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A Game Theory Approach to Multi-Agent Team Cooperation

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Abstract— **The main goal of this work is to design a team of agents that can accomplish consensus over a common value for the agents' output in a cooperative manner. First, a semi-decentralized optimal control strategy introduced recently by the authors is utilized which is based on minimization of individual costs using local information. Cooperative game theory is then used to ensure team cooperation by considering a combination of individual costs as a team cost function. Minimization of this cost function results in a set of Pareto-efficient solutions. The choice of Nash-bargaining solution among the set of Pareto-efficient solutions guarantees the minimum individual cost. The Nash-bargaining solution is obtained by maximizing the product of the difference between the costs achieved through the optimal control strategy and the one obtained through the Pareto-efficient solution. The latter solution results in a lower cost for each agent at the expense of requiring full information set. To avoid this drawback additional constraints are added to the structure of the controller by using the linear matrix inequality (LMI) formulation of the minimization problem. Consequently, although the controller is designed to minimize a unique team cost function, it only uses the available information set for each agent.**

I. INTRODUCTION

Sensor networks (SN), and in general, unmanned system networks (UMSN) are currently one of the strategic areas of research in different disciplines, such as communications, controls, and mechanics. The advantages of wireless UMSN are significant and numerous applications in various fields of research are being considered and developed. Some of these applications are in home and building automation, intelligent transportation systems, health monitoring and assisting, space explorations, and commercial applications [1]. One of the prerequisites for these networks intended to be deployed in different missions is team cooperation and coordination for accomplishing predefined goals and requirements. Cooperation in a network of unmanned systems, known as formation, network agreement, flocking, consensus, or swarming in different contexts, has received extensive attention in the past several years. Several approaches to this problem have been investigated within different frameworks and by considering different architectures [2]-[7].

An optimal approach to team cooperation problem is considered in [8], [9] for formation keeping and in [10], [11] for consensus seeking. The approach in [9] is based on individual cost optimization for sake of achieving team goals under the assumption that the states of the other team members are constant. To solve an optimal consensus problem, the authors in [10] have assumed individual costs for each team member. In evaluating the minimum value of each individual cost, the states of the other agents are assumed to be constant. The work in [11] avoids the above restricting assumptions by decomposing the control input of individual team members into local and global components. In all the above referenced work the optimal problem is based on the individual cost definition for team members. However, to the best of the authors' knowledge, a single team cost function formulation has been proposed in only a few work [8], [12], [13]. In [12], optimal control strategy is applied for formation keeping and a single team cost function is utilized. The authors in [8] assumed a distributed optimization technique for formation control in a leader-follower structure. The design is based on dual decomposition of the local and global constraints. In [13], a centralized solution is obtained by using a game theoretic approach.

It is worth noting that a design-based approach for the purpose of consensus seeking has not been investigated extensively in the literature. In fact, many of the earlier work have focused on only analysis, e.g. [5], [14]. However, the main contribution of this paper is to introduce a novel design-based approach to address the consensus control problem using a single team cost function within a game theoretic framework. The cooperative game theory framework has the advantage of being a multi-objective design tool as well as being able to guarantee a cooperative solution when compared to other design tools. On the other hand, the advantage of minimizing a team cost function is that it can provide a better insight into performance of the entire team when compared to individual performance indices. However, the main disadvantage of this formulation is clearly the requirement of availability of full information set for control design purposes. In the present work this problem is alleviated and the imposed information structure of the team is respected by using linear matrix inequality formulation.

Toward this end, first a decentralized optimal control strategy that was initially introduced in [11] is used to design controllers based on minimization of individual costs. Subsequently, the idea of the cooperative game theory is used to minimize a team cost function, i.e. a linear combination of the cost functions that are used in the optimal approach. This will guarantee that individual cost functions have the minimum possible values for a given team mission. To obtain a solution that is subject to a given information structure as well as to guarantee consensus achievement, a set of LMIs is used to constrain the controller for the entire team.

The organization of the paper is as follows: In Section 2,

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background information is presented. In Section 3, design of a semi-decentralized optimal control is formally presented and in Section 4, the cooperative game theory is introduced. Application of the game theory to the multi-agent team problem is presented in Section 5. Finally, a comparative study and the conclusions are stated in Sections 6 and 7, respectively.

II. Background

Multi-agent teams: Assume a set of agents $\Omega = \{i = 1, \ldots, N\}$, where N is the number of agents. Each member of the team which is denoted by i is placed at a vertex of the network information graph. It has a dynamical representation that is governed by

$$\dot{X}^i = A^i X^i + B^i u^i, \; X^i \in R^n, \; u^i \in R^m, \tag{1}$$

$$Y^i = C^i X^i, \; Y^i \in R^q \tag{2}$$

where X^i, u^i, and Y^i denote the state vector, the input vector, and the output vector of the agent i, respectively, and A^i, B^i and C^i are matrices of appropriate dimensions. *Information structure and neighboring sets:* In order to ensure cooperation and coordination among team members, each member has to know the status (output) of the other members, and therefore members have to communicate with each other. For a given agent i, the set of agents connected to it via communication links is called a neighboring set N^i. The existence of a link between two agents in general may refer to the availability of information of one agent to the other one, in other words $\forall \; i = 1, \ldots, N, \quad N^i = \{j = 1, \ldots, N | (i,j) \in E\}$, where E is the edge set that corresponds to the underlying graph of the network. It is assumed that the graph describing the information structure is connected.

A. Model of interaction between the team members

Assume that the dynamical model of each agent is given by equations (1) and (2). This model defines an isolated agent of the team, but in reality agents have some interactions through the information flow that exists among the neighboring agents. In [11], it was shown that each member's dynamics can be described by the following model that incorporates the interaction terms, namely

$$\begin{cases} \dot{X}^i = A^i X^i + B^i u^i \\ u^i = u^i_l + u^i_g, \; u^i_g = \sum_{j \in N^i} F^{ij} Y^j \\ Y^i = C^i X^i \end{cases} \tag{3}$$

where A^i, B^i and C^i are matrices and vectors of appropriate dimensions, and u^i_l, u^i_g are the decompositions of the input signal into the "local" and the "global" control terms. The local term for each agent is designed by using the agents own output vector whereas the global control utilizes the information received from other agents in its neighboring set. The "global" control term $u^i_g = \sum_{j \in N^i} F^{ij} Y^j$ is also denoted as the interaction term, where F^{ij} is the interaction matrix to ensure compatibility in the agent's input and output channels.

Our main goal in this paper is to ensure that the agents' output, e.g. velocity, converge to the same value, i.e. $Y^i \rightarrow Y^j, \; \forall i, j$. In other words, we require that the team reaches a consensus. For this purpose, we first apply the optimal control strategy introduced in [11]. We then introduce a game theoretic approach to provide a cooperative solution with lower cost values for consensus seeking. A brief description of the optimal control method is presented below as they are needed for formulation of our proposed game theoretic approach. The reader is referred to [11] for further details.

III. Application of semi-decentralized optimal control to a leaderless multi-agent team

A. Definition of a cost function

Let us define the following cost function for each agent,

$$\begin{aligned} J^i &= \int_0^T \{ \sum_{j \in N^i} [(Y^i - Y^j)^T Q^{ij} (Y^i - Y^j)] \\ &+ (u^i_l)^T R^i u^i_l \} dt, \; T > 0 \end{aligned} \tag{4}$$

where Q^{ij}, R^i are symmetric and positive definite (PD) matrices. By minimizing the above cost function for a given controllable and observable system, one may guarantee that all the agents in a neighboring set will have the same output vector in the steady state, i.e. consensus is achieved [11]. In other words, the output vector Y will converge to a vector in the subspace S spanned by the vector $\mathbf{1} = [1 \; 1 \; \ldots \; 1]^T$. This vector is in fact an eigenvector of the Laplacian matrix corresponding to the underlying graph.

B. The dynamical model

For sake of simplicity, and since the design approach based on optimal control is used for comparison purposes only, we assume that the dynamical equation of each agent is a simple double integrator. However, the approach based on the game theory is general enough for an arbitrary linear model of agents. In other words, we assume that the dynamics of each agent is governed by

$$\begin{cases} \dot{r}^i = v^i \\ \dot{v}^i = u^i_l + u^i_g \\ u^i_g = \sum_{j \in N^i} F^{ij} Y^j, \; Y^i = v^i, i = 1, \ldots, N \end{cases} \tag{5}$$

in which $r^i, v^i \in R^m$ are the position and the velocity vectors, respectively.

Our proposed control strategy that results from the minimization of the cost function (4) subject to the above dynamical model is provided in the following lemma.

Lemma 1 Assume a team of agents is given whose dynamics are governed by the double integrator equations in (5) and are embedded with interactions among the agents based on the neighboring sets. Assume that the control input of each agent is decomposed into the local and the global parts as explained previously. Then, the global and local control laws proposed below minimize the cost function (4) so that a consensus on the velocity value is guaranteed, where

$$u_g^i = \sum_{j \in N^i} F^{ij} v^j, \quad F^{ij} = 2K^i(t)^{-1} Q^{ij}, \ \forall i, j \qquad (6)$$

$$u_l^i = -\frac{1}{2}(R^i)^{-1} K^i(t) v^i, \ i = 1, \dots, N \qquad (7)$$

in which K^i satisfies the differential Riccati equation (DRE) $-\dot{K}^i = 2|N^i|Q^{ij} - \frac{1}{2}K^i(R^i)^{-1}K^i, \ K^i(T) = 0$. where $|N^i|$ is the cardinality of the neighboring set defined previously.

Proof: The details may be found in [11].

The above lemma provides a control strategy for consensus seeking using the individual cost function minimization. In the following two sections, the individual cost of each agent as defined in the present section is combined into a team cost function and the cooperative game theory approach will be utilized to "increase" the team cooperation and minimize the individual costs. The cost values that are obtained in the present section are referred to as the "non-cooperative" outcomes in the context of the next two sections.

IV. Cooperative game theory

In this section we give a general description of the "cooperative game theory" and in the next section we modify the formulation introduced here to make it compatible with our specific problem, i.e. consensus seeking problem.

Assume a team of N players with the following dynamical model

$$\dot{x} = Ax + \sum_{i=1}^{N} B^i u^i \qquad (8)$$

where the matrix A has an arbitrary structure. Each player wants to optimize its own cost

$$J^i = \int_0^T (x^T Q^i x + (u^i)^T R^i u^i) dt \qquad (9)$$

in which Q^i and R^i are symmetric matrices and R^i is a PD matrix.

If the players decide to minimize their cost in a non-cooperative manner, a strategy (control input u^i) chosen by the ith player can increase the cost of other players through dynamics of the system that relates different players together. However, if the players decide to cooperate, the individual costs may be minimized if the agents are aware of the others' decisions and can reduce their team cost by selecting a suitable cooperative strategy. Hence, in a cooperative strategy depending on which agent requires more resource the resulting minima can be different. The cooperation ensures that the total cost of the team is less than any other non-cooperative optimal solution.

In a cooperative approach there is no alternative strategy that improves all the members' cost simultaneously. This property can be formally defined by the set of Pareto-efficient solutions as follows.

Pareto efficient strategies [15]: A set of strategies $U^* = [u^{1*}, \dots, u^{N*}]$ is Pareto-efficient if the set of inequalities $J^i(U) \leqslant J^i(U^*), \ i = 1, \dots, N$ with at least one strict inequality does not have a solution for U. The point $J^* = [J^1(U^*), \dots, J^N(U^*)]$ is called a Pareto solution. This solution is never fully dominated by any other solution.

Now consider the following optimization problem and assume that $Q^i \geq 0$, specifically

$$\min_{u^i \in \mathcal{U}^i} \ J^i = \int_0^T (x^T Q^i x + (u^i)^T R^i u^i) dt$$

$$\text{s.t.} \ \ \dot{x} = Ax + \sum_{i=1}^{N} B^i u^i$$

The above is a convex optimization problem for which we try to find a solution. It can be shown that the following set of strategies results in a set of Pareto efficient solutions for this problem. In other words, the solution to the following minimization problem cannot be dominated by any other solution

$$U^*(\alpha) = \arg\min_{U \in \mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U) \qquad (10)$$

where $\alpha \in \mathcal{A}, \ \mathcal{A} = \{\alpha = (\alpha^1, \dots, \alpha^N) | \alpha^i \geq 0 \text{ and } \sum_{i=1}^{N} \alpha^i = 1\}$, U is the set of strategies and the corresponding cost values will be $J^1(U^*(\alpha)), \dots, \ J^N(U^*(\alpha))$. It is worth mentioning that though this minimization is over the set of strategies, the controller parameters (matrices) are in fact optimized. In other words, the control strategies are assumed to be in the form of the state feedback and the coefficient matrices are obtained through the above optimization problem.

The strategies obtained from the above minimization as well as the optimal cost values are functions of the parameter α. Therefore, the Pareto-efficient solution is in general not unique and the set of these solutions, i.e. Pareto frontier, is denoted by $\mathcal{P}$ which is an edge in the space of possible solutions, i.e. Ξ. It can be shown that in both infinite horizon and finite horizon cases, the Pareto frontier will be a smooth function of α [15]. Due to the non-uniqueness of Pareto solutions the next step is to decide how to choose one solution from the set of Pareto solutions (or choose an α from the set of α's). This solution should be selected according to a certain criterion as our final strategy for the team cooperation problem. For this purpose we need to solve the bargaining problem as defined below.

Bargaining problem [15]: In this problem two or more players have to agree on the choice of some strategies from a set of solutions while they may have conflicting interests over this set. However, the players understand that better outcomes may be achieved through cooperation when compared to the non-cooperative outcome (called threat-point). There are two different approaches to bargaining problem, i.e. the axiomatic and the strategic approaches. Some of the well-known axiomatic approaches to this problem are: Nash bargaining, Kalai-Smorodinsky, and Egalitarian.

Applying any of the above mentioned methods to the Pareto efficient solutions, will yield a unique cooperative solution. Due to the interesting properties of the Nash-bargaining solution such as symmetry and Pareto optimality [15], we invoke this method for obtaining a unique solution among the set of Pareto-efficient solutions obtained previously.

Nash bargaining solution (NBS) [15]: In this method a point in Ξ, denoted as Ξ^N, is selected such that the product of the individual costs from d is maximal ($d = [d^i]^T$ is the threat-point or the non-cooperative outcome of team agents), namely

$$\Xi^N(\Xi, d) = \arg\max_{J\in\Xi} \prod_{i=1}^{N} (d^i - J^i), \;\; J \in \Xi \text{ with } J \preceq d$$

in which d^i's are the cost values calculated by using the non-cooperative solution that is obtained by minimizing the cost in (9) individually and constrained to (8) (the threat point). It can be shown that the Nash bargaining solution is on Pareto frontier and therefore the above maximization problem is equivalent to the following problem

$$\alpha^N = \arg\max_{\alpha} \prod_{i=1}^{N} (d^i - J^i(\alpha, U^*)), \quad J \in \mathcal{P}, \; J \preceq d \quad (11)$$

in which $J = [J^i]^T$, and where J^i's are calculated by using the set of strategies given in (10). By solving the maximization problem (11), a unique value for the coefficient α can be found.

Remark 2 Theorem 6.10 in [15] can be used to determine the relationship that exists between the coefficients $\alpha^i, \; i = 1, \dots, N$ and the achievable improvements in the individual costs due to cooperation in the team. According to this theorem the following relationship holds between the value of the cost functions at the NBS, $(J^{1*}(\alpha^*, U^*), \dots, J^{N*}(\alpha^*, U^*))$, the threat-point d, and the optimal weight $\alpha^* = (\alpha^{1*}, \dots, \alpha^{N*})$, that is $\alpha^{1*}(d^1 - J^{1*}(\alpha^*, U^*)) = \dots = \alpha^{N*}(d^N - J^{N*}(\alpha^*, U^*))$ or

$$\alpha^{j*} = \frac{\prod_{i\neq j} (d^i - J^{i*}(\alpha^*, U^*))}{\sum_{i=1}^{N} \prod_{k\neq i} (d^k - J^{k*}(\alpha^*, U^*))} \quad (12)$$

The expression in (12) describes the kind of cooperation that exists among the players. It shows that if during the team cooperation, i.e. minimization of the team cost, a player has improved its cost more, it will receive a lower weight in the minimization scheme (Pareto solution) whereas the one who has not gained a great improvement as a result of participation in the team cooperation receives a greater weight. Therefore, all the players benefit from the cooperation in almost a similar manner, and hence have the incentive to participate in the team cooperation.

V. Application of the Game Theory to Consensus Seeking in a Multi-Agent Team

According to the discussions in previous section, cooperation in a team of N players, e.g. consensus seeking, can be solved in the framework of cooperative games. Our goal is to develop a cooperative solution that utilizes the semi-decentralized cost functions and combine them in a team cost function where improvements in minimizing these functions can be achieved by utilizing the game-theoretic results. For this purpose, we use the individual cost values calculated by utilizing the semi-decentralized optimal control strategy that was introduced in Section III. These values are considered as the non-cooperative outcome of the team, referred to as d^i's in (11). We combine these individual cost functions into a team cost function. Subsequently, we try to find a set of Pareto optimal solutions for minimization of this team cost function through solving (10). Then an NBS can be selected among this set of Pareto-efficient solutions by solving (11).

In this section the above concepts of game theory are applied to a team of agents as described in Section II. The dynamical model of each agent and the related cost functions are described in (3) and (4), respectively. To clearly describe the method we combine the individual cost functions in (4) as a team cost function

$$J^c = \sum_{i=1}^{N} \alpha^i J^i(U) = \int_0^T [X^T Q X + U^T R U] dt \quad (13)$$

in which $\alpha = (\alpha^1, \dots, \alpha^N) \in \mathcal{A}$, $J^i(U)$ is the cost function for the ith player that is defined in (4) and $U(\alpha) = [(u_l^1)^T \; \dots \; (u_l^N)^T]^T$ is the vector of all the agents' local input vectors and

$$\begin{aligned} &R = Diag\{\alpha^1 R^1, \dots, \alpha^N R^N\}, \quad Q = C^T [\delta_{hk}]_{N\times N} C, \\ &\delta_{hh} = \sum_{j\in\mathcal{N}^h} \alpha^j Q^{jh} + \alpha^h \sum_{k\in N^h} Q^{hk}, \\ &\delta_{hk} = \begin{cases} -\alpha^h Q^{hk} - \alpha^k Q^{kh} & \text{for } \; k \in N^h \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (14)$$

The corresponding dynamical model is given by

$$\dot{X} = AX + BU, \quad Y = CX \quad (15)$$

in which X, Y and U are the state, input, and output vectors of the entire team obtained from the concatenation of all the agents' state, output, and input vectors and are given by

$$\begin{aligned} &X = [(X^1)^T \; \dots \; (X^N)^T]^T, \; U = [(u_l^1)^T \; \dots \; (u_l^N)^T]^T, \\ &Y = [(Y^1)^T \quad \dots \quad (Y^N)^T]^T \end{aligned}$$

Furthermore, the matrices A, B and C are defined as

$$\begin{aligned} &A = \begin{bmatrix} A^1, \; 0, \dots, B^1 F^{1j} C^j, \dots, 0 \\ \vdots \\ 0, \dots, B^N F^{Nj} C^j, \dots, 0, A^N \end{bmatrix}, \\ &B = Diag\{B^1, \; \dots, \; B^N\}, C = Diag\{c^1, \dots, c^N\} \end{aligned} \quad (16)$$

where $A^i, \; B^i$, and C^i are defined in (3). The term $B^i F^{ij} C^j$ represents the interaction terms that exist in the dynamical model of each agent, and $\mathcal{N}^h$ is the set of indices of the neighboring clusters to which agent h belongs. Each agent belongs to only those clusters in which at least one of the agent's neighbors exists. Therefore, the total number of these clusters is the same as the number of neighbors of that agent, i.e. $\mathcal{N}^h = N^h$.

The Pareto efficient solution for minimizing this team cost function is achieved through the following strategy

$$U^*(\alpha) = \arg\min_{U\in\mathcal{U}} \sum_{i=1}^{N} \alpha^i J^i(U) = \arg\min_{U\in\mathcal{U}} J^c(\alpha) \qquad (17)$$

in which $U^*(\alpha) = [(u_l^{1*})^T \ \ldots \ (u_l^{N*})^T]^T$ is the vector of all the agents' local input vectors when the total cost function (13) is minimized.

The set of solutions to the minimization problem (17) is a function of the parameter α which provides a set of Pareto-efficient solutions. Among these solutions, a unique solution can be obtained by using one of the methods that was mentioned earlier, e.g., the Nash bargaining solution. Using this method the unique solution to the problem (a unique α) is given by (11) in which J^i's are defined in (4) and are calculated by applying the solution of the minimization problem (17) to the system that is given in (15) and hence are functions of parameter α. The terms d^i's are the values of the cost defined in (4) which is obtained by applying the semi-decentralized solutions that are found in (6)-(7) to the individual subsystems in (5). By solving the maximization problem (11) the parameter α can be found and substituted in the set of control strategies that are obtained in (17). This solution guarantees that the product of the distances between d^i's (non-cooperative solution) and J^i's (cooperative solution) is maximized, implying that the individual costs in the latter case are minimized as much as possible.

In order to solve the minimization problem (17), the cost function (13) should be minimized subject to the dynamical constraint (15). This is a standard linear quadratic regulator (LQR) problem and its solution for an infinite horizon case (i.e. $T \to \infty$) will result in the following control law

$$\begin{aligned} U^*(\alpha, X) &= -R^{-1}B^TPX \\ Q - PBR^{-1}B^TP^T + PA + A^TP &= 0 \end{aligned} \qquad (18)$$

The control U^* can be constructed if the above algebraic Riccati equation (ARE) has a solution for P. Note that the matrix P is not guaranteed to be block-diagonal and hence the control signal U^* yields a centralized strategy in the sense that its components, i.e. u_l^{i*}, are dependent on information from the entire team. Moreover, the solution suggested by (18) does not guarantee consensus for an arbitrary parameter selection. Hence, to ensure that a desirable consensus is obtained that satisfies the constraints on the availability of information, one needs to impose additional constraints on the original minimization problem. For this purpose, the optimization problem is formulated as an LMI problem so that the constraints due to the consensus and the controller structure are incorporated as convex constraints.

A. Solution of the minimization problem: an LMI formulation

In [16], [17], [18], it is pointed out that the LQR problem can be formulated as a maximization or a minimization problem subject to a set of LMIs. In other words, instead of solving the ARE (18), an LMI should be solved. The following is one of the formulations that can be used for this purpose using a semi-definite programming problem framework [18], namely

$$\begin{aligned} &\max trace(P) \ \text{ s.t.} \\ &\mathcal{R}(P) = PA + A^TP - PBR^{-1}B^TP + Q \geq 0, \ P \geq 0 \end{aligned} \qquad (19)$$

where the optimal control law is selected as $U^* = -R^{-1}B^TPX$. This formulation can be translated into an LMI maximization problem by using the Schur complement decomposition, and given that $R > 0$, it can be stated as the following problem.

<u>Problem A</u>

The LQR problem can be formulated as a maximization one subject to a set of LMIs, namely

$$\begin{aligned} &\max trace(P) \ \text{ s.t.} \\ &\begin{bmatrix} PA + A^TP + Q & PB \\ B^TP & R \end{bmatrix} \geq 0, \ P \geq 0 \end{aligned} \qquad (20)$$

It can be shown that the above maximization problem has a solution if and only if the following ARE has a solution

$$Q - PBR^{-1}B^TP^T + PA + A^TP = 0 \qquad (21)$$

Moreover, if $R > 0$ and $Q \geq 0$, the unique optimal solution to the maximization Problem A is the maximal solution to the ARE in (21) [18].

1) Consensus seeking subject to a predefined information structure:
In the above discussion the optimization problem was formulated as a set of LMIs. However, the solutions to this optimization problem does not necessarily satisfy the consensus achieving goal. Therefore, the condition for achieving consensus in the subspace S, i.e. $(A - BR^{-1}B^TP)S = 0$ is now added to Problem A. Note that S is the unity vector, i.e. $S = \mathbf{1}$. Consequently, the solution to the following maximization problem results in an optimal consensus algorithm.

<u>Problem B</u>

The minimization problem for consensus seeking can be formulated as a maximization problem subject to a set of LMIs, namely

$$\begin{aligned} &\max trace(P) \text{ s.t.} \\ &\begin{cases} 1. \begin{bmatrix} PA + A^TP + Q & PB \\ B^TP & R \end{bmatrix} \geq 0, \ P \geq 0 \\ 2. \ (A - BR^{-1}B^TP)S = 0 \end{cases} \end{aligned} \qquad (22)$$

where the optimal control law is selected as $U^* = -R^{-1}B^TPX$ and P is obtained through solving the above set of LMIs.

As discussed previously, the solution to the above problem as well as the one given in (18) requires full network information for each agent. However, each agent has only access to its neighboring set information.

Therefore, one needs to impose a constraint on the controller structure in order to satisfy the limited agent availability of information. For the sake of notational simplicity assume that each agent has a one-dimensional state-space representation, i.e. A^i in (3) is a scalar. The case of a non-scalar system matrix can be treated similarly. The controller coefficient, i.e. $R^{-1}B^TP$ in Problem B should have the same structure as the Laplacian matrix so that the neighboring set constraint holds. However, due to their definitions both R and B are block diagonal. Therefore, it suffices to restrict P to have the same structure as the Laplacian matrix, i.e. $P(i,j) = 0$ if $L(i,j) = 0$, where $L(i,j)$ designates the (i,j) entry of the Laplacian matrix L. We may now solve the following problem to minimize the cost function (13) while simultaneously satisfying all the earlier constraints, namely we have,

Problem C

$$\max trace(P) \text{ s.t.}$$
$$\begin{cases} 1. \begin{bmatrix} PA+A^TP+Q & PB \\ B^TP & R \end{bmatrix} \geq 0, \ P \geq 0 \\ 2. \ (A-BR^{-1}B^TP)S = 0 \\ 3. \ P(i,j) = 0 \text{ if } L(i,j)=0, \ \forall i,j = 1,\ldots,N \end{cases} \tag{23}$$

This problem is an LMI problem in terms of P.

B. Algorithm for finding the Nash bargaining solution

Up to this point we have shown that for any given $\alpha > 0$ the maximization Problem C should first be solved. We now need an algorithm for solving the maximization problem (11) over different values of α so that a suitable and unique α can be found. In [15], two numerical algorithms for solving this maximization problem are given. With minor modifications made to one of these algorithms, the following algorithm will be used in this paper. Namely, we have

Algorithm I

- **Step 1** Start with an initial selection for $\alpha_0 \in \mathcal{A}$ ($\alpha_0 = [1/N,\ldots,1/N]$ is a good choice).
- **Step 2** Compute $U^*(\alpha_0) = \arg\min_{U\in\mathcal{U}} \sum_{i=1}^N \alpha_0^i J^i(U)$ by solving the maximization Problem C.
- **Step 3** Verify if $J^i(U^*) \leq d^i, \ i = 1,\ldots,N$, where d^i is the optimal value of (4) when the control laws (6)-(7) are applied to the dynamical system (5). If this condition is not satisfied, then there is at least one i_0 for which $J^{i_0}(U^*) > d^{i_0}$. In that case, update $\alpha_0^{i_0} = \alpha_0^{i_0} + 0.01, \ \alpha_0^i = \alpha_0^i - \frac{0.01}{N-1}$, for $i \neq i_0$ and return to Step 2 (similarly extend the update rule for more than one i_0).
- **Step 4** Calculate
$$\tilde{\alpha}^j = \frac{\prod_{i\neq j}(d^i - J^i(U^*(\alpha_0)))}{\sum_{i=1}^N \prod_{k\neq i}(d^k - J^k(U^*(\alpha_0)))}, \ j = 1,\ldots,N$$
- **Step 5** Apply the update rule $\alpha_0^i = 0.9\alpha_0^i + 0.1\tilde{\alpha}^i$. If $|\tilde{\alpha}^i - \alpha_0^i| < 0.01, \ i = 1,\ldots,N$, then terminate the algorithm and set $\alpha = \tilde{\alpha}$, else return to Step 2.

The above discussions are now summarized in the following theorem.

Theorem 3 Consider a team of agents with individual dynamical representation (3) or the team dynamics (15), the individual cost function (4), and the team cost function (13) with the corresponding parameters (14) and (16). Furthermore, assume that the desirable value of the parameter α is found by using Algorithm I and the control law U^* is designed as $U^* = -R^{-1}B^TPX$. P is the solution to the following optimization problem

$$\max trace(P) \text{ s.t.}$$
$$\begin{cases} 1. \begin{bmatrix} PA+A^TP+Q & PB \\ B^TP & R \end{bmatrix} \geq 0, \ P \geq 0 \\ 2. \ A_c = (A-BR^{-1}B^TP), \ A_cS = 0 \\ 3. \ P(i,j) = 0 \text{ if } L(i,j)=0, \ \forall i,j = 1,\ldots,N \\ 4. \ A_c(i,j) \neq 0 \text{ if } L_{sub}(i,j) \neq 0, \ \forall i,j = 1,\ldots,N \end{cases} \tag{24}$$

where $S = \mathbf{1}$ and L_{sub} denotes the Laplacian matrix of an arbitrary selected connected subgraph of the original graph describing the information structure. It then follows that

(a) In infinite horizon, i.e. $T \to \infty$, the above controller solves the following min-max problem

$$\begin{aligned} &U^* = \arg\min_{U\in\mathcal{U}} \sum_{i=1}^N \alpha^i J^i(U), \ \alpha \in \mathcal{A}, \\ &\mathcal{A} = \{\alpha = (\alpha^1,\ldots,\alpha^N) | \alpha^i \geq 0 \text{ and } \sum_{i=1}^N \alpha^i = 1\} \\ &\alpha^* = \arg\max_\alpha \prod_{i=1}^N (d^i - J^i(\alpha, U^*)), \quad J \preceq d \end{aligned} \tag{25}$$

(b) The optimal value of the cost function (13) has a finite infimum of $X^T(0)PX(0) - \xi^2 \sum_i \sum_j P(i,j)$ where ξ is a constant coefficient of the consensus value, i.e. $X_{ss} \to \xi\mathbf{1}$.

(c) In addition, the suggested control law guarantees stable consensus of agents output to a common value subject to the dynamical and information structure constraints of the team in a cooperative manner.

Proof: Omitted due to space limitations.

Using the above results, the team consensus goal can be obtained in a cooperative manner and subject to the given information constraints.

VI. Comparison Study

Due to space limitations the simulation results are omitted and only a comparative table is included. This comparison corresponds to a team of four agents that are controlled by using two control strategies, namely a semi-decentralized optimal control law given by Lemma 1 versus a game theoretic-based control law given by Theorem 3. The simulation parameters for both approaches are selected as follows: $A^i = 0_{2\times2}, \ R^i = I_{2\times2}, \ C^i = I_{2\times2}, \ B^i = \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}$, and

$Q^{ij} = \begin{pmatrix} 10 & 3 \\ 3 & 4 \end{pmatrix}$. In the second control strategy the initial value for the parameter α is selected as $\alpha_0 = [1/4, \ldots, 1/4]$ and its optimal value $\alpha = [.1589\ .1229\ 0.263\ 0.4552]$ is obtained by using the procedure in Algorithm I. The interaction gains are selected as $F^{ij} = 1.6 I_{2\times 2}$. Table I compares the cost obtained for the four agents under the two proposed control approaches for a period of 10 sec. As expected the cost values for the game theory approach are less than those obtained from the optimal control approach. However, it should be noted that this is achieved at the expense of an increased computational complexity in the game-theoretic approach. In fact, in this method two optimization problems, i.e. a maximization and a minimization problem should be solved as compared to the semi-decentralized approach where only a single minimization problem needs to be solved. Therefore, there is a tradeoff between the control computational complexity and the achievable control performance.

VII. Conclusions

A control strategy based on cooperative game theory is applied to the problem of cooperation in a team of unmanned systems. The main goal of the team is to reach at a common output, i.e. to have consensus. To achieve this goal, first a semi-decentralized optimal control strategy is designed following the results that the authors have recently developed in [11] for a team of agents with individual cost functions. Next, cooperative game theory approach is applied to the cooperation problem where linear combination of locally defined cost functions is considered as the team cost function. The Nash bargaining solution is chosen as "the best solution" among the Pareto efficient ones found by optimizing the team cost function. Unfortunately, to implement this strategy one requires to have access to full team information measurements. To remedy this major obstacle and shortcoming the corresponding optimization problem is formulated as a set of LMIs where the available information structure is enforced onto the control structure. Consequently, one can now guarantee consensus achievement with minimum individual cost by maximizing the difference between the cost obtained through cooperative and non-cooperative (that is, the semi-decentralized optimal controller) approaches. Moreover, the consensus achievement condition is added as a constraint to the set of LMIs. By performing a comparative study between the game theory and the optimal control strategies, it is concluded that the former approach results in lower individual as well as team cost values as predicted. Moreover, the game theory approach results in a global optimal solution that is subject to the imposed constraints. In future work, a quantified index will be presented to measure the effects of decentralization of information on the increase of the team cost. This quantization may provide an insight into the tradeoffs that exist between the availability of information and team cost based on the two proposed methods.

TABLE I

A comparative evaluation of the performance index corresponding to the two control design strategies.

Control Scheme	Performance Index			
	Agent 1	Agent 2	Agent 3	Agent 4
Optimal control	24,996	17,597	14,624	29,513
Game theory	19,838	10,868	11,805	27,855

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