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Optimal Path Planning for Uncertain Exploration

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Abstract— **Exploration always occurs in the presence of uncertainty. In this paper, we consider path planning for autonomous vehicles equipped with range-based sensors and traveling in an uncertain area. The mission of the vehicles is to explore a set of objects of interest while reducing uncertainty in object position, visibility and state. A connection is shown between the Kalman filter (used to reduce uncertainty) and the so-called Shannon model for exploration through the use of a range-based covariance. This connection is exploited to estimate states and to travel between objects of interest. A bound on the covariance error and several illustrative examples are provided.**

I. INTRODUCTION

A. Overview

A vehicle can explore a given area by traveling amongst specific objects of interest and collecting information with range based sensors. The quality of information collected by the onboard sensors can thus be improved by optimizing the vantage point of the explorer. Thus an optimal exploratory path requires an understanding of the coupling between the kinematics of the vehicle and the informatics of the sensor. But the objects of interest can be obscured, their locations may not be known with great accuracy and they may have an unknown state. This paper is devoted to the problem of planning the paths of vehicles to explore a given area regardless of these uncertainties.

The key idea of this work is to exploit properties of the Kalman filter to reduce uncertainty while accounting for the coupling between informatics and kinematics. Specifically we use Shannon's channel capacity equation to represent the maximum rate that information can be transmitted over a noisy channel and a range-based covariance to connect estimation and exploration. Accounting for these joint couplings in the design of optimal paths for exploration is the main conceptual contribution of this paper.

B. Motivation

Autonomous vehicles are often employed to explore an area and investigate objects of interest. The Air Force, Navy and NASA all have many examples of autonomous vehicles being used to collect information [1] [2] [3]. As these vehicles are used in the real world, sensor noise and measurement error are always present. Traditionally a Kalman filter is used to estimate a true state and minimize uncertainty. But the standard Kalman filter does not account for range based sensors where the uncertainty due to noise can be improved by changing position. Thus we have a coupled problem between estimation, information collection and kinematics. In this paper we will show an equivalence between estimation and information collection when kinematics are considered and exploit this connection to find optimal paths and true states.

C. Literature Review

A large body of research has been published in recent years about motion control of autonomous vehicles. Although an exhaustive overview of the state of the art is beyond the scope of this paper, a brief review of the most relevant literature is as follows.

Many methods exist for solving the basic trajectory-planning problem [4]. However, not all of them solve the problem in its full generality. For instance, some methods require the workspace to be two-dimensional and the obstacles, if any, to be polygonal. Despite many external differences, the methods are based on few different general approaches: roadmap [4], [5], [6], cell decomposition [7], [8], [9], [5], [10], potential field [11], [12], [13] and probabilistic [14], [15]. Optimal control approaches have also been studied in [16] and [17].

Information based exploration has been discussed in a few papers in recent years, most notably in Reference [18]. Other methods have used information collection to conduct area searches [19], decentralized sensor control [20] and optimal sensor placement [21]. Kalman first proposed his filter in 1960 [22] and many other formulations have appeared since, such as a state-based covariance [23]. The formulation of an optimal estimator that can affect the uncertainty of its measurements by direct control of its path, however, has been discussed infrequently, first as combined sensing and control [24] and most recently by [18].

Although the current literature discusses various methods of planning optimal paths for exploration and even account for the coupling between informatics and kinematics, few papers account for the additional tri-coupling between estimation, information and kinematics through a range-based covariance. The current paper addresses this issue.

D. Original Contributions

Based on a generic integrated system model, the problem of exploration for autonomous vehicles in the presence of uncertainty is formulated as an optimal path planning problem where the states are the Cartesian coordinates of the vehicles and the amounts of information collected about each

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object of interest, the objective function is the total mission time, and the boundary conditions are subject to inequality constraints that reflect the uncertainty reduction. The present paper studies this optimization problem and provides the following original contributions:

- An exploration problem is solved when the position of the object of interest is uncertain.
- An equivalence is shown between the "Shannon model for exploration" and a Kalman filter.
- This equivalence allows for previously derived properties to be used when uncertainty is present.
- A Kalman filter for exploration is formulated for path planning when the visibility of the state is unknown or when the visibility is affected by Gaussian noise. A bound is given on the lower limit on the uncertainty of the visibility.

E. Paper Outline

The remainder of the paper is as follows. In Section II, the optimization problem is formulated first in terms of Shannon's Information theory and secondly in terms of a Kalman filter while in Section III these formulations are shown to be equivalent. In Section IV, state estimation in the presence of uncertainty and a range-based covariance are addressed. Finally, conclusions and future work are discussed in Section V.

II. Problem Formulation

The problem treated in this paper is to minimize the total mission time required for an autonomous vehicle to collect a specified amount of information about m objects of interest in a given area. The vehicle is assumed to have onboard sensors. Each sensor has a channel of limited bandwidth over which information is collected and the signal-to-noise ratio is dependent upon range.

The vehicle begins at a given initial location with free headings and must collect at least a specified amount of information about each object of interest.

A. Modeling

We seek to explore a given area, by which we mean to collect a specified amount of information about each of m objects of interest, at known locations in the area. To collect information, we use onboard active, energy-based sensors, e.g., radar.

Shannon [25] wrote that information is produced when "one message is chosen from a set of possible messages" and he introduced a theory to quantify this information. The earliest application of this theory was in the engineering of communication systems, which convey messages over a distance. As Shannon stated, "Frequently the messages have meaning; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem" [25]. In this work, the message is the state of a particular object of interest or area. Here, a state is a quantitative measure of an object, i.e., the size of an object, its position, its visibility, the value of its scientific interest, etc, described by real numbers or numbers of bits.

According to [25], the maximum rate at which information can be transmitted over a noisy communication channel (i.e., the channel capacity) is:

$$\dot{I} = w \log_2(1 + \text{SNR}), \tag{1}$$

where w is the channel bandwidth and SNR is the signal-to-noise ratio.

Moreover, according to [26], a radar sensor located at Cartesian coordinates (X, Y) and observing an object at Cartesian coordinates (A_j, B_j) (where j represents a particular object of interest) will provide a reading with signal-to-noise ratio of the form:

$$\text{SNR} = \frac{k^4}{((X - A_j)^2 + (Y - B_j)^2)^2}, \tag{2}$$

where the parameter k depends on the object.

Combining (1) and (2), the information collection model is as follows. Let I_j denote the amount of information that the vehicle has collected about the jth object of interest, $1 \le j \le m$. Then,

$$\dot{I}_j = w \log_2(1 + \frac{k_j^4}{((X - A_j)^2 + (Y - B_j)^2)^2}), 1 \le j \le m, \tag{3}$$

where we assume that all radar sensing processes, viewed as communication channels, have the same bandwidth, and the parameters k_j depend on the jth object of interest, which is located at Cartesian coordinates (A_j, B_j). The signals in exploration missions usually have a low signal-to-noise-ratio so we can simplify (3) with a first-order Taylor series expansion as:

$$R_v^{-1} = \frac{k^4}{((X - A_j)^2 + (Y - B_j)^2)^2}, \tag{4}$$

This model takes neither the uncertainty of the state of the object of interest (interesting or uninteresting) nor noise in the signal into account. A typical Kalman filter can be used to estimate the state of the object, but has no dependence on the sensor range (and thus, on kinematics). This estimator can be expressed as:

$$\dot{X} = V \cos\psi, \tag{5}$$
$$\dot{Y} = V \sin\psi, \tag{6}$$
$$\dot{Z} = 0, \tag{7}$$
$$\dot{P} = AP + PA^T - PC^T R_v^{-1} CP + R_w, \tag{8}$$
$$\dot{\hat{Z}} = A\hat{Z} + K(\tilde{Y} - C\hat{Z}), \tag{9}$$
$$\tilde{Y} = CZ + v, \tag{10}$$
$$K = PC^T R_v^{-1}, \tag{11}$$

where the kinematics of vehicle represented by the change in Cartesian coordinates (X, Y) are defined by a simple unicycle model, the speed of the vehicle is defined by V, the heading of the vehicle by ψ and the state of the object of interest, Z, is unchanging. Here P is the covariance matrix,

A is a linear matrix of state dynamics, C is a linear output matrix of the sensors, R_v is the variance of the measurement noise, R_w is the variance of the model uncertainty, v is Gaussian noise with variance R_v, $\hat{Z}$ is the estimate of the state of the object of interest and $\tilde{Y}$ is the measurement output.

B. Previous Properties

From previous work [27] [28] we have shown that several properties of optimal paths for information collection exist for vehicles with range-based sensors. The proofs of these properties are shown in [28]:

Proposition 1: *If the objects of interest are isolated, then the optimal paths consist of sequences of straight lines (far from the objects of interest) connected by short turns (near the objects of interest).*

Corollary 1: *If in addition to being isolated, the objects of interest are poorly visible, then the problem becomes a multi-vehicle traveling salesman problem (MTSP) [29].*

Proposition 2: *When the visibility of all the objects of interest approaches infinity, $t_f \to 0$ and the lengths of paths traveled by the vehicles approach zero.*

III. Estimation

The Kalman filter model has no dependence upon range to the object of interest, a needed element for the range-based sensors in use and to use the previously derived properties. Here we add a covariance based on range and compare it to the Shannon information collection model shown earlier.

A. Comparison of the Shannon Channel Capacity Equation and the Kalman Filter

$$\dot{I} = \frac{wk^4}{((X-A_j)^2 + (Y-B_j)^2)^2}. \quad (12)$$

As before, the state of Z is fixed, however this state is observable only with the corruption of noise. The variance of the measurement noise, R_v is now dependent upon range through the SNR derived from the radar equation. In this formulation, we assume that k is known.

From the new model, if we consider that the state of the object of interest is "hidden" in the noise with an unchanging state, $A = 0$, $B = 0$, $C = 1$ and $R_w = 0$ yielding:

$$\dot{X} = V\cos\psi, \quad (13)$$
$$\dot{Y} = V\sin\psi, \quad (14)$$
$$\dot{Z} = 0, \quad (15)$$
$$\dot{P} = -PC^T R_v^{-1} CP, \quad (16)$$
$$\dot{\hat{Z}} = K(\tilde{Y} - C\hat{Z}), \quad (17)$$
$$\tilde{Y} = CZ + v, \quad (18)$$
$$R_v^{-1} = \frac{k^4}{((X-A_j)^2 + (Y-B_j)^2)^2}, \quad (19)$$
$$K = PC^T R_v^{-1} \quad (20)$$

The boundary conditions of our optimization problem do not require a specific state of the object of interest or estimated state, rather, they require that the covariance of the estimated state approach 0. The estimated state is independent from the optimization problem so the dynamics that the optimization is subject to are:

$$\dot{X} = V\cos\psi, \quad (21)$$
$$\dot{Y} = V\sin\psi, \quad (22)$$
$$\dot{P} = -PC^T R_v^{-1} CP, \quad (23)$$
$$R_v^{-1} = \frac{k^4}{((X-A_j)^2 + (Y-B_j)^2)^2}, \quad (24)$$

B. Information Filter

An equivalent form of the Kalman filter is the Information filter, derived in [30]. The Kalman filter from (21) - (24) can be reformed as:

$$\dot{P} = AP + PA^T - PC^T R_v^{-1} CP + R_w, \quad (25)$$
$$\zeta = P^{-1}, \quad (26)$$
$$\zeta P = I, \quad (27)$$
$$0 = \dot{\zeta} P + \zeta \dot{P}, \quad (28)$$

$$\dot{\zeta} = -P^{-1}\dot{P}P^{-1}, \quad (29)$$
$$= -P^{-1}(AP + PA^T - RC^T R_v^{-1} CP + R_w)P^{-1}, \quad (30)$$
$$= -\zeta A - A^T P^{-1} + C^T R_v^{-1} C - P^{-1} R_w P^{-1}, \quad (31)$$
$$= -\zeta A - A^T \zeta + C^T R_v^{-1} C - \zeta R_w \zeta \quad (32)$$

where I is the identity matrix and $C = \sqrt{w}$.

Referring back to (21)-(24), we can simplify (32) as:

$$\dot{\zeta} = C^T R_v^{-1} C, \quad (33)$$
$$= \frac{wk^4}{((X-A_j)^2 + (Y-B_j)^2)^2} \quad (34)$$

Therefore our final simplified model is:

$$\dot{X} = V\cos\psi, \quad (35)$$
$$\dot{Y} = V\sin\psi, \quad (36)$$
$$\dot{\zeta} = \frac{wk^4}{((X-A_j)^2 + (Y-B_j)^2)^2}. \quad (37)$$

which is exactly the same as the information collection model formulated from Shannon's channel capacity equation. This equivalence of the Kalman filter and the Shannon formulation shows that information collection is the same as the reduction of uncertainty. Knowledge of the covariance (or information collected) can now be used to estimate the final state as:

$$\dot{\hat{Z}} = K(\tilde{Y} - C\hat{Z}), \quad (38)$$
$$\tilde{Y} = CZ + v, \quad (39)$$
$$K = \zeta^{-1} C^T R_v^{-1} \quad (40)$$

Thus the Kalman filter with a range dependent covariance utilizes Shannon's channel capacity equation for the information state. The previously derived properties can therefore be used to find optimal paths while estimating a state corrupted with measurement noise.

IV. Uncertain State Measurement

The state of the object of interest is not the only state that can be corrupted by noise. In this section we address the measurement of the visibility of the object of interest and its position. Both of these states are needed to plan optimal paths [27] [28] .

A. Corrupted Visibility

The utilization of the exploration method presented in this paper assumes prior knowledge of the location of the object of interest and its reflectivity / emissivity value, k. While aerial surveys or flyovers will often reveal the location of objects of interest, the k value must usually be estimated when planning optimal paths. In this section, we consider the estimation of k in a clear and in an obscured environment.

As motivation, consider an autonomous underwater vehicle (AUV) tasked with identifying underwater mines. The suspected mines have been located by a surface ship quickly surveying the area with a towed array sonar. The ship can only identify objects in the water, but cannot confirm if they are mines or not. An AUV is deployed to ascertain the state of the object of interest (mine or not) given only their initial location. The underwater mines lie in clear water, but over time, growths on the object have changed their accoustical signature leading to an unknown reflectivity (visibility) constant. Thus the correct k value must be estimated in order to correctly predict an optimal path, but it is known to be static throughout the mission period.

The k value can be estimated along with the state of the object of interest. Closer examination, i.e. traveling to reduce the sensor range, will reduce the uncertainty of the measurements. Here the uncertainty depends on k but not on $\hat{k}$, the estimated value of k. Only our prediction model depends on $\hat{k}$. Figure 1 shows the reduction of uncertainty as the vehicle approaches the object of interest and the corresponding error reduction in the estimate for k. The exploration equations to steer the vehicle and estimate the state of k and Z (represented for space by α) are:

$$\dot{X} = V\cos\psi, \quad (41)$$
$$\dot{Y} = V\sin\psi, \quad (42)$$
$$\dot{Z} = 0, \quad (43)$$
$$\dot{k} = 0, \quad (44)$$
$$\dot{P}_\alpha = -P_\alpha C_\alpha^T R_v^{-1} C_\alpha P_\alpha, \quad (45)$$
$$\dot{\hat{\alpha}} = K_\alpha(\tilde{Y}_\alpha - C_\alpha\hat{\alpha}), \quad (46)$$
$$\tilde{Y}_\alpha = C_\alpha \alpha + v, \quad (47)$$
$$R_v^{-1} = \frac{k^4}{((X-A_j)^2 + (Y-B_j)^2)^2}, \quad (48)$$
$$K_\alpha = P_\alpha C_\alpha^T R_v^{-1}. \quad (49)$$

It should be noted here that R_v is always a positive value so the covariance can only be reduced. Passing close to the object of interest and then farther away will not minimize and then increase the uncertainty about its state.

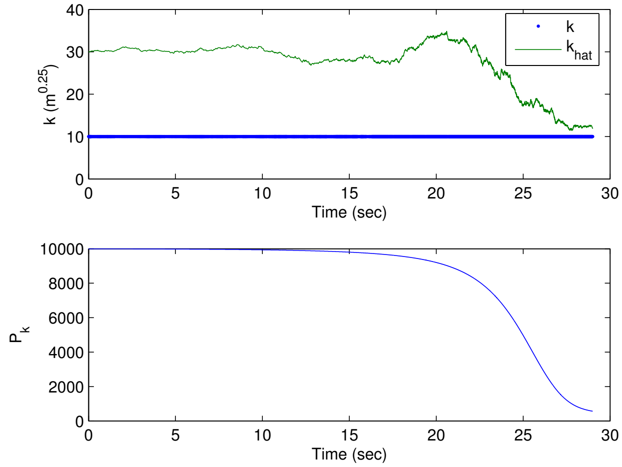

Fig. 1. Convergence of the estimate of k as a function of time.

A more complicated (and realistic) scenario is that of acoustically noisy waters. Here, ships in the area and uncertain terrain can acoustically obscure the waters between the vehicle and the object of interest. While the state itself remains unchanged ($A = 0$), Gaussian noise, W, with variance R_w is added to the model as the so-called process noise as:

$$\dot{k} = W, \quad (50)$$
$$\dot{P}_k = -P_k C_k^T R_v^{-1} C_k P_k + R_w, \quad (51)$$
$$(52)$$

The time history of the covariance and state estimate is shown in figure 2.

In this formulation, a steady state covariance error exists at the final time due to the model noise, W. At the final time, $\dot{P} = 0$. We can find the final covariance error by examining the change in covariance in the single state, P_k.

$$\dot{P}_k = -P_k C_k^T R_v^{-1} C_k P_k + R_w, \quad (53)$$
$$0 = -P_k C_k^T R_v^{-1} C_k P_k + R_w, \quad (54)$$
$$-R_w = -P_k C_k^T R_v^{-1} C_k P_k, \quad (55)$$
$$P_k = \sqrt{\frac{R_w R_v(X,Y)}{C_k^2}} \quad (56)$$

If an object of interest is directly visited, the steady state uncertainty will go to zero. Otherwise the steady-state value of P_k is limited by the range at which the object of interest is approached.

B. Uncertain Object Location

We next remove the assumption that the location of the objects of interest are exactly known. It is often the case that our apriori knowledge of the position of these these objects is marked with uncertainty. While their actual position is stationary, the exact range to the object of interest (or heading

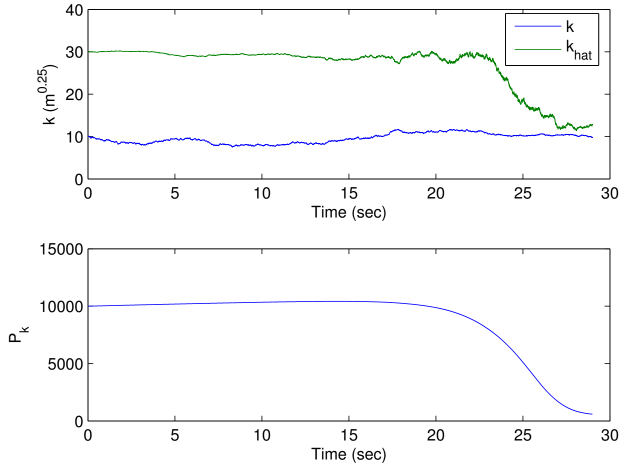

Fig. 2. Convergence of the estimate of k as a function of time. Here $R_w = 40$

to reduce this range) may be unreliable. While similar to the above development (where the position of the object is unchanging but is observed with noise), here multiple states are uncertain.

The state equations are:

$$\dot{X} = V\cos\psi, \qquad \dot{Y} = V\sin\psi, \tag{57}$$

$$\dot{Z} = 0, \qquad \dot{A} = 0, \tag{58}$$

$$\dot{B} = 0, \qquad \dot{k} = W, \tag{59}$$

where the object state and position is fixed while the visibility is dependent upon random environmental changes.

For conciseness, we consider the subscript α to represent Z, A, B, k. The covariance equations are:

$$\dot{P_\alpha} = -P_\alpha C_\alpha^T R_v^{-1} C_\alpha P_\alpha, \tag{60}$$

where each depends upon the range to the object of interest but have no input.

The estimate equations are:

$$\dot{\hat{\alpha}} = K_\alpha(\tilde{Y_\alpha} - C_\alpha\hat{\alpha}). \tag{61}$$

The output equations are:

$$\tilde{Y_\alpha} = C_\alpha \alpha + v, \tag{62}$$

where each output is corrupted by white Gaussian noise.

The variance of the noise:

$$R_v^{-1} = \frac{k^4}{((X - A_j)^2 + (Y - B_j)^2)^2}, \tag{63}$$

is dependent on the range to the object of interest.

The Kalman gains are:

$$K_\alpha = P_\alpha C_\alpha^T R_v^{-1}, \tag{64}$$

Rather than plan a path to the actual object of interest, here, since the object position is unknown, we plan a path to the estimated location of the object of interest $(\hat{A}, \hat{B})$ with:

$$\psi(t) = \arctan\frac{\hat{B}_j - Y}{\hat{A}_j - X}. \tag{65}$$

Figures 4 and 5 shows the time history of the state estimation and Figure 3 shows the vehicle path with uncertain object of interest placement.

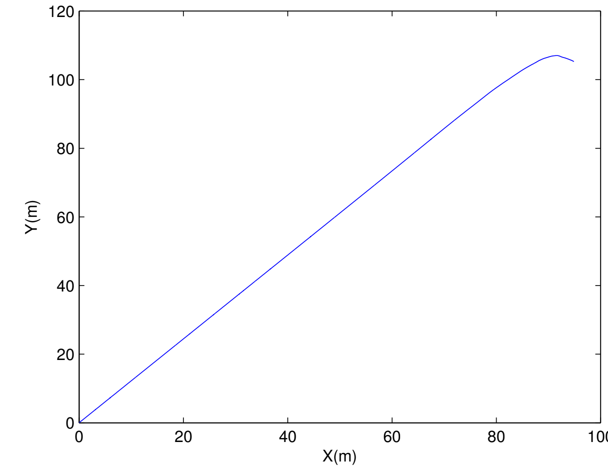

Fig. 3. Flight path with uncertain object of interest location

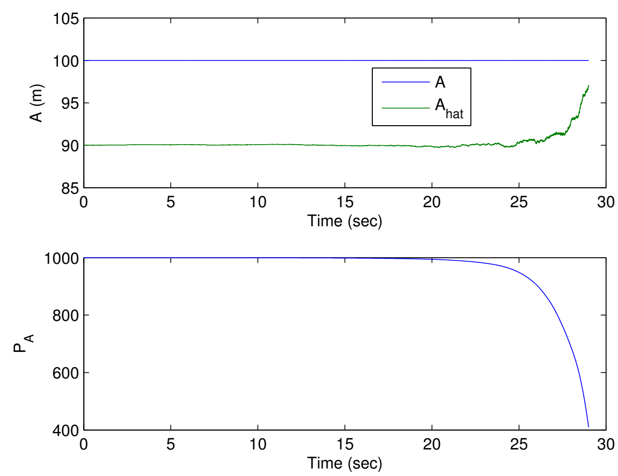

Fig. 4. Time history of the convergence of the A estimate

In the neighborhood of each object of interest there exists a visibility disk, within which information can be collected at appreciable rates. Outside of the visibility disk for a particular object, no information can be collected about that object. The visibility disk is assumed isotropic, i.e., the rate at which information is collected depends only upon the range from the object to the explorer, not on their relative azimuth. As shown in each of the above figures, there is a definite range at which uncertainty is reduced, which outlines the radius of the visibility disk.

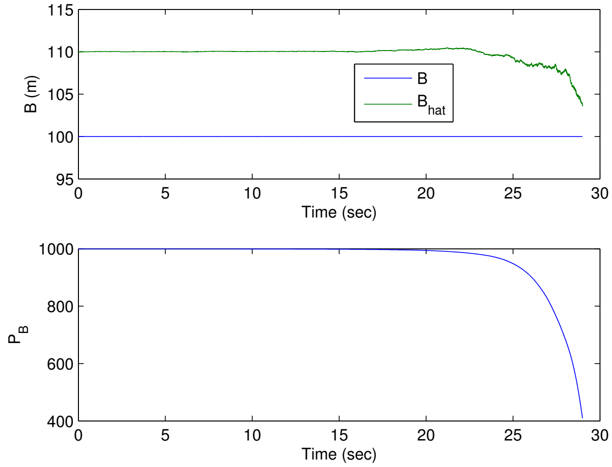

Fig. 5. Time history of the convergence of the B estimate

We define D_j as the isotropic visibility disk centered on the jth object of interest. When the vehicle is within the visibility disk of the jth object, i.e., $(X, Y) \in D_j$, the object is considered visible. Otherwise the object is invisible to the vehicle. Define D to be the union of all visibility disks $D_j, 1 \leq j \leq m$, i.e., $D = \cup_{j=1}^{m} D_j$.

Futher definitions will simplify the remaining analysis. Objects are said to be clustered if their visibility discs are pathwise connected. Furthermore, clusters are said to be isolated if they are not pathwise connected. Finally, an object is said to be isolated if it is not in any cluster.

We can return to the properties of optimal paths discussed earlier. A vehicle will travel straight if it is outside of D and will turn when it is within D.

V. Conclusions

Parameters, such as object visibility and position, can be estimated and accounted for in the synthesis of optimal paths. Estimation is only effective, however, when information can be collected, inside the visibility disc.

Future work will examine the use of the clustering heuristic with non-isotropic sensors as well as applying uncertain visibility estimation to optimal non-isotropic path planning.

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