2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Stability Analysis of Heat Exchanger Dynamics

Tiejun Zhang, John T. Wen, Juan Catano, and Rongliang Zhou

Abstract— **In the study of vapor compression cycle, momentum balance equation is often ignored in the heat exchanger model. In this paper, we investigate the effect of the momentum balance through a systematic study of the open loop stability of a heat exchanger. We consider 1-D fluid flow in a pipe in four cases of increasing complexity the most general case corresponds to the heat exchanger model: 1. incompressible flow without heat transfer; 2. incompressible flow with heat transfer; 3. compressible flow without heat transfer; 4. compressible flow with heat transfer. Among the three balance equations, mass, momentum, and energy, case 1 involves only the momentum, case 2 involves both momentum and energy, case 3 involves mass and momentum, and case 4 requires all three equations. It is shown that in cases 1, which corresponding to the incompressible flow without heat input, the system is lumped and always stable, and in cases 2, 3 and 4, the system is stable if and only if the equilibrium flow velocity is sufficiently high. Finite difference approximation and linearization of the dynamic models are used for local stability evaluation in case 3 and 4. The overall cycle analysis as well as a simulation example is also included. The result of this study now forms the foundation to investigate the open loop stability and closed loop control design for vapor compression cycles used in HVAC and electronic cooling systems.**

I. INTRODUCTION

Vapor compression cycles (VCC) have been widely applied in various industrial and household thermal-fluid processes, such as air-conditioning and refrigeration systems [1], [2]. Recently, the refrigeration cycle has been used for the purpose of high-power microelectronics cooling [3]. In general, the highly-integrated chips are compactly assembled with high-density thermal energy generation, and their peak heat fluxes are expected to reach up to 1 kW/cm^2. This will lead to critical heat dissipation problems. Transient simulation is another critical concern for plant operation and optimization [1], [2], [4], [5]. To improve the cooling capability and efficiency of the whole refrigeration system, it is desirable to investigate novel dynamic thermal management methods at the system level. These can be used to obtain insightful understanding of the vapor compression cycle operation. A reliable dynamic model can also be used to predict large transient behavior during system start-up/shutdown [6], [7] and to avoid potentially harmful operating conditions.

Due to the large inertia of heat and mass transfer processes, the evaporator and condenser are generally treated as dynamic components. These transient heat exchangers play dominant roles in the overall system since they usually retain most of the active refrigerant charge in the cycle. To develop a low-order heat exchanger model with appropriate accuracy, the lumped-parameter moving-boundary (MB) method is employed [8]. It assumes mean void fraction in heat exchangers to be time-invariant, simplifying the challenging problem from the transient momentum conservation. It should be noted that the assumption of time-invariant mean void fraction is only demonstrated to be valid for complete evaporation or condensation. A moving-boundary (MB) formulation with a simple set of ordinary differential equations is given in [9] for dynamic simulation of refrigeration system, including the superheat vapor, two-phase, and subcooled liquid zones. However, the three-zone evaporator model is only valid over small variations in operating conditions. The advanced moving-boundary models with switching policy may obtain better accuracy and robustness in wide transient operation range [10]. During the vapor compression system start-up or shut-down processes, the drastic dynamics changes will drive the above three zones to form or disappear correspondingly [4]. Instead of the traditional moving boundary methods, the finite volume (FV) formulation is another kind of dynamic modeling approach, which is verified to be more robust through start-up and all load-change transients of a centrifugal chiller. In addition, the FV model has better charge prediction ability compared with the homogeneous two-phase assumption of MB model. The disadvantage is that FV model executes three-times slower than MB with similar accuracy [6].

Nevertheless, none of the above dynamic models has included the full conservation equations, especially the momentum balance. The pressure drop in heat exchangers is neglected in both the MB and FV models. In fact, it is important to include the momentum balance in electronics cooling, where microchannels are commonly used and significant pressure drop is observed in microscale heat exchangers [11]. In this paper, we try to study the flow system with increasing complexity, from the simplest incompressible flow without heat input to the complex compressible flow with heat input. Eventually, the comprehensive heat exchanger dynamics, including the full mass balance, energy balance, and momentum balance equations, is under investigation. A set of analytical stabilizing conditions can be explicitly obtained. With the inlet boundary conditions, the open-loop stability of heat exchanger is shown to be relevant to its fluid flow velocity. A numerical simulation example is finally presented to demonstrate the corresponding stability results.

II. GENERAL HEAT EXCHANGER DYNAMICS

Heat exchangers, including evaporator and condenser, are the key dynamic components in a vapor compression refrig-

Th authors are with Center for Automation Technologies and Systems, Rensselaer Polytechnic Institute, 110 8th Street, Troy, NY 12180, USA {zhangt6,wenj,catanj,zhour}@rpi.edu

eration cycle. Without loss of generality, the comprehensive conservation principles for refrigerant flow can be formulated as the following set of partial differential equations.

Mass Balance:

$$\frac{\partial \rho A}{\partial t} + \frac{\partial \dot{m}}{\partial z} = 0 \tag{1}$$

Momentum Balance:

$$\frac{\partial \dot{m}}{\partial t} + \frac{1}{A}\frac{\partial(\dot{m}^2/\rho)}{\partial z} + \frac{\partial PA}{\partial z} + F_{visc} = 0 \tag{2}$$

Energy Balance:

$$\frac{\partial[\rho A(u+v^2/2)]}{\partial t} + \frac{\partial[\dot{m}(u+v^2/2)+PAv]}{\partial z} - \frac{q}{L} = 0 \tag{3}$$

where $\dot{m}$ is the mass flowrate, $\dot{m} = \rho A v$, q is the refrigerant heat gain from the hot side, F_{visc} is the friction force due to fluid viscosity, $F_{visc} = \pi D \tau$ for circular channels.

For a horizontal heat exchanger, the specific total energy $e = u + v^2/2$, the mass flux $w = \rho v$, and defining $\xi = \rho e$, the full refrigerant dynamics can be characterized by the following equations:

Mass Balance:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial w}{\partial z} \tag{4}$$

Momentum Balance:

$$\frac{\partial w}{\partial t} = -\frac{\partial(w^2/\rho + P)}{\partial z} - \frac{4\tau}{D} \tag{5}$$

Energy Balance:

$$\frac{\partial \xi}{\partial t} = q' - \frac{\partial[w(\xi+P)/\rho]}{\partial z} \tag{6}$$

where the specific heat gain from outside, $q' = \frac{q}{AL}$, the kinetic energy term $v^2/2$ compared with the enthalpy is usually negligible and thus ignored in most cases [1], [2], [4], [5]. Here for the simplicity of analysis, assuming there exists the homogeneous friction stress $\tau = \frac{f}{2}\rho v^2 = \frac{fw^2}{2\rho}$ for two-phase flow. The detailed friction stress correlations for both the single and two-phase flows are given in the Appendix or the reference [11].

Since there are three first order spatial derivatives, there needs to be three boundary conditions in general

$$g\left(\begin{bmatrix} \rho(t,0) \\ w(t,0) \\ \xi(t,0) \end{bmatrix}, \begin{bmatrix} \rho(t,L) \\ w(t,L) \\ \xi(t,L) \end{bmatrix}\right) = 0. \tag{7}$$

In addition, the wall energy balance may also be included in the comprehensive heat exchanger dynamics,

$$C_w M_w \frac{\partial T_w}{\partial t} - \frac{\partial q_{in}}{\partial z} + U_r \pi D (T_w - T_r) = 0. \tag{8}$$

where U_r is the heat transfer coefficient between the refrigerant and wall, then q' in (6) reads $4U_r(T_w - T_r)/D$. Here the Boelter, Gnielinski and Petukhov correlations are employed for the single-phase boiling, while Kandlikar correlation is used for two-phase nucleate and convective boiling [11]. As for the condensation, the Shah's correlation is used for the prediction of heat transfer coefficient [12].

A. Linearized Heat Exchanger Dynamics

To analyze the stability of the heat exchanger about an operating point, consider a specific operating condition (ρ^*, w^*, ξ^*) corresponding to a constant heat input q'. Since the system is at a steady state, the equilibrium condition satisfies:

$$\begin{aligned} \frac{\partial w^*}{\partial z} &= 0 \\ \frac{\partial(w^{*2}/\rho^* + P^*)}{\partial z} + \frac{4\tau^*}{D} &= 0 \\ \frac{\partial(w^*(\xi^*+P^*)/\rho^*)}{\partial z} &= q'. \end{aligned} \tag{9}$$

with the linearized boundary condition from (7):

$$D_0 \begin{bmatrix} \rho^*(0) \\ w^* \\ \xi^*(0) \end{bmatrix} + D_1 \begin{bmatrix} \rho^*(L) \\ w^* \\ \xi^*(L) \end{bmatrix} = c \tag{10}$$

where D_0 and D_1 are two constant 3×3 matrices, and c is a 3×1 constant vector. Note that w^* is constant throughout the heat exchanger, but ρ^* and ξ^* are possibly spatially varying (i.e., dependent on z).

Linearize the 3 balance equations (4)–(6) about (ρ^*, w^*, ξ^*), and write the equations in terms of the deviation from the equilibrium, $\widetilde{\rho} := \rho - \rho^*$, $\widetilde{w} := w - w^*$, $\widetilde{\xi} := \xi - \xi^*$, we get the third order convective equation with spatially varying coefficients and coupled boundary conditions:

$$\begin{aligned} &\frac{\partial \mathcal{X}(t,z)}{\partial t} = \frac{\partial A(z)\mathcal{X}(t,z)}{\partial z} + E(z)\mathcal{X}(t,z) \\ &D_0\mathcal{X}(t,0) + D_L\mathcal{X}(t,L) = 0 \end{aligned} \tag{11}$$

where

$$\mathcal{X} = \begin{bmatrix} \widetilde{\rho} & \widetilde{w} & \widetilde{\xi} \end{bmatrix}^T,$$

$$A = \begin{bmatrix} 0 & -1 & 0 \\ (\frac{w^{*2}}{\rho^{*2}} - \frac{\partial P^*}{\partial \rho}) & -2\frac{w^*}{\rho^*} & -\frac{\partial P^*}{\partial \xi} \\ \frac{w^*}{\rho^*}(\frac{\xi^*+P^*}{\rho^*} - \frac{\partial P^*}{\partial \rho}) & -\frac{\xi^*+P^*}{\rho^*} & -\frac{w^*}{\rho^*}(1+\frac{\partial P^*}{\partial \xi}) \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{4}{D}\frac{\partial \tau^*}{\partial \rho} & -\frac{4}{D}\frac{\partial \tau^*}{\partial w} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

B. Finite Difference Discretization

We now return to the original equation (11). There are many possible discretization schemes, from the Galerkin approximation to finite element method. In this paper, we approximate $\frac{\partial}{\partial z}$ using a simple finite differencing scheme. Let $\mathcal{X}_i = \mathcal{X}(i\Delta z)$, $\Delta z = L/N$. Using backward differencing, the approximation of (11) becomes:

$$\dot{\mathcal{X}}_i = A_i\mathcal{X}_i - A_{i-1}\mathcal{X}_{i-1} + E_i\mathcal{X}_i \tag{12}$$

where $A_i = A(i\Delta z)/\Delta z$, $E_i = E(i\Delta z)$ and the boundary condition is $D_0\mathcal{X}_0 + D_L\mathcal{X}_N = 0$. In general, let $D^\perp$ be a matrix whose columns form a basis for the null space of $\begin{bmatrix} D_0M & D_LM \end{bmatrix}$:

$$\begin{bmatrix} D_0M & D_LM \end{bmatrix} D^\perp = 0. \tag{13}$$

More explicitly, the spatial finite difference heat exchanger model mentioned above can be described as follows

Finite Difference Approximate Dynamic Model:

For $z \in [0, L], \Delta z = L/N, i = 0, 1, \ldots, N$, it follows that

Mass Balance:

$$\frac{d\rho_i}{dt} = \frac{1}{\Delta z}\left(w_{i-1} - w_i\right) \tag{14}$$

Momentum Balance:

$$\frac{dw_i}{dt} = \frac{1}{\Delta z}\left(\frac{w_{i-1}^2}{\rho_{i-1}} + P_{i-1} - \frac{w_i^2}{\rho_i} - P_i\right) - \frac{4\tau_i}{D} \tag{15}$$

Energy Balance:

$$\frac{d\xi_i}{dt} = q' + \frac{1}{\Delta z}\left[\frac{w_{i-1}}{\rho_{i-1}}(\xi_{i-1} + P_{i-1}) - \frac{w_i}{\rho_i}(\xi_i + P_i)\right] \tag{16}$$

III. STABILITY ANALYSIS OF HEAT EXCHANGER SYSTEMS

With the above finite-difference modeling method, the linearization of heat exchanger model (14)-(16) can be represented in the general form (12). Notice that when the inlet condition is specified, $\mathcal{X}_0 = 0$, the stability is determined by the stability of all the discretized matrices A_i.

A case of particular interest is when the output of the heat exchanger is connected back to the input with a gain of γ:

$$\mathcal{X}_0 = -\gamma \mathcal{X}_N. \tag{17}$$

And it follows from (13) that

$$D^{\perp} = \begin{bmatrix} \gamma I \\ -I \end{bmatrix}. \tag{18}$$

If all A_i are Hurwitz (i.e., all eigenvalues are in the open left half plane, $M^{-1}AM = \Lambda, \Lambda < 0$), the necessary and sufficient condition for stability is

$$\gamma^2 < 1. \tag{19}$$

This condition is not particularly surprising, it follows that the L_2 gain of the heat exchanger (with input $\mathcal{X}(t, 0)$ and output $\mathcal{X}(t, L)$) is bounded by 1. Hence (19) is simply the classic small gain stability condition.

However, when the outlet condition is specified, $\mathcal{X}_N = 0$, the backward differencing scheme needs to be replaced by forward differencing, resulting in the stability condition that A_k matrices need to be anti-stable, which also agrees with the previous analysis.

We could also transform (12) to the Laplace domain and view the heat exchanger as a matrix transfer function from $\mathcal{X}_0$ to $\mathcal{X}_N$:

$$\mathcal{X}_N = \underbrace{(-1)^N \prod_{i=1}^{N} (sI - (A_i + E_i))^{-1} A_i}_{\mathcal{F}} \mathcal{X}_0. \tag{20}$$

If $(A_i + E_i)$ is a stable matrix, the gain (or the H_∞-norm) of $\mathcal{F}$ satisfies the small gain condition:

$$\|\mathcal{F}\|_{H_\infty} = \sup_{\omega} \|\mathcal{F}(j\omega)\| \leq 1. \tag{21}$$

This is an important property when we consider the full refrigeration cycle as the interconnection of heat exchangers.

A. Incompressible Flow without Heat Input (Case 1)

For incompressible flow, such as liquid, the governing equations (4)-(6) can be significantly simplified. The mass density always remains constant with respect to both time and location, so the mass balance equation or continuity equation degenerates as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial w}{\partial z} = 0 \tag{22}$$

It can be implied that the mass flux also remains constant, $w_i = w_0 = \bar{w}$, $\rho_i = \rho_0 = \bar{\rho}$ in the finite difference model. Since there is no heat input here, so the fluid pressure is only dependent on density and also kept constant, $P_i = P_0 = \bar{P}$. The fluid in heat exchanger is homogeneous and characterized by a simple lumped model. As a result, only the momentum balance is involved in this case, which is the well-known Navier-Stokes equation.

Momentum Balance:

$$\frac{d\bar{w}}{dt} = -\frac{4\bar{\tau}}{D} \tag{23}$$

Let $\delta\bar{w} = \bar{w} - \bar{w}^*$ around the equilibrium $\bar{w}^*$, one may obtain the linearized model

$$\frac{d(\delta\bar{w})}{dt} = -\left(\frac{4}{D}\frac{\partial \bar{\tau}^*}{\partial \bar{w}}\right)\delta\bar{w}^* \tag{24}$$

The incompressible flow system without heat input is locally stable if and only if the condition (25) is satisfied

$$\frac{\partial \bar{\tau}^*}{\partial \bar{w}} > 0. \tag{25}$$

In fact, this condition holds definitely since the friction $\bar{\tau}^*$ always increases with the velocity value or the mass flux w.

B. Incompressible Flow with Heat Input (Case 2)

As shown above, for incompressible flow, the density and the mass flux are lumped along the length, $\rho_i = \rho_0 = \bar{\rho}$, $w_i = w_0 = \bar{w}$. When the heat is transfered into the fluid, its specific energy changes correspondingly, thus the pressure P only changes with ξ. Then the fluid in heat exchanger will be subject to two balances below.

Momentum Balance:

$$\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z} - \frac{4\tau}{D} = -\frac{\partial P}{\partial \xi}\frac{\partial \xi}{\partial z} - \frac{4\tau}{D} \tag{26}$$

Energy Balance:

$$\frac{\partial \xi}{\partial t} = q' - \frac{w}{\rho}\frac{\partial(\xi + P)]}{\partial z} = q' - \frac{w}{\rho}\left(1 + \frac{\partial P}{\partial \xi}\right)\frac{\partial \xi}{\partial z} \tag{27}$$

Then by resorting to the linearizing method mentioned in Section II, it is easy to get the following local linear model

$$\begin{bmatrix} \frac{\partial(\delta w)}{\partial t} \\ \frac{\partial(\delta \xi)}{\partial t} \end{bmatrix} = \begin{bmatrix} -\frac{4}{D}\frac{\partial \bar{\tau}^*}{\partial \bar{w}^*} & -\frac{\partial P^*}{\partial \xi} \\ -\frac{q'}{\bar{w}^*} & -\frac{\bar{w}^*}{\bar{\rho}^*}\left(1 + \frac{\partial P^*}{\partial \xi}\right) \end{bmatrix} \begin{bmatrix} \frac{\partial(\delta w)}{\partial z} \\ \frac{\partial(\delta \xi)}{\partial z} \end{bmatrix} \tag{28}$$

where $\delta w = w - w^*,\ \delta\xi = \xi - \xi^*,\ (w^*, \xi^*)$ is the steady-state mass flux and specific energy at the equilibrium point.

The incompressible flow system with heat input is locally stable if and only if the conditions (29)-(30) are satisfied

$$\frac{4}{D}\frac{\partial \bar{\tau}^*}{\partial \bar{w}} + \frac{\bar{w}^*}{\bar{\rho}^*}\left(1 + \frac{\partial P^*}{\partial \bar{\xi}}\right) > 0, \tag{29}$$

$$\frac{4\bar{w}^*}{\bar{\rho}^* D}\frac{\partial \bar{\tau}^*}{\partial \bar{w}}\left(1 + \frac{\partial P^*}{\partial \bar{\xi}}\right) - \frac{q'}{\bar{w}^*}\frac{\partial P^*}{\partial \bar{\xi}} > 0. \tag{30}$$

It should be noted that $\frac{\partial P}{\partial u} = \rho \frac{\partial P}{\partial \xi}$ since the specific energy variable $\xi = \rho(u + \upsilon^2/2)$. In general, the friction τ increases with the mass flux w, $\frac{\partial \tau}{\partial w} > 0$, and the pressure P increases with the specific internal energy u and ξ, $\frac{\partial P}{\partial \xi} > 0$, therefore the condition (29) holds for the positive flow velocity $\upsilon = \frac{w}{\rho}$.

C. Compressible Flow without Heat Input (Case 3)

As a matter of fact, incompressible flow is only an idealization to simplify our analysis. In reality, all materials are compressible to some extent, especially the gas flow. So it is more desirable to investigate the stability of compressible flow systems, here we start from the isothermal heat exchanger with mass and momentum equations (14)-(15).

With the local linearizing method in Section II, one may get

$$A_i = \begin{bmatrix} 0 & -\frac{1}{\Delta z} \\ \frac{1}{\Delta z}\left(\frac{w_i^{*2}}{\rho_i^{*2}} - \frac{\partial P_i^*}{\partial \rho_i}\right) - \frac{4}{D}\frac{\partial \tau_i^*}{\partial \rho_i} & -\frac{2w_i^*}{\rho_i^*\Delta z} - \frac{4}{D}\frac{\partial \tau_i^*}{\partial w_i} \end{bmatrix} \tag{31}$$

The eigenvalues of $\mathcal{A}_i$ can be obtained from its characteristic equation $\det(\lambda I - \mathcal{A}_i) = 0$, that is,

$$\lambda^2 + \left(\frac{2w_i^*}{\rho_i^*\Delta z} + \frac{4}{D}\frac{\partial \tau_i^*}{\partial w_i}\right)\lambda + \frac{1}{\Delta z^2}\left(\frac{w_i^{*2}}{\rho_i^{*2}} - \frac{\partial P_i^*}{\partial \rho_i}\right) - \frac{4}{\Delta z D}\frac{\partial \tau_i^*}{\partial \rho_i} = 0 \tag{32}$$

The compressible flow system without heat input is locally stable if and only if both of the conditions (33)-(34) are satisfied

$$\frac{w_i^*}{\rho_i^*} + \frac{2\Delta z}{D}\frac{\partial \tau_i^*}{\partial w_i} > 0, \tag{33}$$

$$\left(\frac{w_i^{*2}}{\rho_i^{*2}} - \frac{\partial P_i^*}{\partial \rho_i}\right) - \frac{4\Delta z}{D}\frac{\partial \tau_i^*}{\partial \rho_i} > 0. \tag{34}$$

The condition (33) definitely holds since the friction τ increases with the mass flux w.

D. Compressible Flow with Heat Input (Case 4)

Linearizing the full finite difference dynamics (14)-(16) yields the model (12) with (35) (shown at the top of the next page). The eigenvalues of A_i can be obtained from its characteristic equation, $\det(\lambda I - A_i) = 0$. Then the stability of the discretized system is determined by the the system matrix in (35) by using the Routh-Hurwitz stability criterion [13]. Defining the mean flow velocity $\upsilon = w/\rho$, and

$$a = \frac{\xi + P}{\rho},\ b = \frac{\partial P}{\partial \rho},\ c = \frac{\partial P}{\partial \xi},\ d = \frac{f\Delta z}{D}$$

the system matrix (35) is equivalently as follows

$$\begin{bmatrix} 0 & -1 & 0 \\ (1+2d_i)\upsilon_i^{*2} - b_i & -(2+4d_i)\upsilon_i^* & -c_i \\ (a_i - b_i)\upsilon_i^* & -a_i & -(1+c_i)\upsilon_i^* \end{bmatrix} \tag{36}$$

Then the stabilizing conditions of the compressible flow system with heat input become as follows

$$(3 + c_i + 4d_i)\,\upsilon_i^* > 0, \tag{37}$$

$$(3 + 2c_i + 6d_i + 4c_i d_i)\,\upsilon_i^{*2} > a_i c_i + b_i, \tag{38}$$

$$(1 + c_i + 2d_i + 2c_i d_i)\,\upsilon_i^{*2} > a_i c_i + b_i, \tag{39}$$

$$\begin{aligned}\big(8 + 8c_i + 28d_i + 24c_i d_i + 2c_i^2 + 16c_i d_i^2 \\ + 4c_i^2 d_i + 24d_i^2\big)\,\upsilon_i^{*2} > 2a_i c_i + a_i c_i^2 \\ + 4a_i c_i d_i + 2b_i + b_i c_i + 4b_i d_i.\end{aligned} \tag{40}$$

These conditions are thus available for stability evaluation of the heat exchanger system around equilibrium.

IV. STABILITY ANALYSIS OF REFRIGERATION CYCLE

The heat exchanger dynamics (4)-(6) is applicable to both the evaporator and condenser. We now consider the entire refrigeration cycle with the evaporator heat input q_e, condenser heat extraction q_c, compressor speed ω, and valve opening A_v. For the linearized analysis, first consider the cycle in a steady state under a constant operating condition $(q_e^*, q_c^*, \omega^*, A_v^*)$. The linearized dynamics for the evaporator and condenser are both of the form in (11), and may be combined into a model of the same structure:

$$\frac{\partial \mathcal{X}(t,z)}{\partial t} = \frac{\partial \mathcal{A}(z)\mathcal{X}(t,z)}{\partial z} + \mathcal{E}(z)\mathcal{X}(t,z) \tag{41}$$

$$\mathcal{X} = \begin{bmatrix} \mathcal{X}_e \\ \mathcal{X}_c \end{bmatrix},\ \mathcal{A} = \begin{bmatrix} A_e & 0 \\ 0 & A_c \end{bmatrix},\ \mathcal{E} = \begin{bmatrix} E_e & 0 \\ 0 & E_c \end{bmatrix}.$$

The boundary conditions of the evaporator and condenser are connected through the expansion valve and compressor. Linearized about the operating point, we have

Linearized Valve Model (v):

$$\mathcal{X}_e(t,0) = G_v \mathcal{X}_c(t, L_c) + H_v \delta A_v \tag{42}$$

Linearized Compressor Model (m):

$$\mathcal{X}_c(t,0) = G_m \mathcal{X}_e(t, L_e) + H_m \delta\omega \tag{43}$$

where $\delta A_v = A_v - A_v^*$ and $\delta\omega = \omega - \omega^*$ are the control variables.

The same finite difference discretization scheme may be applied to approximate the convective type partial differential equation (41) by a set of ordinary differential equations as in (12). Let $\mathbf{X}$ be the discretized $\mathcal{X}_e$ and $\mathcal{X}_c$:

$$\mathbf{X} = \begin{bmatrix} \mathcal{X}_{e_1} & \dots & \mathcal{X}_{e_{N_e}} & \mathcal{X}_{c_1} & \dots & \mathcal{X}_{c_{N_c}} \end{bmatrix}^T.$$

Then

$$\begin{aligned}\dot{\mathbf{X}} &= \begin{bmatrix} A_e & 0 \\ 0 & A_c \end{bmatrix}\mathbf{X} + \begin{bmatrix} E_e & 0 \\ 0 & E_c \end{bmatrix}\mathbf{X} \\ &\quad + E_0(GE_N\mathbf{X} + Hu).\end{aligned} \tag{44}$$

$$A_i = \left[\begin{array}{ccc} 0 & -\frac{1}{\Delta z} & 0 \\ \frac{1}{\Delta z}\left(\frac{w_i^{*2}}{\rho_i^{*2}} - \frac{\partial P_i^*}{\partial \rho_i}\right) - \frac{4}{D}\frac{\partial \tau_i^*}{\partial \rho_i} & -\frac{2w_i^*}{\rho_i^* \Delta z} - \frac{4}{D}\frac{\partial \tau_i^*}{\partial w_i} & -\frac{1}{\Delta z}\frac{\partial P_i^*}{\partial \xi_i} \\ \frac{1}{\Delta z}\left(\frac{w_i^* \xi_i^*}{\rho_i^{*2}} + \frac{w_i^* P_i^*}{\rho_i^{*2}} - \frac{w_i^*}{\rho_i^*}\frac{\partial P_i^*}{\partial \rho_i}\right) & -\frac{\xi_i^* + P_i^*}{\rho_i^* \Delta z} & -\frac{w_i^*}{\rho_i^* \Delta z}\left(1 + \frac{\partial P_i^*}{\partial \xi_i}\right) \end{array} \right] \tag{35}$$

where A_e and A_c are block subdiagonal matrices, E_e and E_c are diagonal matrices, and

$$E_0 = \left[\begin{array}{cccccc} -A_{e_1}^T & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & -A_{c_1}^T & \dots & 0 \end{array} \right]^T,$$

$$E_N = \left[\begin{array}{cccccc} 0 & \dots & I & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & I \end{array} \right],$$

$$G = \left[\begin{array}{cc} 0 & G_v \\ G_m & 0 \end{array} \right], H = \left[\begin{array}{cc} H_v & 0 \\ 0 & H_m \end{array} \right], u = \left[\begin{array}{c} \delta A_v \\ \delta \omega \end{array} \right].$$

This is in the standard state space form of a linear control system, and standard linear system analysis and design tools may be applied.

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}u \tag{45}$$

Note that the interconnection matrices, the linearized gains of the compressor and valve, G_m and G_v, directly affects the property of the system. The full cycle may also be viewed as the interconnection between the evaporator and condenser through the gain matrices G_m and G_v. We have shown that if the evaporator and condensers are both stable, then the inlet to outlet transfer function is small gain. Hence, a sufficient condition for the full cycle stability is that the interconnection gains G_m and G_v preserves the small gain condition.

V. CASE STUDY

The refrigeration system parameters are acquired from the testbed component manufacturers for simulation purposes. Refrigerant R-134a is chosen as the working fluid, whose thermophysical property data are provided by the NIST[1]. In the following simulations, the heat exchanger is divided into $N = 5$ sections along its flow path for finite difference models. The detailed component models of the refrigeration system have been developed in [14].

With loss of generality, we consider four input variables of the refrigeration system: the evaporator heat input q_e (this is to simulate the electronics cooling power), the compressor speed ω, the opening position A_v of electronic expansion valve, and the condenser cooling water flowrate m_{dc}. In this example, we choose the initial system inputs as follows q_e=7.5 (kW), ω=3000 (rpm), A_v=15 (%), m_{dc}=0.3 (kg/s). Then by solving a set of nonlinear equations, we may obtain the corresponding steady states, which are presented in Fig. 1. It shows the pressure-enthalpy ($P - h$) diagram of vapor compression cycle, where 'C','V','E','M' denote the exit points of the condenser, expansion valve, evaporator and compressor, respectively. Notice that the pressure drop is relatively small, e.g. the steady pressure profile along the condenser length P_c=[1162.3,1161.9,1161.6,1161.4,1161.3] (kPa). This is because convectional-scale evaporator and condenser are used in our current first-stage testbed.

[1]National Institute of Standards and Technology, http://webbook.nist.gov/chemistry/fluid/

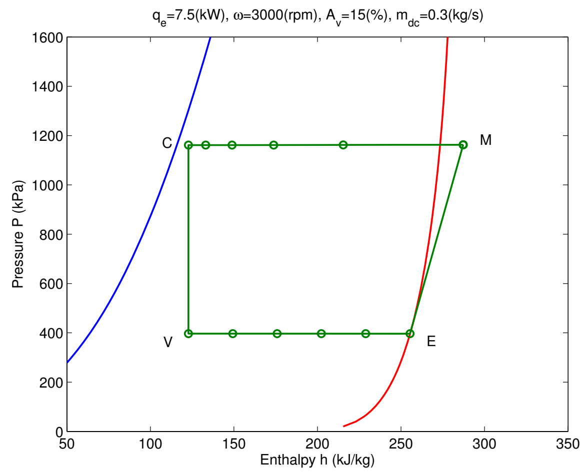

Fig. 1. Pressure-Enthalpy Diagram from Finite Diff. Heat Exchanger Model

Rather than just the steady-state operation, the transient dynamics and stability of the refrigeration system are the main concerns in this paper. To confirm the cycle stability, one may investigate the open-loop responses of the refrigeration system to the inputs q_e, ω, A_v, respectively. Starting from the above steady states, we impose a step increase on the evaporator heat q_e from 7.5 (kW) to 8.5 (kW) at the 5th second then let the refrigeration system settle down to new steady states. Afterwards, we change compressor speed stepwisely from 3000 (rpm) to 4000 (rpm) at the 30th second while q_e is fixed at 8.5 (kW) and A_v is fixed at the initial value of 15 (%). The next step is to change A_v separately at the 55th second, both evaporator and condenser settle down to new equilibriums as shown in Fig. 2 and Fig. 3. For the safe operation purpose, we should make sure the evaporator exit flow is vapor only all the time, otherwise it would damage the compressor. The transient evaporator exit flow quality is depicted in Fig. 4. Also in this figure, the cycle of performance (COP), superheat degree (SH) and cycle energy gain ($E_{CR} = W_m + q_e - q_c$) responses in transient are included, where W_m is the compressor power and q_c the condenser dumping energy.

VI. CONCLUSIONS

In this paper, systematic studies have been performed to investigate the effect of momentum balance on the heat exchanger dynamics and its open-loop stability. With respect to different flow conditions, four cases are taken into account

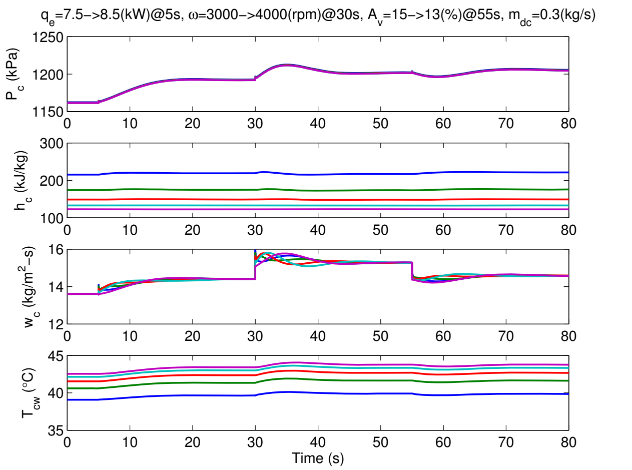

Fig. 2. Dynamic responses of refrigerant pressure, enthalpy, mass flux, and wall temperature in condenser

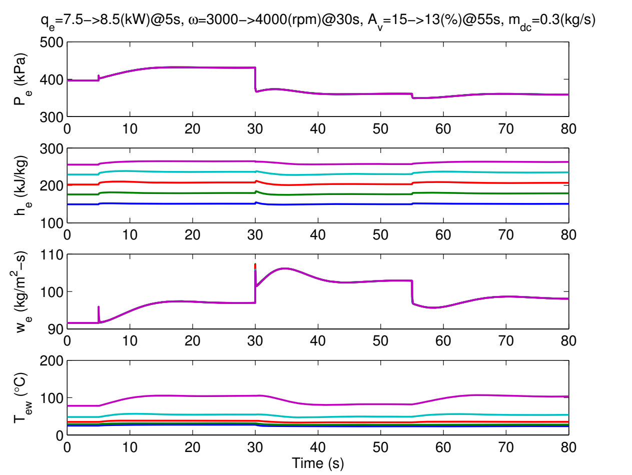

Fig. 3. Dynamic responses of refrigerant pressure, enthalpy, mass flux, and wall temperature in evaporator

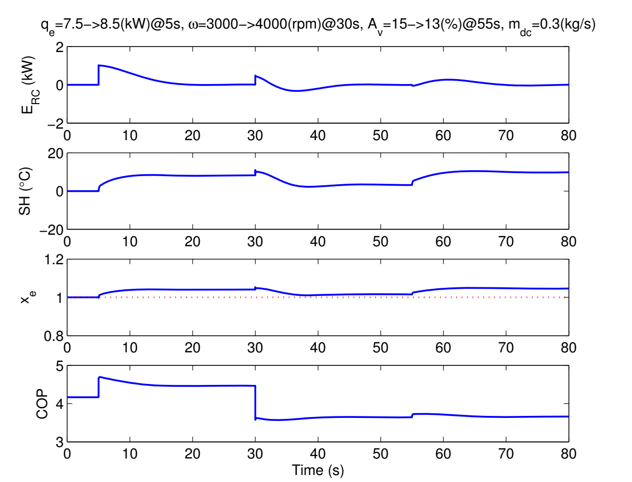

Fig. 4. Dynamic responses of cycle energy gain, superheat, evaporator exit quality, and cycle of performance

from incompressible flow to compressible flow, and from isothermal flow to non-isothermal flow with heat transfer. General sufficient and necessary stabilizing conditions are obtained to ensure the local heat exchanger system stability. Moreover, we also include some stability analysis about the overall refrigeration cycle. The proposed analysis method may offer some guide to the transient operation and design of vapor compression cycle systems.

VII. ACKNOWLEDGMENTS

This work is supported in part by the Office of Naval Research (ONR) under the Multidisciplinary University Research Initiative (MURI) Award GG10919 entitled "System-Level Approach for Multi-Phase, Nanotechnology-Enhanced Cooling of High-Power Microelectronic Systems." This work is also supported in part by the Center for Automation Technologies and Systems (CATS) under a block grant from the New York State Foundation for Science, Technology and Innovation (NYSTAR). John Wen is supported in part by the Outstanding Overseas Chinese Scholars Fund of Chinese Academy of Sciences (No. 2005-1-11).

REFERENCES

[1] X.-D. He, "Dynamic Modeling and Multivariable Control Vapor Compression Cycles in Air Conditioning Systems". PhD Thesis, Massachusetts Institute of Technology, Cambridge, MA, Feb. 1996.

[2] B. P. Rasmussen, "Dynamic Modeling and Advanced Control of Air Conditioning and Refrigeration Systems". PhD Thesis, University of Illinois at Urbana-Champaign, Urbana, IL, Aug. 2005.

[3] D. C. Price, "A review of selected thermal management solutions for military electronic systems". *IEEE Trans. on Components and Packaging Technologies*, **26(1)**, pp. 26–39, March, 2003.

[4] M. Willatzen, N. B. O. L. Pettit, and L. Ploung-Sorensen, "A general dynamic simulation model for evaporators and condensers in refrigeration. Part I-II". *International Journal of Refrigeration*, **21**(5), pp. 398–414, 1998.

[5] D. Leducq, J. Guilpart, and G. Trystram, "Low order dynamic model of a vapor compression cycle for process control design". *Journal of Food Process Engineering*, **26**, pp. 67–91, 2003.

[6] S. Bendapudi, J. E. Braun, and E. A. Groll, "A comparison of moving boundary and finite volume formulations for transients in centrifugal chillers". *International Journal of Refrigeration*, **31**(8), pp. 1437–1452, 2008.

[7] C. J. L. Hermes and C. Melo, "A first-principles simulation model for the start-up and cycling transients of household refrigerators". *International Journal of Refrigeration*, in press, 2008.

[8] G. L. Wedekind, B. L. Bhatt, , and B. T. Beck, "A system mean void fraction model for predicting various transient phenomena associated with two-phase evaporating and condensing flows". *International Journal of Multiphase Flow*, **4**(1), pp. 97–114, 1978.

[9] E. W. Grald, and J. W. MacArthur, "A moving-boundary formulation for modeling time-dependent two-phase flows". *International Journal of Heat and Fluid Flow*, **13**(3), pp. 266–272, 1992.

[10] T. L. McKinley and A. G. Alleyne, "An advanced nonlinear switched heat exchanger model for vapor compression cycles using the moving-boundary method". *International Journal of Refrigeration*, 31(7), pp. 1253–1264, Nov. 2008.

[11] S. Kandlikar, S. Garimella, D. Li, S. Colin, and M. R. King, *Heat Transfer and Fluid Flow in Minichannels and Microchannels*, Elsevier, 2006.

[12] M. M. Shah, "A general correlation for heat transfer during film condensation in tubes". *International Journal of Heat and Mass Transfer*, **22**(4), pp. 547–556, 1979.

[13] K. Ogata, *Modern Control Engineering, 5th Edition*, Prentice Hall, Upper Saddle River, N.J., 2008.

[14] T. J. Zhang, J. Catano, R. L. Zhou, and J. T. Wen, "Dynamic Modeling Of Refrigeration Cycle For Electronics Cooling". *Proceedings of the 2008 ASME International Mechanical Engineering Congress & Exposition (IMECE08)*, Boston, Oct.31–Nov.6, 2008.