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Optimal Input Shaping Filters for Non-Zero Initial States

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Abstract— **This paper presents an approach to design optimal vibration reduction input shapers for systems with non-zero initial conditions. The problem is first formulated as an optimal control problem and the optimal solution is shown to be bang-bang. Once the structure of the optimal shaper is known, a parametric problem formulation is presented for the computation of the switching times. For digital implementation, discrete time approximate solutions are derived by solving a quasi convex Linear Program. Simulation results are shown for closed-form implementation of these filters on flexible structures. The digital solutions are experimentally verified on a portable bridge crane.**

I. Introduction

Vibration reduction of flexible structures has long been the subject of active research [1]–[6]. The control is required to cause a finite rigid-body motion with low residual vibrations of the flexible modes [3]. The design problem has been approached in primarily two ways. In the first approach, an optimal shaped control profile is designed for the rest-to-rest motion of the entire flexible system. Here the shaped command is obtained by formulating a constrained optimal control problem that is solved for each reference move [7]. The cost criteria include time [3], robustness [5], fuel cost [8], power [9], deflection [10], move vibrations [11], jerk [9], higher mode excitations [12], etc. The optimal solution is obtained by Pontryagin's minimum principle [13] and Euler-Lagrange necessary conditions [14].

In this approach, the problem needs to be re-solved for every change in reference move. An alternate approach is the idea of using vibration reduction filters [15]–[18] where the filter is designed beforehand for canceling the flexible mode dynamics. This approach is illustrated in Fig.1 where a step reference is modified using a three impulse filter. Optimal profiles of such filters are obtained by formulating a constrained time-optimal control problem where only the flexible mode dynamics are considered [7]. Any kind of robustness and over-excitation constraints can be formulated as terminal state inequality constraints [12], [19]. The optimal solution for the filter design problem with terminal state inequality constraints is derived as bang-bang [7], [12].

Since the structure of the optimal solution is known, analytical expressions can be derived for representing vibration energy [20], robustness and excitations of the flexible modes [3], [16], [17]. The control parameters can be computed by formulating a nonlinear parametric optimization problem [18].

These filters can be implemented either as pure convolution, or by closed-form methods [6], [21]–[25]. In the convolution based input-shaping technique, the reference command is computed for pure rigid-body and is later convolved with a single filter. In the closed-form method, the input is modified for each on-off switch in the rigid-body command using different filters designed for different transitions [6], [21]–[24], [26]. The rigid-body switch times are required to be corrected in order to satisfy the rest-to-rest motion conditions [6]. The correction term is defined as Preloading [22], and is a quadratic function of the filter parameters and the reference input [22].

The designs reported in the literature considered zero initial conditions where the switched control is designed for rest-to-rest motion. As a result, when the solutions are applied to systems where the states have finite non-zero initial conditions, the residual vibrations will not be cancelled. The non-zero initial conditions can result from a disturbance or error in implementation of previous commands where the next reference command appears before the states have settled to zero velocities. Also, in the closed-form implementation of shaping filters, any error in filter parameters can cause residual vibrations at the start of subsequent transitions. As a result, there is a need for transition filters that cancel vibrations in the presence of non-zero initial states. Since the states can be estimated by a state estimator or measured by sensors, the initial values may be available to the designer.

An approach is presented for including non-zero initial conditions in the design of vibration reduction optimal con-

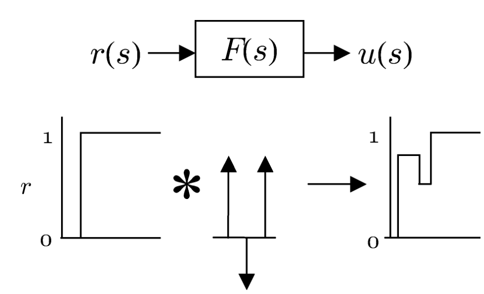

Fig. 1: Three impulse vibration reduction filter based on switching control. The impulse amplitudes and switching times are selected to cancel the flexible poles while maintaining the actuator limitations and unit DC gain.

trol. The proposed approach provides the optimal solution of a shaping filter, and a shaped command. The design problem is formulated as a constrained time-optimal problem whose solution is a bang-bang function. A discrete-time formulation is also presented for implementation on digital controllers. A scheme is presented for computing the filter parameters using a parametric nonlinear optimization formulation. Simulation results are presented where such filters are used to cancel residual vibrations in the closed-form implementation. Experimental results from a portable bridge crane verify the effectiveness of the discrete-time formulation.

II. Problem Formulation

We consider the dynamics of flexible modes represented by $\{\omega, \zeta\}$ which can be modeled as [3]

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u(t)\,, \tag{1}$$

where $-1 \leq u(t) \leq 1$. A non-zero initial condition is considered such that $x(0) = [x_1(0),\ x_2(0)]$. Vibration cancellation requires the states at final time T to be $x(T) = [1/\omega^2,\ 0]$. The design of an optimal shaping filter for non-zero initial states can be formulated as the contrained time-optimal control problem

$$\begin{aligned} &\text{minimize} && T \\ &\text{subject to} && \dot{x}(t) = Ax(t) + bu(t) \\ &&& x(0) = [x_1(0),\ x_2(0)]\,, \quad x(T) = [1/\omega^2,\ 0] \\ &&& -1 \leq u(t) \leq 1\,. \end{aligned} \tag{2}$$

Since the states are known at the initial and final times, the optimal control can be shown to be *bang-bang* from Pontryagin's minimum principle [13]. The optimal control can be computed from Euler-Lagrange necessary conditions [14]. Additional robustness constraints like zero vibration and derivative (ZVD) [16], [17], specified insensitivity (SI) [17], [27], or reduced higher-mode excitations [12], [28] can be included as terminal state inequality constraints. The optimal solution of such constrained time-optimal control problems with terminal state inequality constraints can also be shown to have a *bang-bang* form [12]. The solution $u(t)$ provides the time-optimal input-shaping filter that cancels the vibrations in the presence of non-zero initial conditions.

An approximate solution can be derived from the discrete time formulation of (2) as follows

$$\begin{aligned} &\text{minimize} && N \\ &\text{subject to} && \widehat{x}(k+1) = \widehat{A}\widehat{x}(k) + \widehat{b}\widehat{u}(k) \\ &&& x(0) = [x_1(0),\ x_2(0)]\,, \quad x(T) = [1/\omega^2,\ 0] \\ &&& -1 \leq \widehat{u}(k) \leq 1\,, \quad k = 1, \ldots, N\,, \end{aligned} \tag{3}$$

which is a quasi-convex problem where each iteration is a convex Linear Program (LP) [29]. Here $\{\widehat{x}, \widehat{A}, \widehat{b}\}$ represents the discrete time state dynamics. Optimization routines like MOSEK [30], CVX [31], and Matlab [32] can be used to solve this problem. The solution provides a shaping filter designed for non-zero initial conditions that can be implemented on digital controllers.

III. Parametric Formulation

Since the optimal solution of (2) is bang-bang, the structure of the resulting optimal shaping filter can be written as [5], [17]

$$U(s) = \frac{I_0 + I_1 e^{-t_1 s} + \ldots + I_p e^{-t_p s}}{s}\,, \tag{4}$$

where

$$\boldsymbol{I} = \begin{bmatrix} 1 & -2 & 2 & \ldots & 2 \end{bmatrix}. \tag{5}$$

The switching times $\boldsymbol{t}$ can be computed using a parametric optimization problem [18] formulated next. From (1), the states corresponding to the mode $\{\omega, \zeta\}$ are given in the Laplace domain as [33]

$$\begin{aligned} x_1(s) &= \frac{sU(s) + sx_2(0) + (s^2 + 2\zeta\omega s)x_1(0)}{s(s^2 + 2\zeta\omega s + \omega^2)} \\ x_2(s) &= \frac{sU(s) + sx_2(0) - \omega^2 x_1(0)}{(s^2 + 2\zeta\omega s + \omega^2)}\,. \end{aligned} \tag{6}$$

For vibration cancellation, the roots of the shaper (4) should be placed such that the numerator of (6) cancels the poles of the flexible modes located at $-\zeta\omega \pm j\omega\sqrt{1-\zeta^2}$ [3], [15], [16]. In other words

$$\begin{aligned} &sU(s) + sx_2(0) + (s^2 + 2\zeta\omega s)x_1(0)\big|_{s=-\zeta\omega \pm j\omega\sqrt{1-\zeta^2}} = 0 \\ &sU(s) + sx_2(0) - \omega^2 x_1(0)\big|_{s=-\zeta\omega \pm j\omega\sqrt{1-\zeta^2}} = 0\,, \end{aligned} \tag{7}$$

which simplifies to

$$sU(s) + sx_2(0) - \omega^2 x_1(0)\big|_{s=-\zeta\omega \pm j\omega\sqrt{1-\zeta^2}} = 0\,. \tag{8}$$

For the shaping filter defined in (4), the pole-zero cancellation condition (8) becomes

$$\begin{aligned} C(\omega,\zeta) - C_0(\omega,\zeta,x_1(0),x_2(0)) &= 0 \\ S(\omega,\zeta) - S_0(\omega,\zeta,x_1(0),x_2(0)) &= 0\,, \end{aligned} \tag{9}$$

where

$$\begin{aligned} C(\omega,\zeta) &= I_0 + \sum_{k=1}^{p} I_k\, e^{t_k \zeta\omega}\, \cos(\omega_d t_k) \\ S(\omega,\zeta) &= \sum_{k=1}^{p} I_k\, e^{t_k \zeta\omega}\, \sin(\omega_d t_k)\,, \end{aligned} \tag{10}$$

and

$$\begin{aligned} C_0(\omega,\zeta,x_1(0),x_2(0)) &= \omega^2 x_1(0) + \zeta\omega x_2(0) \\ S_0(\omega,\zeta,x_1(0),x_2(0)) &= \omega_d x_2(0)\,, \end{aligned} \tag{11}$$

where $\omega_d = \omega\sqrt{1-\zeta^2}$.

The conditions (9) can also be obtained from the time response of the LTI system with transfer function (6) subjected to the control (4). The amplitude $V(\omega,\zeta,x_1(0),x_2(0))$ of the residual vibrations [16] at time $t = t_p$ can be derived as

$$V = \frac{e^{-\zeta\omega t_p}}{\omega_d}\sqrt{(C - C_0)^2 + (S - S_0)^2}\,. \tag{12}$$

The conditions (9) therefore ensure zero residual vibrations.

Another criterion of evaluating the performance of vibration cancellation filters is the residual vibration sensitivity S that is defined as the ratio of the amplitudes of residual vibrations for the shaped to unshaped command [27]. From (4) and (12), the vibration sensitivity function $S(\omega, \zeta, x_1(0), x_2(0))$ is written as

$$S = \frac{\sqrt{(C - C_0)^2 + (S - S_0)^2}}{\sqrt{(I_0 - C_0)^2 + (S_0)^2}} . \tag{13}$$

The filter parameters can now be obtained by the following nonlinear parametric optimization problem

$$\begin{aligned} \underset{\mathbf{t}}{\text{minimize}} \quad & t_p \\ \text{subject to} \quad & C(\omega, \zeta) = C_0(\omega, \zeta, x_1(0), x_2(0)) \\ & S(\omega, \zeta) = S_0(\omega, \zeta, x_1(0), x_2(0)) . \end{aligned} \tag{14}$$

where the variables are the switching times $\boldsymbol{t}$. Nonlinear optimization routines like SNOPT [34], GAMS [35], and Matlab [32] can be used to solve (14). The solution provides an optimal zero vibration (ZV) [16] filter that cancels the vibrations for systems with non-zero initial conditions.

The shaping filter can be made robust to parameter variations by formulating zero vibration and derivative (ZVD) [16], [17] constraints as

$$\begin{aligned} C_d(\omega, \zeta) - C_{0d}(\omega, \zeta, x_1(0), x_2(0)) = 0 \\ S_d(\omega, \zeta) - S_{0d}(\omega, \zeta, x_1(0), x_2(0)) = 0 , \end{aligned} \tag{15}$$

where

$$\begin{aligned} C_d(\omega, \zeta) &= \sum_{k=1}^{p} I_k\, t_k\, e^{t_k \zeta \omega} \cos(\omega_d t_k) \\ S_d(\omega, \zeta) &= \sum_{k=1}^{p} I_k\, t_k\, e^{t_k \zeta \omega} \sin(\omega_d t_k) , \end{aligned} \tag{16}$$

and

$$\begin{aligned} C_{0d}(\omega, \zeta, x_1(0), x_2(0)) &= \left(\frac{2\omega\zeta}{2\omega^2 - 1}\right) x_1(0) + x_2(0) \\ S_{0d}(\omega, \zeta, x_1(0), x_2(0)) &= \left(\frac{2\omega_d}{2\omega^2 - 1}\right) x_1(0) . \end{aligned} \tag{17}$$

The ZVD conditions are obtained by setting the derivative of (9) with respect to ω to zero [16]. Similar expressions can derived for specified insensitivity (SI) robustness constraints [17], [27] and reduced higher-mode excitation constraints [12], [28], that limits the sensitivity (13) in specified bands of frequencies and damping ratios

$$S(\omega, \zeta) \leq S_{\max} , \quad \forall\, \omega_1 \leq \omega \leq \omega_2 , \;\; \zeta_1 \leq \zeta \leq \zeta_2 . \tag{18}$$

The nonlinear problem (14) is non-convex in the variable $\boldsymbol{t}$, as a result KKT conditions only guarantee the local optimality of the solution [29]. The global optimality can, however, be verified for such problems using Euler-Lagrange necessary condition [36]. The methodology is based on the approach presented in [37] to verify the global optimality of solution where the initial states are given, and final states are either specified, or are constrained by terminal state inequalities [7].

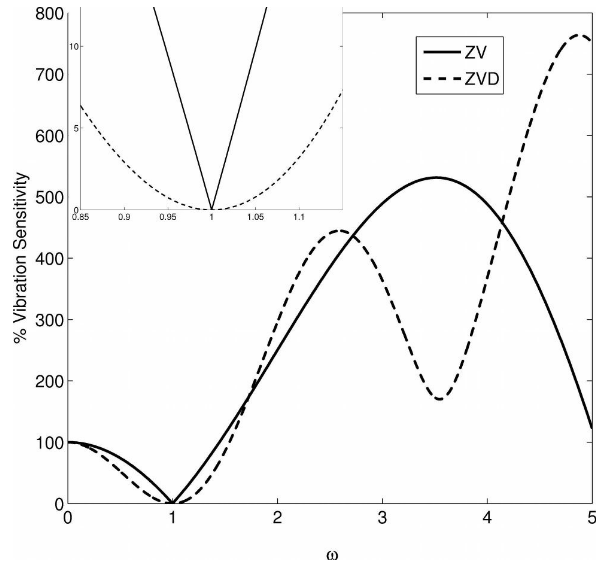

Fig. 2: Vibration sensitivity for optimal ZV and ZVD filters designed for $\omega = 1$, initial condition $x(0) = [0.1, -0.1]$, and final condition $x(T) = [1, 0]$. The inset is the zoomed-in plot of the vibration sensitivity around the nominal mode $\omega = 1$.

For implementation on digital controllers, a discrete time shaping filter can be obtained from (3). Efficient interior point based algorithms exist for solution of such convex problems [38]. Since (3) is convex, the solution provides the global optimal solution to the problem [29]. The discrete time solution can also be used to obtain initial conditions for nonlinear optimization routines employed to solve (14). The approach leads to faster convergence of the nonlinear routine [39]. The discrete time solution also provides an approximation to the number of switchings p in the true time-optimal solution [12].

An example is considered where ZV and ZVD filters are designed for canceling the vibrations of an undamped system with $\omega = 1$. The initial conditions are taken as $x(0) = [0.1, -0.1]$. The solution is obtained by solving the nonlinear parametric problem using SNOPT [34] with an initial point obtained by solving the corresponding LP using MOSEK [30]. The vibration sensitivities S of the resulting filters are shown in Fig. 2.

For implementing these filters on flexible systems, the reference command is first designed for the rigid-body motion alone, and is later shaped by convolving with a shaping filter designed separately based on the approach described above [16], [17]. In this scheme, a single filter is used to modify the reference command. An alternate scheme is the Closed-form implementation, where different filters are utilized for different rigid-body transitions [6], [21]–[25]. Since the filter design can be performed offline, and the online computation of the rigid-body reference command can be done very easily, the shaping approach is very efficient as compared to the approach where optimal control is required to be recomputed for each reference move [16]. However, the filtered reference control only provides an approximation to the true time-optimal control.

IV. Optimal Shaped Control

For flexible systems where a true time-optimal shaped control is required to be computed for a given reference command, an optimal control formulation similar to (2) can be written as

$$\begin{aligned} \text{minimize} \quad & T \\ \text{subject to} \quad & \dot{x}(t) = Ax(t) + bu(t) \\ & x(0) = [x_1(0),\ x_2(0)]\,, \quad x(T) = [1/\omega^2,\ 0] \\ & \dot{y}_r(t) = A_r y_r(t) + b_r u(t) \\ & y_r(0) = [y_1(0),\ y_2(0)]\,, \quad y_r(T) = [y_{ref},\ 0] \\ & -1 \le u(t) \le 1\,. \end{aligned} \tag{19}$$

Here, the rigid-body and the flexible mode dynamics are represented by $\{y_r, A_r, b_r\}$ and $\{x, A, b\}$, respectively. The non-zero initial conditions are specified as $\{x(0), y_r(0)\}$ and the control is required to move the system to y_{ref}, cancel the vibrations, and bring the system to rest. Similar to (2), the optimal control can be shown to be bang-bang using Pontryagin's minimum principle [7], [14], [40]. The shaped command for $-1 \le u(t) \le 1$ is given by (4) with

$$\boldsymbol{I} = \left[\ 1 \quad -2 \quad 2 \quad \ldots \quad 1\ \right]. \tag{20}$$

A parametric optimization problem can now be formulated where in addition to vibration cancellation conditions (9), the control needs to satisfy the rigid-body conditions obtained by the final-value theorem [7], [33]. The resulting non-linear optimization problem becomes

$$\begin{aligned} \underset{\mathbf{t}}{\text{minimize}} \quad & t_p \\ \text{subject to} \quad & C(\omega,\zeta) = C_0(\omega,\zeta,x_1(0),x_2(0)) \\ & S(\omega,\zeta) = S_0(\omega,\zeta,x_1(0),x_2(0)) \\ & \sum_{k=1}^{p} I_k t_k = -y_2(0) \\ & \sum_{k=1}^{p} I_k t_k^2 = 2y_{ref} - y_1(0)\,, \end{aligned} \tag{21}$$

with $\boldsymbol{I}$ given by (20). The solution of (21) provides an optimal shaped command for vibration cancellation of flexible system with non-zero initial conditions. As mentioned earlier, the optimal control needs to be recomputed for each reference input y_{ref}.

V. Experimental Verification

To test the effectiveness of the input shapers designed for non-zero initial states, experiments were conducted on the portable bridge crane shown in Fig. 3 [41]. The crane has a workspace of approximately 1m×1m×1.6m. The overhead bridge and trolley are driven using Siemens AC servo motors and are controlled using a Siemens PLC. The crane is also equipped with a vision system to measure payload position.

ZV and ZVD shapers were designed for an initial payload deflection of 5 degrees (*i.e.*, 10 degrees peak-to-peak payload swing) and a suspension cable length of 1.15m. Using these shapers, the crane trolley was moved 30cm and the residual payload swing was measured. As shown in Fig. 4, the ZV and ZVD shaped commands reduced the initial 10 degree payload swing down to 3.12 and 2.73 degrees peak-to-peak oscillation at the end of the move, respectively. This is a dramatic reduction from the unshaped commands, which increased the vibration amplitude from the initial conditions. Note that there are several souces of experimental error such as inaccuracies in the initial conditions, nonlinearities in the drive systems, and uncertainty in the suspension length. In light of these inaccuracies the experimental results are quite promising.

Figure 4 also shows the effects of errors in the measurement of the initial conditions. The effectiveness of both the ZV and ZVD shapers degrades as a function of the error, but

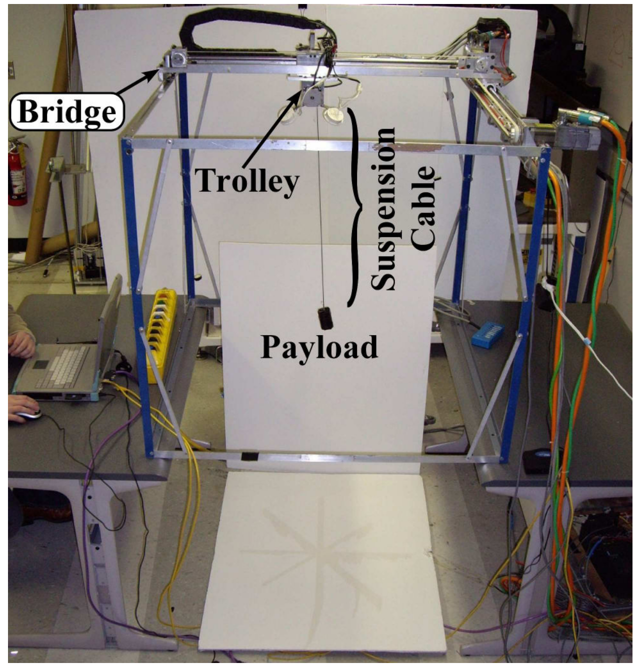

Fig. 3: Portable Bridge Crane

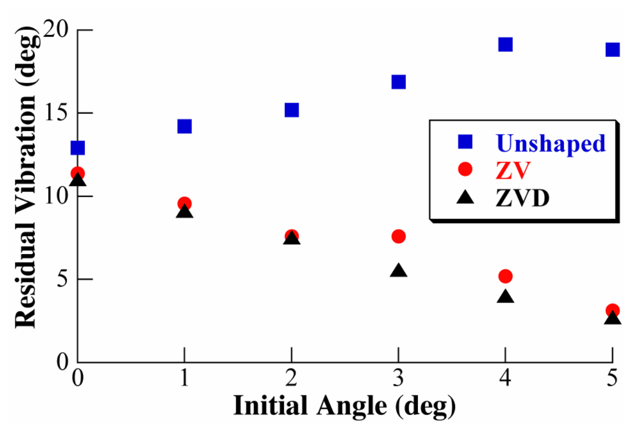

Fig. 4: Residual Vibration Amplitude for Unshaped, ZV-shaped, and ZVD-shaped Moves

neither command excites more vibration than the unshaped case over the range of initial conditions tested.

VI. Discussion

In the closed-form implementation of vibration reduction shaping filters, different filters are utilized for each rigid-body transition [6], [21]–[25]. These filters are called the transition shapers [6], and are designed to cancel the vibrations during each rigid-body transition. The rigid-body switch timings are later corrected using a term which is quadratic in reference point and filter parameters [22]. However, error in computation of any filter parameters can lead to cumulative residual vibrations that cannot be cancelled by subsequent transition shapers. For example, an error in the switching times corresponding to the first transition shaper leads to residual vibrations. If the vibrations remain in the system until the time when the second transition shaper is implemented, then the subsequent TS filters will not be able to completely cancel the vibrations.

In order to improve the performance, the residual vibrations are required to be estimated, and subsequent TS filters must be designed with these estimated non-zero initial conditions. An example is shown in Fig. 5 where the closed-form control is derived for a flexible system with transfer function

$$\frac{Y(s)}{U(s)} = \frac{\omega^2}{s^2(s^2+\omega^2)}, \quad \frac{a(s)}{U(s)} = \frac{\omega^2}{(s^2+\omega^2)}, \tag{22}$$

where y and a denote the position and acceleration, respectively. The transition filters are initially designed for $\omega = 10$ with zero initial conditions, and the rigid-body switching times are computed for $y_{ref} = 100$. A 10% error is introduced in the last switching time of the first TS filter. This error may arise due to error in precise implementation of TS switching times, or due to a disturbance input. As a result, the cumulative vibrations can be seen from the acceleration plot if traditional designs are used. The use of vibration reduction filters may actually aggravate the resonance if not designed properly by taking into account the initial conditions. If the second TS filter is designed with non-zero initial conditions at the time of second rigid-body switchings, then the modified TS filter perfectly cancels the vibrations, as seen in the plot.

In addition to the closed-form implementation of shaping filters, the proposed approach is useful for flexible systems where the next reference command arrives before the residual vibrations have subsided. An estimate of state variables is usually available that can be considered in the computation of the switched control parameters.

The approach presented above can be easily extended to any number of flexible modes that can be considered in the state matrices $\{A, b\}$ in (2), (3), and (19). For each flexible mode, analytical expressions similar to (11) can be derived. Since the problem is formulated as a constrained time-optimal control problem, other constraints, for example specified fuel usage [42]–[46], move vibration [11], specified deflection [10], [47], velocity limit, etc, can also be included

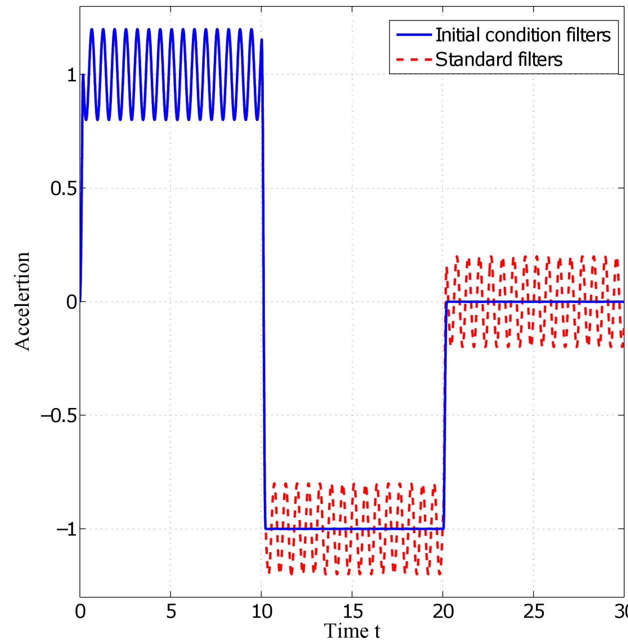

Fig. 5: Acceleration response for closed-form implementation of TS filters designed for cancelling $\omega = 10$ mode and $y_{ref} = 100$.

in the original problem as state constraints. Depending upon the kind of state constraints, the optimal solution might be bang-bang, bang-off-bang, or might containt singular arcs [7].

VII. Conclusion

A new approach was presented for designing vibration reduction shaping filters and shaped control for systems with non-zero initial conditions. The design of vibration reduction shaping filters considers only the flexible modes, while optimal shaped commands are designed for the entire system, including the rigid-body dynamics. The amplitude of residual vibrations and the vibration sensitivity are derived to include the vibrations caused by non-zero initial conditions and different designs are evaluated according to these performance criteria. The true time-optimal solution to such problems was shown to be bang-bang. A simpler parametric formulation was presented that provides the switching times by solving a nonlinear optimization problem. The problem formulation was also considered in discrete time in order to obtain solutions that can be implemented on digital controllers. Simulation results were presented to demonstrate the effectiveness of the proposed approach and experimental results from a portable crane verified the proposed approach.

VIII. Acknowledgment

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