorem, we recall a lemma which describes a necessary condition for Problem 1.

Lemma 4 (Theorem 2 in [10]): Suppose that the following constraint qualification holds true: There exist some $r \in \mathcal{R}$ and some SOS polynomials $\phi_i(\theta)$ $(i = 0, 1, \cdots, k)$ such that

$$r^2 - \sum_{i=1}^{k} \theta_i^2 = \phi_0(\theta) + \sum_{i=1}^{k} \phi_i(\theta) g_i(\theta). \quad (13)$$

Under this hypothesis, if the polynomial $f(\theta)$ is positive over Ω, then there exist $\varepsilon > 0$ and SOS polynomials $s_0(\theta), s_1(\theta), \cdots, s_k(\theta)$ such that

$$f(\theta) - 2 \sum_{i=1}^{k} g_i(\theta) s_i(\theta) = s_0(\theta) + \varepsilon. \quad (14)$$

Remark 6: Lemma 4 shows only its necessity for Problem 1. However, the described condition is also a sufficient condition for the problem, because SOS polynomials are non-negative for all θ.

Remark 7: Constraint qualification condition (13) always holds for $g_i(\theta)$ in (2). In particular, let $\phi_i(\theta)$ $(i = 0, 1, \cdots, k)$ be set all 1. Then inequality (13) holds with $r = \sqrt{1 + \sum_{i=1}^{k} \delta_i^2}$.

We are ready to give the proof of Theorem 3.

Proof: [Proof of Theorem 3] Suppose that $f(\theta)$ is positive over Ω. Then, from Lemma 4, Lemma 3, and Remark 4, for sufficiently large m_i $(i = 1, \cdots, k)$, there exist a positive number ε, k skew-symmetric matrices $\hat{N}_i$, k^2 skew-symmetric matrices $\bar{N}_{j_i}$, and k vectors $\hat{h}_i \in \mathcal{R}^{\sigma}$ such that (6) and (7) hold. From Theorem 2, for the same m_i $(i = 1, \cdots, k)$ and the same ε, there always exist k skew-symmetric matrices $\hat{M}_i$, k^2 skew-symmetric matrices $\bar{M}_{i_j}$ and k vectors $\hat{l}_i$ such that (8) holds for all the vertices of $\Omega^{[2]}$. This completes the proof. ∎

Theorem 3 shows that Theorem 1 is a sufficient condition which becomes a necessary and sufficient condition for Problem 1 when monomials in $\breve{\theta}$ are sufficiently high-order, which consequently means the increase of the sizes of PDSVs, i.e. the sizes of $\hat{M}_i$, $\bar{M}_{i_j}$ and $\hat{l}_i$.

IV. NUMERICAL EXAMPLE

Let us consider the following Motzkin form which is borrowed from [33].

$$f(\theta) = \theta_1^2 \theta_2^4 + \theta_1^4 \theta_2^2 + 1 - 3\theta_1^2 \theta_2^2. \quad (15)$$

This polynomial function is confirmed to be non-negative; that is, the global minimum 0 is obtained at $(\theta_1^2, \theta_2^2) = (1, 1)$; however, it is not SOS. For this polynomial, we apply Lemma 3 and Theorem 1 with various m_i and various indeterminate region $(\theta_1, \theta_2) \in [-\delta,\ \delta]^2$ setting $\varepsilon = 10^{-9}$. The assured lower bounds of $f(\theta)$ are given in Tables I, II, and III.

Table I shows that both methods, i.e. Lemma3 and Theorem 1, almost give the exact analysis regardless of m_i. On the other hand, Tables II and III indicate that setting small m_i does not give the exact analysis for large hyper-rectangles. In addition, Tables II and III show that Theorem 1 gives more precise analysis than Lemma 3, which indicates that Theorem 2 holds; that is, even when Lemma 3 does not hold Theorem 1 may hold.

TABLE I
ASSURED LOWER BOUNDS OF $f(\theta)$ IN (15) WITH LEMMA 3 AND THEOREM 1 SETTING $\delta = 1$

m_i $(i = 1, 2)$	Lemma 3	Theorem 1
2	-0.026	0.000
3	-0.024	0.000
4	0.000	0.000
5	0.000	0.000

TABLE II
ASSURED LOWER BOUNDS OF $f(\theta)$ IN (15) WITH LEMMA 3 AND THEOREM 1 SETTING $\delta = 5$

m_i $(i = 1, 2)$	Lemma 3	Theorem 1
2	-13.486	-2.864
3	-12.491	-2.292
4	-0.140	-0.072
5	-0.004	0.000

TABLE III
ASSURED LOWER BOUNDS OF $f(\theta)$ IN (15) WITH LEMMA 3 AND THEOREM 1 SETTING $\delta = 10$

m_i $(i = 1, 2)$	Lemma 3	Theorem 1
2	-55.661	-13.386
3	-47.381	-9.846
4	-1.383	-0.463
5	-0.016	-0.005

V. CONCLUSIONS

On the positivity check of polynomials in which the indeterminate region is given as a hyper-rectangle, a new tractable sufficient condition is proposed via Parameter-Dependent Slack Variable (PDSV) approach. In our method, the parameter-dependency of PDSVs is structurally restricted, which leads to that our derived condition has only constant and quadratically parameter-dependent terms. This parameter-dependency equivalently converts the derived condition to a parametrically affine LMI condition, which can be easily checked using some software for SDP problems without introducing any conservatism. It is also proved that the derived condition always holds if the condition via SOS approach holds. In addition, since the SOS approach with sufficiently high-order monomials is a necessary and sufficient condition for the addressed problem, our derived condition is proved to be a sufficient condition which becomes a necessary and sufficient condition for the addressed problem with increase of the sizes of PDSVs.

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