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Singular value decomposition for a class of linear time-varying systems and its application to switched linear systems

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Abstract— **This paper deals with singular value decomposition (SVD) for a class of linear time-varying systems. The class considered here describes switched linear systems with periodic switching. Based on an appropriate input-output description, the calculation method of singular values and singular vectors is derived. The SVD enables us to characterize the dominant input–output signals using singular vectors, which form orthogonal systems in input and output spaces. Then SVD is applied to switched linear systems to improve the transient response. A numerical example is provided to demonstrate the proposed method.**

I. INTRODUCTION

For linear dynamical systems, singular value decomposition (SVD) plays an important role in analysis and control. Various studies of the subject have been made to date. For example, model reduction methods for finite-dimensional linear systems have been developed [1](Chaps. 7–8) and the finite-dimensional approximation problem of a class of infinite-dimensional systems has been considered in [2]. In [3], a compensation signal design method for improving the transient response of linear systems has been derived based on the SVD for linear dynamical systems. For constrained systems, the compensation signal design problem has been reported in [4]. Model predictive control for constrained continuous-time linear systems has also been developed in [5].

In this paper, we first consider the SVD of a Hankel-like operator describing the input-output relation of a class of linear time-varying systems and derive a method of calculating singular values and singular vectors. The class of systems we consider here represents switched linear systems with periodic switching. The SVD provides orthogonal input and output sequences that enable us to approximate the original infinite-dimensional input and output spaces using a finite number of singular vectors. Then, we specifically examine a switched linear system and consider the compensation signal design problem using the newly established SVD for switched linear systems to improve the transient response. The compensation signal design we consider in this paper is based on a feedforward method and the resulting compensation input over the entire time interval of interest is computed off-line using the desired and uncompensated responses.

The paper is organized as follows. In section II, the SVD of the Hankel-like operator for a linear time-varying system is considered and the calculation method of singular values and singular vectors is derived. In section III, the obtained SVD is applied to a switched linear system with a periodic switching law and the compensation law for improving the transient response is considered. A numerical example is given in section IV to illustrate the fundamental properties of the proposed method.

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II. SINGULAR VALUE DECOMPOSITION FOR A LINEAR TIME-VARYING SYSTEM

Consider a class of linear time-varying systems defined over a finite horizon $[0, h]$:

$$\Sigma : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)v(t), \ x(0) = 0 \\ z(t) = E(t)x(t) \end{cases} \tag{1}$$

$$(A(t), B(t), E(t)) := \begin{cases} (A_1, B_1, E_1) & 0 \le t < t_1 \\ (A_2, B_2, E_2) & t_1 \le t < t_2 \\ \vdots & \vdots \\ (A_N, B_N, E_N) & t_{N-1} \le t \le t_N \end{cases}$$

$$0 < t_1 < t_2 < \cdots < t_N = h$$

where $x(t) \in \mathbb{R}^{n_x}$, $v(t) \in \mathbb{R}^{n_v}$, $z(t) \in \mathbb{R}^{n_z}$ denote the state, input, and output, respectively. This system represents a time-dependent switched linear system. Switched linear systems with the periodic switching law can be described by this particular form (additional details are presented in section III). In this section, starting with introduction of appropriate generalized input and output spaces, the singular value decomposition for this system will be derived and used for the transient improvement of switched linear systems.

First, define Hilbert spaces $\mathcal{V} := L_2(0, h; \mathbb{R}^{n_v})$ and $\mathcal{Z} := \mathbb{R}^{n_x} \times L_2(0, h; \mathbb{R}^{n_z})$ with the inner products

$$\langle f_1, f_2 \rangle_{\mathcal{V}} := \int_0^h f_1^{\mathrm{T}}(\beta) f_2(\beta) d\beta, \qquad f_1, f_2 \in \mathcal{V}, \tag{2}$$

$$\langle g_1, g_2 \rangle_{\mathcal{Z}} := g_1^{0^{\mathrm{T}}} g_2^0 + \int_0^h g_1^{1^{\mathrm{T}}}(\beta) g_2^1(\beta) d\beta,$$

$$g_1 = \begin{bmatrix} g_1^0 \\ g_1^1 \end{bmatrix}, g_2 = \begin{bmatrix} g_2^0 \\ g_2^1 \end{bmatrix} \in \mathcal{Z} \tag{3}$$

and denote the input and output in $\mathcal{V}, \mathcal{Z}$ as

$$v \in \mathcal{V}, \ \hat{z} := \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in \mathcal{Z}, \tag{4}$$

$$z^0 := Fx(h), \ F \in \mathbb{R}^{n_x \times n_x}, \tag{5}$$

$$z^1(t) := z(t), \ 0 \le t \le t_N = h \tag{6}$$

where F is an appropriate weighting matrix for the terminal state. The relation between v and $\hat{z}$ is written by a linear operator $\Gamma \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$:

$$\hat{z} = \Gamma v, \quad v \in \mathcal{V}, \ \hat{z} \in \mathcal{Z}, \tag{7}$$

$$\Gamma v := \begin{bmatrix} (\Gamma v)^0 \\ (\Gamma v)^1 \end{bmatrix},$$

$$\begin{aligned}(\Gamma v)^0 =& F \sum_{s=1}^{N-1} \Phi_{(N,s)}(h) \int_{t_{s-1}}^{t_s} e^{A_s(t_s-\tau)} B_s v(\tau) d\tau \\ &+ F \int_{t_{N-1}}^{h} e^{A_N(h-\tau)} B_N v(\tau) d\tau, \end{aligned} \tag{8}$$

$$\begin{aligned}&(\Gamma v)^1_{[t_k, t_{k+1}]}(t) \\ &= E_{k+1} \sum_{s=1}^{k} \Phi_{(k+1,s)}(t) \int_{t_{s-1}}^{t_s} e^{A_s(t_s-\tau)} B_s v(\tau) d\tau \\ &\quad + E_{k+1} \int_{t_k}^{t} e^{A_{k+1}(t-\tau)} B_{k+1} v(\tau) d\tau, \end{aligned} \tag{9}$$

$$t_k \le t \le t_{k+1}, \ k = 0, \dots, N-1,$$

$$\Phi_{(\ell,m)}(t) := e^{A_\ell(t-t_{\ell-1})} e^{A_{\ell-1}(t_{\ell-1}-t_{\ell-2})} \cdots e^{A_{m+1}(t_{m+1}-t_m)}, \tag{10}$$

$$\ell, m \in \mathbb{Z}^+ : \ell > m \ge 0,$$

$$t_{\ell-1} \le t \le t_\ell.$$

For the operator Γ, we consider the following singular value problem:

$$\Gamma f = \sigma g, \ \Gamma^* g = \sigma f, \tag{11}$$

$$\sigma \in \mathbb{R}, f \in \mathcal{V}, g \in \mathcal{Z}, (f \ne 0, g \ne 0).$$

The singular vectors $f \in \mathcal{V}$ and $g \in \mathcal{Z}$ represent the input and output signals. The pairs (f, g) corresponding to the larger singular values σ characterize the dominant input–output behavior of system Σ. The following theorem provides a calculation method of the singular values σ and the explicit characterization of the singular vectors f and g which satisfy the relation (11).

Theorem 1: The singular values are given by the roots of the following transcendental equation:

$$\det\{M(\sigma)\} = 0, \tag{12}$$

$$M(\sigma) := \begin{bmatrix} -\frac{1}{\sigma} F^{\mathrm{T}} F & I \end{bmatrix} e^{J_N(\sigma)\bar{d}_N} e^{J_{N-1}(\sigma)\bar{d}_{N-1}} \cdots e^{J_1(\sigma)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$J_m(\sigma) := \begin{bmatrix} A_m & \frac{1}{\sigma} B_m B_m^{\mathrm{T}} \\ -\frac{1}{\sigma} E_m^{\mathrm{T}} E_m & -A_m^{\mathrm{T}} \end{bmatrix},$$

$$\bar{d}_m := t_m - t_{m-1}, \ m = 1, 2, \dots, N.$$

Let σ_i be a singular value. Then, the corresponding singular vectors $f_i \in \mathcal{V}$ and $g_i \in \mathcal{Z}$ are given as follows:

$$\begin{aligned} f_{i[t_{k-1},t_k]}(\beta) = \frac{1}{\sigma_i} \begin{bmatrix} 0 & B_k^{\mathrm{T}} \end{bmatrix} e^{J_k(\sigma_i)(\beta - t_{k-1})} e^{J_{k-1}(\sigma_i)\bar{d}_{k-1}} \\ \cdots e^{J_1(\sigma_i)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix} q_i, \end{aligned} \tag{13}$$

$$k = 1, 2, \dots, N,$$

$$g_i^0 = \frac{1}{\sigma_i} \begin{bmatrix} F & 0 \end{bmatrix} e^{J_N(\sigma_i)\bar{d}_N} e^{J_{N-1}(\sigma_i)\bar{d}_{N-1}} \cdots e^{J_1(\sigma_i)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix} q_i, \tag{14}$$

$$\begin{aligned} g^1_{i\,[t_{k-1},t_k]}(\beta) = \frac{1}{\sigma_i} \begin{bmatrix} E_k & 0 \end{bmatrix} e^{J_k(\sigma_i)(\beta - t_{k-1})} e^{J_{k-1}(\sigma_i)\bar{d}_{k-1}} \\ \cdots e^{J_1(\sigma_i)\bar{d}_1} \begin{bmatrix} 0 \\ I \end{bmatrix} q_i, \end{aligned} \tag{15}$$

$$k = 1, 2, \dots, N,$$

$$q_i \ne 0 \ : \ M(\sigma_i) q_i = 0. \tag{16}$$

Proof: For the operator Γ , the adjoint $\Gamma^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ is calculated as follows (see the Appendix for details):

$$(\Gamma^* \hat{z})_{[t_{k-1},t_k]}(\tau)$$

$$= \begin{cases} B_k^{\mathrm{T}} e^{A_k{}^{\mathrm{T}}(t_k-\tau)} \Phi^{\mathrm{T}}_{(N,k)}(h) F^{\mathrm{T}} \hat{z}^0 \\ \quad + \sum_{s=k}^{N-1} \int_{t_s}^{t_{s+1}} B_k^{\mathrm{T}} e^{A_k{}^{\mathrm{T}}(t_k-\tau)} \Phi^{\mathrm{T}}_{(s+1,k)}(\beta) E_k^{\mathrm{T}} \hat{z}^1(\beta) d\beta \\ \quad + \int_{\tau}^{t_k} B_k^{\mathrm{T}} e^{A_k{}^{\mathrm{T}}(\beta-\tau)} E_k^{\mathrm{T}} \hat{z}^1(\beta) d\beta \\ \qquad t_{k-1} \le \tau \le t_k, \ k = 1, 2, \dots, N-1 \\ B_N^{\mathrm{T}} e^{A_N{}^{\mathrm{T}}(h-\tau)} F^{\mathrm{T}} \hat{z}^0 + \int_{\tau}^{t_N} B_N^{\mathrm{T}} e^{A_N{}^{\mathrm{T}}(\beta-\tau)} E_N^{\mathrm{T}} \hat{z}^1(\beta) d\beta \\ \qquad t_{N-1} \le \tau \le t_N, \ k = N \end{cases} \tag{17}$$

$$\hat{z} = \begin{bmatrix} \hat{z}^0 \\ \hat{z}^1 \end{bmatrix} \in \mathcal{Z}.$$

By introducing the auxiliary variables

$$p_1(t) := \int_0^t e^{A_1(t-\tau)} B_1 v(\tau) d\tau, \ 0 \le t \le t_1, \tag{18}$$

$$p_k(t) := e^{A_k(t-t_{k-1})} p_{k-1}(t_{k-1}) + \int_{t_{k-1}}^{t} e^{A_k(t-\tau)} B_k v(\tau) d\tau,$$

$$t_{k-1} \le t \le t_k, \ k = 2, 3, \dots, N, \tag{19}$$

$$q_N(t) := e^{A_N^{\mathrm{T}}(t_N-t)} F^{\mathrm{T}} \hat{z}^0 + \int_t^{t_N} e^{A_N^{\mathrm{T}}(\beta-t)} E_N^{\mathrm{T}} \hat{z}^1(\beta) d\beta, \tag{20}$$

$$q_k(t) := e^{A_k^{\mathrm{T}}(t_k-t)} q_{k+1}(t_k) + \int_t^{t_k} e^{A_k^{\mathrm{T}}(\beta-t)} E_k^{\mathrm{T}} \hat{z}^1(\beta) d\beta, \tag{21}$$

$$t_{k-1} \le t \le t_k, \ k = 1, 2, \dots, N-1$$

into (8), (9) and (17), the relation (11) is rewritten as the following set of equations

$$\dot{p}_k(t) = A_k p_k(t) + B_k v(t), \tag{22}$$

$$\sigma \hat{z}^1(\beta) = E_k p_k(\beta), \tag{23}$$

$$t_{k-1} \le t \le t_k, \ k = 1, 2, \dots, N,$$

$$\sigma \hat{z}^0 = F p_N(t_N), \tag{24}$$

$$p_1(0) = 0, \tag{25}$$

$$\dot{q}_k(t) = -A_k^{\mathrm{T}} q_k(t) - E_k^{\mathrm{T}} \hat{z}^1(t), \tag{26}$$

$$t_{k-1} \le t \le t_k, \ k = 1, 2, \dots, N,$$

$$\sigma v(t) = B_k^{\mathrm{T}} q_k(t), \tag{27}$$

$$q_N(t_N) = F^{\mathrm{T}} \hat{z}^0. \tag{28}$$

By eliminating v and $\hat{z}^1$ from the differential equations (22) and (26) using (23) and (27), (22) and (26) yield the following differential equation

$$\begin{bmatrix}\dot{p}_k(t)\\ \dot{q}_k(t)\end{bmatrix} = \begin{bmatrix} A_k & \frac{1}{\sigma}B_kB_k^{\mathrm{T}} \\ -\frac{1}{\sigma}E_k^{\mathrm{T}}E_k & -A_k^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}p_k(t)\\ q_k(t)\end{bmatrix}. \quad (29)$$

The solution to this differential equation on $[t_{k-1}, t_k]$ is given by

$$\begin{bmatrix}p_k(t_k)\\ q_k(t_k)\end{bmatrix} = e^{J_k(\sigma)(t_k - t_{k-1})}\begin{bmatrix}p_{k-1}(t_{k-1})\\ q_{k-1}(t_{k-1})\end{bmatrix}. \quad (30)$$

From equations (24) and (28), the boundary condition $q_N(t_N) = \frac{1}{\sigma}F^{\mathrm{T}}Fp_N(t_N)$, which implies $[-\frac{1}{\sigma}F^{\mathrm{T}}F, I]\begin{bmatrix}p_N(h)\\ q_N(h)\end{bmatrix} = 0$, is obtained. Then, we have

$$\begin{aligned}
&\begin{bmatrix}-\frac{1}{\sigma}F^{\mathrm{T}}F & I\end{bmatrix}\begin{bmatrix}p_N(h)\\ q_N(h)\end{bmatrix} \quad (31)\\
&= \begin{bmatrix}-\frac{1}{\sigma}F^{\mathrm{T}}F & I\end{bmatrix} e^{J_N(\sigma)\bar{d}_N}\begin{bmatrix}p_{N-1}(t_{N-1})\\ q_{N-1}(t_{N-1})\end{bmatrix}\\
&= \cdots\\
&= \begin{bmatrix}-\frac{1}{\sigma}F^{\mathrm{T}}F & I\end{bmatrix} e^{J_N(\sigma)\bar{d}_N}e^{J_{N-1}(\sigma)\bar{d}_{N-1}}\cdots e^{J_1(\sigma)\bar{d}_1}\begin{bmatrix}0\\ I\end{bmatrix}q_1(0)\\
&= M(\sigma)q_1(0)\\
&= 0. \quad (32)
\end{aligned}$$

Because it is shown that $q_1(0) \neq 0$ iff $f \neq 0,\ g \neq 0$, the matrix $M(\sigma)$ in (32) must be singular for the singular values σ. Therefore, the singular values are given by the roots of the equation (12). The singular vectors corresponding to the singular values are constructed by expressing v and z^0, z^1 using the auxiliary variables p_k and q_k.

■

Remark 2: The singular vectors $\{f_i\},\ \{g_i\}$ form orthogonal sequences in spaces $\mathcal{V}$ and $\mathcal{Z}$. The singular vectors f_i corresponding to the large singular values describe the input signals over $[0, h]$ that have a large effect on the input–output dynamics of the linear time-varying system (1) because output $\hat{z}$ is given as $\sigma_i \cdot g_i$, which indicates that g_i is magnified by σ_i when f_i is applied to the system.

Remark 3: Once the singular values are calculated from the determinant equation (12), the computation of the singular vectors (13)–(15) is easy as they have the form of the autonomous response of switched linear systems. The singular values are calculated by using general methods such as bisection algorithm. Although we will only employ larger singular values and corresponding singular vectors for the design of compensation signal, the computation of smaller singular values tend to demanding as the transcendental equation (12) involves the exponential matrix of the Hamiltonian matrix $J_m(\sigma)$.

Based on the SVD for switched linear systems with periodic switching obtained in this section, we derive a feedforward compensation signal design problem of the switched linear systems.

In the following, we normalize the singular vectors as $\|f_i\|_{\mathcal{V}} = 1, \|g_i\|_{\mathcal{Z}} = 1$ and denote singular values by $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots$ in decreasing order and the corresponding singular vectors by f_i and g_i $(i = 1, 2, \dots)$.

III. TRANSIENT IMPROVEMENT OF A SWITCHED LINEAR SYSTEM

A switched system consists of several subsystems and a switching signal that determines the transition of the dynamics among the subsystems. Various studies related to stabilizing switching law, system structures (controllability, observability) and optimal control have been investigated [6], [7], [8], [9], [10], [11]. In the optimal control of the switched systems, if one pursues optimality exclusively, then the switching frequency might become very high or the switching law might require infinitely many switchings on finite interval [10], [11], which is not desirable from a practical viewpoint. Instead of designing the switching function to achieve good performance, we consider the feedforward compensation signal design problem.

In this section, based on the SVD for the class of linear time-varying systems developed in the previous section, we will derive a method for improving the transient response of a switched linear system with a periodic switching law. The compensation signal is computed off-line beforehand based on uncompensated and desired transient responses over the finite time interval of interest.

First, we consider a switched linear system

$$\Sigma_s : \begin{cases} \dot{x}(t) = A_{s(t)}x(t) + B_{s(t)}u(t) \\ z(t) = E_{s(t)}x(t) \end{cases} \quad (33)$$

where $x(t), u(t)$, and $z(t)$ respectively denote the state, input, and output. Subscript $s(t) \in \mathcal{S}$ signifies the switching signal, where $\mathcal{S}$ is an index set. When we apply

$$u(t) = K_{s(t)}x(t) + v(t) \quad (34)$$

to Σ_s, the resulting system can be described as follows.

$$\tilde{\Sigma}_s : \begin{cases} \dot{x}(t) = \tilde{A}_{s(t)}x(t) + B_{s(t)}v(t) \\ z(t) = E_{s(t)}x(t) \end{cases} \quad (35)$$

$$\tilde{A}_{s(t)} := A_{s(t)} + B_{s(t)}K_{s(t)} \quad (36)$$

In (34), $v(t)$ denotes the compensation signal that we will design to improve the transient response. If feedback gain matrices K_i and coefficients ω_i exist such that the matrix

$$A_0 = \sum_{i\in\mathcal{S}} w_i\tilde{A}_i \ (\omega_i \geq 0, \sum_{i\in\mathcal{S}}\omega_i = 1) \quad (37)$$

is Hurwitz, then the system can be stabilized via periodic switching signals as follows [6], [7]. Let K_i and w_i be one pair satisfying (37). Then, for a small $T > 0$, the periodic switching function

$$s(t) = \begin{cases} 1, & t \in [kT, (k+\omega_1)T) \\ 2, & t \in [(k+\omega_1)T, (k+\omega_1+\omega_2)T) \\ \vdots & \vdots \\ m, & t \in [(k+1-\omega_m)T, (k+1)T) \end{cases} \quad (38)$$
$$k = 0, 1, \dots$$

stabilizes the system exponentially because matrix $A_d := e^{\omega_m A_m T} \cdots e^{\omega_1 A_1 T} = e^{A_0 T + f(T)T^2}$ (f is an analytic and bounded matrix-valued function) can be stable and $x((k+1)T) = A_d x(kT)$ holds (refer to [6] for details). Consequently, a system with the periodic switching signal can be represented by the form of the linear time-varying system (1).

For the system stabilized by the periodic switching function, we consider the transient improvement by the compensation input $v(t), (0 \le t \le h)$ in the sense that the resulting system response approaches a certain prescribed response. Let

$$\hat{y}_d = \begin{bmatrix} x_d(h) \\ y_{d[0,h]} \end{bmatrix} \in \mathcal{Z}$$

be a pair of a desired terminal state and output and let

$$\hat{y} = \begin{bmatrix} x(h) \\ y_{[0,h]} \end{bmatrix} \in \mathcal{Z}$$

be the nominal response (without compensation input v). Here two responses are assumed to have the condition $y_{d[0,h]}(0) = y_{[0,h]}(0)$ as the initial condition is not compensable. Define the error $\hat{e} := \hat{y}_d - \hat{y}$ ($\in \mathcal{Z}$). The desired response $\hat{y}_d$ can be chosen from the response of a certain reference model that has a good transient property. We design compensation input $v(t)$ such that the resulting system response closely approximates the desired response $\hat{y}_d$. We construct the compensation input $v(t)$ by the linear combination of N_s dominant singular vectors $f_i \in \mathcal{V}$, which represent the input signals over $[0, h]$ in $\mathcal{V}$. Although the SVD is defined for the system (1) with zero initial condition, it is applicable since the singular values and singular vectors are used to generate the error signal between the desired and uncompensated responses (the error at initial time: $y_{d[0,h]}(0) - y_{[0,h]}(0) = 0$). Because N_s singular vectors $g_1, g_2, \ldots, g_{N_s}$ form the orthonormal system in $\mathcal{Z}$, the closest element of $\mathrm{span}\{g_1, g_2, \ldots, g_{N_s}\}$ to $\hat{e}$ is given as

$$\hat{e}' := \sum_{i=1}^{N_s} \langle \hat{e}, g_i \rangle_{\mathcal{Z}} \cdot g_i. \tag{39}$$

Consequently, by applying the compensation input

$$v_{[0,h]} = -\sum_{i=1}^{N_s} \frac{1}{\sigma_i} \langle \hat{e}, g_i \rangle_{\mathcal{Z}} \cdot f_i \tag{40}$$

the error is minimized in the input consisting of the N_s singular vectors f_i because the input (40) yields the error signal with the opposite sign in the output. The number N_s is a design parameter for approximating the error $\hat{e}$ and the method for choosing a suitable value of N_s beforehand has not been established. It is noteworthy that although the error $\hat{e}$ decreases monotonically as we increase N_s, the compensation signal might become larger.

Remark 4: For standard linear time-invariant systems, the SVD-based compensation method using the orthogonal expansion technique has already been examined in [3]. Herein, we have derived a compensation method for switched linear systems using the newly derived SVD for linear time-varying systems.

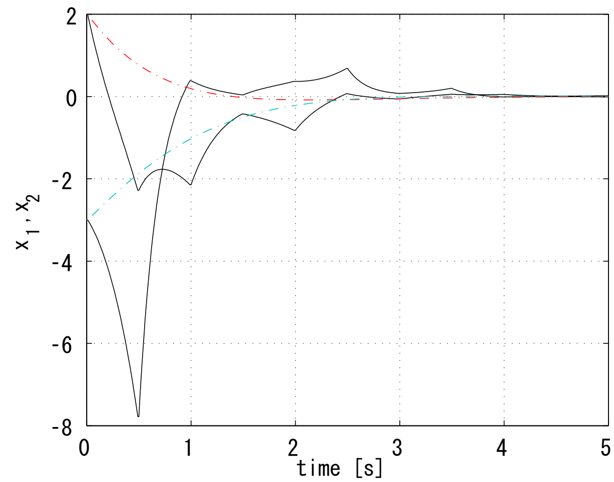

Fig. 1. State response (solid) and response of average system (dashed)

TABLE I
SINGULAR VALUES OF $\Gamma \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$

σ_1	1.8353	σ_9	0.2695
σ_2	1.7834	σ_{10}	0.2563
σ_3	1.3738	σ_{11}	0.2401
σ_4	0.6928	σ_{12}	0.2291
σ_5	0.5808	σ_{13}	0.1909
σ_6	0.5062	$\vdots$	$\vdots$
σ_7	0.4395		
σ_8	0.4187		

IV. NUMERICAL EXAMPLE

Here we provide a simple numerical example to illustrate the fundamental properties of the proposed SVD-based compensation method.

First, consider a pre-compensated switched linear system.

$$\begin{cases} \dot{x}(t) = \tilde{A}_{s(t)} x(t) + B_{s(t)} v(t) \\ z(t) = x(t) \end{cases} \tag{41}$$

$$\mathcal{S} = \{1, 2\},\ \tilde{A}_1 = \begin{bmatrix} -3 & 2 \\ 1 & 2 \end{bmatrix},\ \tilde{A}_2 = \begin{bmatrix} 1 & -1 \\ -3 & -5 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, (E_1 = E_2 = I) \tag{42}$$

Both subsystems are unstable, and have the eigenvalues $(-3.37, 2.37)$ and $(-5.46, 1.46)$. We first design the stabilizing periodic switching law according to the method described in section III. For this system, we choose the coefficients $w_1 = 0.5$, $w_2 = 0.5$ in (37). Then, the matrix $A_0 = w_1 \tilde{A}_1 + w_2 \tilde{A}_2$ becomes Hurwitz. Therefore, it can be stabilized by periodic switching. When we choose $T = 1$, the stable matrix $A_d = e^{0.5 \cdot A_2 \cdot 1} \cdot e^{0.5 \cdot A_1 \cdot 1}$ is obtained (eigenvalues: $0.135 \pm 0.2527i$), and the periodic switching

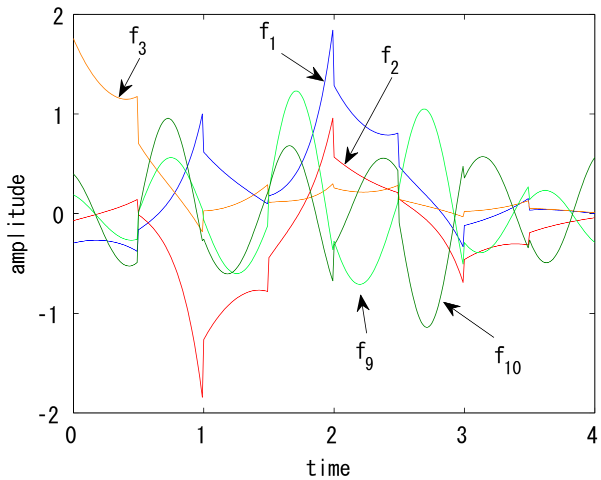

Fig. 2. Normalized singular vectors $f_i \in L_2(0,h;\mathbb{R})$

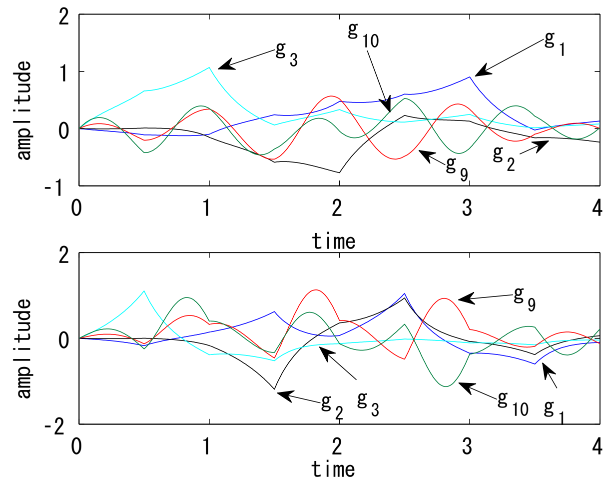

Fig. 3. Singular vectors $g_i^1 \in L_2(0,h;\mathbb{R}^2)(\|g_i\|_{\mathcal{Z}} = 1)$, upper:$x_1$, lower:$x_2$

law

$$s(t) = \begin{cases} 1, & t \in [kT, (k+0.5)\cdot 1) \\ 2, & t \in [(k+0.5)\cdot 1, (k+1)\cdot 1) \end{cases} \quad k = 0,1,2,\ldots \tag{43}$$

exponentially stabilizes the system. Fig. 1 shows the state response (x_1, x_2) for the initial condition $x(0) = [2, -3]^{\mathrm{T}}$ with the periodic switching law (solid line). The response of the average system ($\dot{x}(t) = A_0 x(t)$) is also shown as a dashed line. Although the response converges to zero, the behavior is not smooth at switching instants and exhibits sharp changes. We will design compensation signal v, which improves this response in the sense that the resulting response of the system comes to resemble that of the average system. Although we use the response of the average system here, not only the response of the average system but also any other response can be used. It should be noted that the response comes to resemble that of the average system as T in (43) decreases. However, in such a case, much more frequent switching between subsystems A_1 and A_2 is required.

To design the compensation signal, we first compute the singular values and singular vectors. For parameters $h = 4$ (compensation period) and $N = 8$, the singular values are computed from Theorem 1; they are presented in Table I (the matrix F in (5) is chosen by the solution to the Lyapunov equation $F^{\mathrm{T}}F = P : A_0^{\mathrm{T}}P + PA_0 + I = 0$). Fig. 2 and Fig. 3 depict the normalized singular vectors $f_i \in \mathcal{V}$ and $g_i^1 \in L_2(0,h;\mathbb{R}^2)(i = 1,2,3,9,10)$. They are obtained by first computing the singular vectors based on (13)–(15) and then multiplying them by the value of either $1/\|f_i\|_{\mathcal{V}}$ or $1/\|g_i\|_{\mathcal{Z}}$. The singular vectors corresponding to smaller singular values (σ_9, σ_{10}) tend to exhibit the oscillating shape. As we addressed in Remark 2, when the singular vector f_1 corresponding to the largest singular value is applied to the system (41) ($v(t) = f_1(t)$), the output response $z = x$ is given by $\sigma_1 \cdot g_1^1(t) = 1.8353 \cdot g_1^1(t)$, which implies that the system generates magnified signal of $g_1^1(t)$. Also, note that the singular value σ_i represents the value of the norm of the output g_i corresponding to the unit energy input $v = f_i$. In contrast, when the normalized singular vector f_{10} corresponding to smaller singular value σ_{10} is applied to the system, $\sigma_{10} \cdot g_{10}^1(t) = 0.2563 \cdot g_{10}^1(t)$ is generated in the output. Therefore, if we employ the singular vectors corresponding to larger singular values, the compensation signal that has a dominant effect on the input–output relation is obtained. Using $7(= N_s)$ dominant singular values $(\sigma_1, \sigma_2, \cdots, \sigma_7)$, we design compensation signal v according to the method addressed in section III. Fig. 4 shows the compensation signal. It is observed that the amplitude around the initial time is larger to improve the large deviation of the original response. Fig. 5 shows the state response with compensation. It is observed that the response approaches that of the average system.

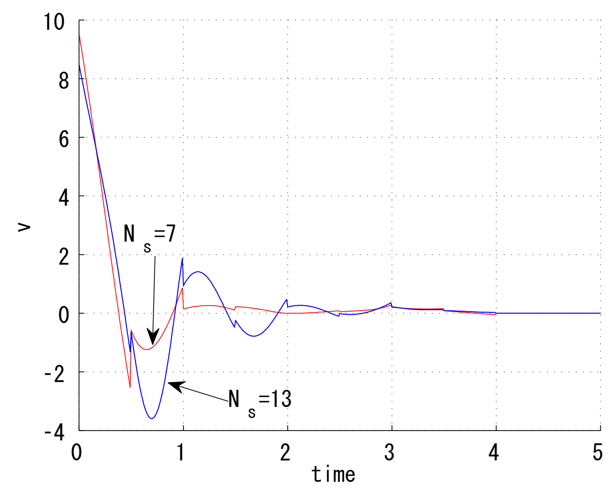

Fig. 4. Compensation signal v

The responses for $N_s = 13$ are also shown in Fig. 4 and Fig. 5. Although the compensation signal for $N_s = 13$ exhibits similar behavior near the initial time, the amplitude

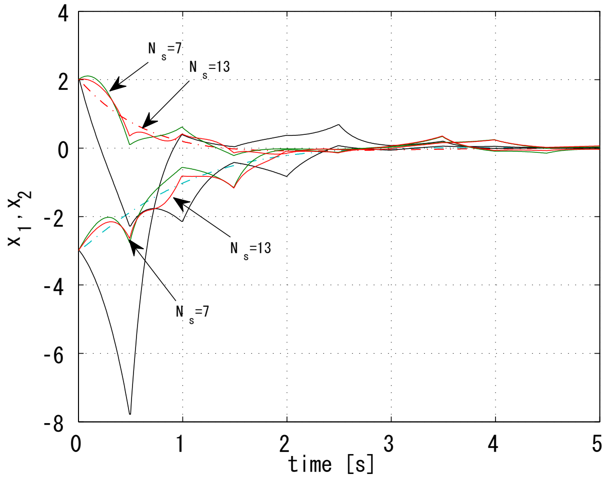

Fig. 5. State response with compensation

is larger than that of the compensation signal for $N_s = 7$ after $t = 0.5$ [s]. Even if we use more singular values, the amplitude would become larger and eliminate only small deviations.

V. CONCLUSION

In this paper, we have considered the SVD of the Hankel-like operator representing the input-output relation of a class of linear time-varying systems. The class of the systems describes switched linear systems with periodic switching. We have derived a computation method for singular values and singular vectors of the operator. Based on this SVD, the compensation signal design for switched linear systems with a periodic switching law is discussed. A numerical example was presented to illustrate the fundamental properties of the proposed method.

As described here, we have considered the design problem for pre-compensated systems, it would be important to design the switching signal and the compensation signal simultaneously to achieve good performance without requiring excessive frequent mode transition between subsystems. Also, it is necessary to evaluate the relationship between approximation error and the number of employed singular values and vectors N_s. These are the directions of future research.

REFERENCES

[1] K. Zhou, J. Doyle and K. Glover: *Robust and Optimal Control*, Prentice Hall, 1995.

[2] K. Glover, J. Lam and J.R. Partington: Rational approximation of a class of infinite-dimensional systems I: Singular values of Hankel operators, *Mathematics of Control, Signals, and Systems*, Vol.3, No.4, pp.325–344, 1990.

[3] T. Izumi, A. Kojima and S. Ishijima: Improving transient with compensation law, *Proc. of the 15th IFAC World Congress*, pp.181–186, Barcelona, Spain, 2002.

[4] N. Hara and A. Kojima: An off-line design method of compensation law for constrained linear systems, *Int. J. Robust and Nonlinear Control*, Vol.18, No.1, pp.51–68, 2008.

[5] A. Kojima and M. Morari: LQ control for constrained continuous-time systems, *Automatica*, Vol.40, No.7, pp.1143–1155, 2004.

[6] Z. Sun and S.S. Ge: *Switched Linear Systems*, Springer, 2005.

[7] I. Masubuchi and G. Zhai: Control of Hybrid Systems V – Analysis and control of switched systems, *Systems, Control and Information*, Vol.52, No.1, pp.25–31, 2008. (in Japanese)

[8] D. Liberzon: *Switching in Systems and Control*, Birkhäuser, 2003.

[9] X. Xu and P.J. Antsaklis: Optimal control of switched systems based on parameterization of the switching instants, *IEEE Trans. on Automatic Control*, Vol.49, No.1, pp.2–16, 2004.

[10] S.C. Bengea and R.A. DeCarlo: Optimal control of switching systems, *Automatica*, Vol.41, No.1, pp.11–27, 2005.

[11] T. Das and R. Mukherjee: Optimally switched linear systems, *Automatica*, Vol.44, No.5, pp.1437–1441, 2008.

APPENDIX

Derivation of the adjoint $\Gamma^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ in (17): First, we compute the following inner product:

$$\langle \hat{z}, \Gamma v \rangle_{\mathcal{Z}} = \langle \begin{bmatrix} \hat{z}^0 \\ \hat{z}^1 \end{bmatrix}, \begin{bmatrix} (\Gamma v)^0 \\ (\Gamma v)^1 \end{bmatrix} \rangle_{\mathcal{Z}}$$
$$= \hat{z}^{0\mathrm{T}} (\Gamma v)^0 + \int_0^h \hat{z}^{1\mathrm{T}}(\beta)(\Gamma v)^1(\beta) d\beta. \quad (44)$$

The first and second terms in (44) are calculated as shown below.

(i) first term

$$\hat{z}^{0\mathrm{T}}(\Gamma v)^0$$
$$= \hat{z}^{0\mathrm{T}} F \sum_{s=1}^{N-1} \Phi_{(N,s)}(h) \int_{t_{s-1}}^{t_s} e^{A_s(t_s-\tau)} B_s v(\tau) d\tau$$
$$+ \hat{z}^{0\mathrm{T}} F \int_{t_{N-1}}^{h} e^{A_N(h-\tau)} B_N v(\tau) d\tau$$

$$= \sum_{k=1}^{N-1} \int_{t_{k-1}}^{t_k} \left(B_k^{\mathrm{T}} e^{A_k^{\mathrm{T}}(t_k-\tau)} \Phi_{(N,k)}^{\mathrm{T}}(h) F^{\mathrm{T}} \hat{z}^0 \right)^{\mathrm{T}} v(\tau) d\tau$$
$$+ \int_{t_{N-1}}^{t_N} \left(B_N^{\mathrm{T}} e^{A_N^{\mathrm{T}}(h-\tau)} F^{\mathrm{T}} \hat{z}^0 \right)^{\mathrm{T}} v(\tau) d\tau \quad (45)$$

(ii) second term

$$\int_0^h \hat{z}^{1\mathrm{T}}(\beta)(\Gamma v)^1(\beta) d\beta$$
$$= \sum_{k=1}^{N-1} \sum_{s=1}^{k} \int_{t_k}^{t_{k+1}} \int_{t_{s-1}}^{t_s} \hat{z}^{1\mathrm{T}}(\beta) E_k \Phi_{(k+1,s)}(\beta) e^{A_s(t_s-\tau)} \times B_s v(\tau) d\tau d\beta$$
$$+ \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{\beta} \hat{z}^{1\mathrm{T}}(\beta) E_k e^{A_k(\beta-\tau)} B_k v(\tau) d\tau d\beta$$

$$= \sum_{k=1}^{N-1} \int_{t_{k-1}}^{t_k} \sum_{s=k}^{N-1} \int_{t_s}^{t_{s+1}} \hat{z}^{1\mathrm{T}}(\beta) E_k \Phi_{(s+1,k)}(\beta) e^{A_k(t_k-\tau)} \times B_k d\beta v(\tau) d\tau$$
$$+ \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\tau}^{t_k} \hat{z}^{1\mathrm{T}}(\beta) E_k e^{A_k(\beta-\tau)} B_k d\beta v(\tau) d\tau \quad (46)$$

For the transformation presented above, we used the reversal of the order of integration and change of the order of summation. Consequently, by definition (2) of the inner product in $\mathcal{V}$, the adjoint $\Gamma^* \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$, which satisfies $\langle \hat{z}, \Gamma v \rangle_{\mathcal{Z}} = \langle \Gamma^* \hat{z}, v \rangle_{\mathcal{V}}$, is given by (17). ■