2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Minimum Initial Marking Estimation in Labeled Petri Nets

Lingxi Li and Christoforos N. Hadjicostis

Abstract— **This paper develops an algorithm for estimating the *minimum initial marking* based on the observation of a sequence of labels that is produced by underlying transition activity in a given labeled Petri net. We assume that the structure of the net is completely known while the initial marking of the net is unknown. Given the observation of the sequence of labels, we aim to estimate the minimum initial marking of the net, i.e., an initial marking that (i) allows for the firing of at least one sequence of transitions that is consistent with both the observed sequence of labels and the net structure; and (ii) has the least total number of tokens (i.e., the minimum number of tokens summed over all places). We develop a recursive algorithm that can be used online to find the minimum initial marking with complexity that is polynomial in the length of the observed label sequence. Such minimum initial markings are useful for characterizing the minimum number of resources required at initialization for a variety of systems.**

Index terms: **Labeled Petri nets, Initial marking estimation, Observed label sequence.**

I. Introduction

Petri nets are a graphical and mathematical tool that can be used to model a variety of dynamical systems [1], [2]. As the size and complexity of practical systems increase, significant attention is devoted to problems of state/event estimation using Petri net models. One of the most well-studied estimation problems in Petri nets is that of estimating the marking (state) of a given Petri net based on the observation of its event sequence. For instance, in [3], [4] the authors present an algorithm for obtaining an estimate (and a corresponding error bound) for the marking of a given Petri net based on complete knowledge of the observed firing sequence but without knowledge of the initial state; these works also discuss how this marking estimate may be used to design a controller. In [5], the authors consider the problem of marking estimation based on the observation of a sequence of labels in a given labeled Petri net with known initial marking. They show that, under some conditions on the structure of the Petri net, the set of markings consistent with the observed label sequence can be captured by a linear system whose size does not depend on the length of the observed label sequence. This approach was further extended in [6] to handle nets with silent transitions (i.e., transitions whose firing cannot be observed). Generalizations and analyses of these ideas to arbitrary labeled Petri nets were developed in [7], [8] where the authors showed that the number of markings (and firing vectors) that are consistent with a sequence of labels of length k is bounded by a polynomial function in k.

In this paper, we consider a setting where we have complete knowledge of the structure of a *labeled* Petri net but no information about its initial marking. Given an observed sequence of labels (that is generated by unknown underlying transition activity in the net), we aim at finding the minimum initial marking, i.e., an initial marking that (i) allows for the firing of at least one sequence of transitions that is consistent with both the observed sequence of labels and the Petri net structure, and (ii) has the minimum total number of tokens (the total number of tokens of a particular marking is taken to be the sum of the number of tokens at each place in the net). We obtain a solution to this problem by developing a dynamic programming algorithm that is able to find the minimum initial marking with complexity that is polynomial[1] in the length of the observed label sequence. Note that finding an initial marking that satisfies constraint (i) becomes trivial if constraint (ii) is removed; however, constraint (ii) makes the solution of this problem important for minimum resource allocation problems in manufacturing systems. For example, in part production settings, minimum resource allocation involves the determination of the least resources required to complete pre-specified machine operations, e.g., in order to make a part (a final functional workpiece) from raw materials [9]. In our setup, the given sequence of labels could represent a sequence of (possibly different) tasks, each of which may be accomplished via a sequence of transitions (different alternatives for finishing a specific task). The structure of the given labeled Petri net represents the ways in which different tasks can be accomplished and the interactions/constraints among them (as imposed by the given manufacturing system). The recursive algorithm that we develop resembles the one in [10] but is based on different principles. Our problem formulation generalizes the work in [3], [4] where the authors consider initial marking estimation given a known transition firing sequence. In our setup, the observed sequence of labels may correspond to multiple transition firing sequences whose number will in general be

This material is based upon work supported in part by the National Science Foundation under NSF ITR Award 0426831. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of NSF.

Lingxi Li was with the Coordinated Science Laboratory, and the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign; he is now with the Department of Electrical and Computer Engineering, Indiana University-Purdue University Indianapolis. C. N. Hadjicostis is with the Department of Electrical and Computer Engineering, University of Cyprus, and also with the Coordinated Science Laboratory, and the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign. Corresponding author: C. N. Hadjicostis, 110 Green Park, 75 Kallipoleos Avenue, P.O. Box 20537, 1678 Nicosia, Cyprus. Emails: {lingxili, chadjic}@illinois.edu.

[1]The complexity of our dynamic programming algorithm will in general be exponential in certain parameters of the Petri net; nevertheless, our focus on the growth of algorithmic complexity with respect to the length of the observation sequence is justified because the size of the net is fixed.

exponential in the length of the label sequence. Using the recursive algorithm that we develop, we are able to avoid the explicit calculation of the initial markings that correspond to each possible transition firing sequence, and keep the complexity of the algorithm polynomial in the length of the label sequence.

Crucial to our algorithmic complexity analysis is the fact that the number of firing vectors that correspond to valid sequences of transitions (i.e., sequences of transitions that are consistent with the observed sequence of labels) is upper bounded by a polynomial function in the length of the observed sequence of labels [7], [8]. Our techniques, however, are quite distinct from the ones in [7], [8] since what constraints the number of possible initial markings is the minimality requirement imposed by constraint (ii) and not the number of firing vectors (indeed there could be multiple minimal initial markings associated with each firing vector—because each such firing vector could correspond to multiple transition firing sequences).

II. Petri Net Notation

In this section, we review basic definitions and terminology that will be used throughout the paper. More details about Petri nets can be found in [1], [2].

A Petri net structure is a directed weighted bipartite graph $N = (P,T,A,W)$ where $P = \{p_1, p_2, \ldots, p_n\}$ is a finite set of *places* (drawn as circles), $T = \{t_1, t_2, \ldots, t_m\}$ is a finite set of *transitions* (drawn as bars), $A \subseteq (P \times T) \cup (T \times P)$ is a set of arcs (from places to transitions and from transitions to places), and $W : A \to \{1,2,3,\ldots\}$ is the *weight function* on the arcs.

Let b^-_{ij} denote the integer weight of the arc from place p_i to transition t_j, and b^+_{ij} denote the integer weight of the arc from transition t_j to place p_i ($1 \le i \le n$, $1 \le j \le m$). Note that b^-_{ij} (or b^+_{ij}) is taken to be zero if there is no arc from place p_i to transition t_j (or vice versa). We define the *input incident matrix* $B^- = [b^-_{ij}]$ (respectively the *output incident matrix* $B^+ = [b^+_{ij}]$) to be the $n \times m$ matrix with b^-_{ij} (respectively b^+_{ij}) at its i^{th} row, j^{th} column position. The *incident matrix* of the Petri net is defined to be $B \equiv B^+ - B^-$.

A *marking* is a vector $M : P \to (Z^+_0)^n$ that assigns to each place in the Petri net a nonnegative integer number of tokens (drawn as black dots). We use $M(p)$ to denote the marking of place p (i.e., the number of tokens in place p). We also use $|M|$ to denote the total number of tokens summed over all places of marking M (i.e., $|M| = \sum_{i=1}^n M(p_i)$). A transition t is said to be *enabled* if each of its input places has at least $B^-(p,t)$ tokens, where $B^-(p,t)$ is the weight of the arc from an input place p to transition t. We use $M[t\rangle$ to denote the fact that transition t is enabled at marking M. An enabled transition t may *fire* and, if it fires, it removes $B^-(p_{in},t)$ tokens from each input place p_{in} of t, and deposits $B^+(p_{out},t)$ tokens to each output place p_{out} of t to yield a new marking $M' = M + B(:,t)$, where $B(:,t)$ denotes the column of B that corresponds to t. This is also denoted by $M[t\rangle M'$, and we say marking M' is *reachable* from marking M via the firing of transition t.

Let $\sigma = t_{i1}t_{i2}\ldots t_{ik}$ ($t_{ij} \in T$) be a transition firing sequence of length k. We say σ is enabled with respect to M if $M[t_{i1}\rangle M_1[t_{i2}\rangle \ldots [t_{ik}\rangle$; this is denoted by $M[\sigma\rangle$. The set of all transition firing sequences enabled under marking M of a net N is denoted by $E(N,M)$. Let $M[\sigma\rangle M'$ denote that M' is reachable via the firing of transition sequence σ from M and let $\overline{\sigma}(t)$ be the total number of occurrences of transition t in σ. More specifically, $\overline{\sigma} = [\overline{\sigma}(t_1)\ldots\overline{\sigma}(t_m)]^T$ is the *firing vector* that corresponds to σ. For a transition firing sequence that contains a single transition t, we use $\bar{t}$ to denote its firing vector. Note that if $M[\sigma\rangle M'$, we can express M' as $M' = M + B\overline{\sigma}$. Furthermore, we use $|\sigma|$ to denote the total number of transition firings in sequence σ (i.e., $|\sigma| = \sum_{i=1}^m \overline{\sigma}(t_i)$, which is equal to k in this case).

In a labeled Petri net structure $N_L = (P,T,A,W,L,\Sigma)$ (refer to Fig. 1), $N = (P,T,A,W)$ is a Petri net structure and the *labeling function* $L : T \to \Sigma \cup \{\lambda\}$ assigns to each transition in the net a label from a given alphabet Σ, or the empty label λ if the transition is unobservable. Note that two or more transitions may correspond to the same label. For a label $l \in \Sigma \cup \{\lambda\}$, we use T_l to denote the set of transitions with label l, and $|T_l|$ to denote cardinality of set T_l. In this paper, we assume that the labeled Petri net is λ-free, i.e., all transitions are observable ($T_\lambda = \emptyset$). Thus, given a transition firing sequence $\sigma = t_{i1}t_{i2}\ldots t_{ik}$ of length k, the observed label sequence is $\omega = L(\sigma) = L(t_{i1})L(t_{i2})\ldots L(t_{ik})$, i.e., a string in Σ^k.

We use $\vec{\mathbf{1}}_n$ (respectively $\vec{\mathbf{0}}_n$) to denote the $n \times 1$ vector of all ones (respectively zeros). If A and B are two sets, we use $A - B$ to denote the set of elements that are in A but not in B.

III. Problem Formulation

The problem we deal with in this paper is the following. Consider a λ-free labeled Petri net structure $N_L = (P,T,A,W,L,\Sigma)$. Given an observed label sequence $\omega = l_1 l_2 \ldots l_k$ (where $l_j \in \Sigma, j \in \{1,2,\ldots,k\}$) that has been generated by an underlying (unknown) firing sequence $t_{i1}t_{i2}\ldots t_{ik}$ (i.e., $l_j = L(t_{ij})$), we need to find the (set of) initial marking(s) that: (i) allows (allow) for the firing of at least one sequence of transitions that is consistent with both ω and the structure of the net, and (ii) is (are) minimum (i.e., the marking(s) has (have) the minimum total number of tokens).

Clearly, the set of minimum initial markings $Z_{minimum}(\omega)$ is the set of solutions to the following problem:

$$Z_{minimum}(\omega) = arg \min_M |M| \quad s.t. \quad M[\sigma\rangle \quad \& \quad L(\sigma) = \omega. \quad (1)$$

Definition 1 Given a set of distinct markings $S = \{M_1, M_2, \ldots, M_q\}$, marking $M_i \in S$ is said to be a *minimal marking* of S if $\nexists M_j$ such that $M_j \le M_i$ where $i,j \in \{1,2,3,\ldots,q\}$, $i \ne j$, and the inequality is taken elementwise. In other words, there is no other (distinct) marking M_j that has token numbers smaller than or equal to those of M_i at all places.

In general, given a set of markings, the minimal marking of the set is not necessarily unique because the ordering relation is only partial.

Definition 2 Given an observed sequence of labels ω, the set of *initial marking estimates* with respect to ω is given by $Z(\omega) = \{M' \in (Z_0^+)^n \mid \exists \sigma \in T^* : M'[\sigma\rangle \text{ and } L(\sigma) = \omega\}$.

Definition 3 Given an observed sequence of labels ω, the set of *minimal initial marking estimates* with respect to ω is given by $Z_{minimal}(\omega) = \{M \in Z(\omega) \mid \nexists M' \in Z(\omega) : M' \leq M \text{ and } M' \neq M\}$.

Definition 4 Given an observed sequence of labels ω, the set of *minimum initial marking estimates* with respect to ω is given by $Z_{minimum}(\omega) = \{M \in Z(\omega) \mid |M| \leq |M'| \text{ for all } M' \in Z(\omega)\}$.

Remark 1 It is not hard to argue that $Z_{minimum}(\omega) = \{M \in Z_{minimal}(\omega) \mid |M| \leq |M'| \text{ for all } M' \in Z_{minimal}(\omega)\}$.

Definition 5 Given an observed label sequence $\omega = l_1 l_2 \ldots l_k$ ($l_j \in \Sigma$, $j \in \{1,2,\ldots,k\}$), $\omega_{k-1} = l_1 l_2 \ldots l_{k-1}$ is the *prefix* of ω with length $k-1$. Similarly, given a transition firing sequence $\sigma = t_{i1} t_{i2} \ldots t_{ik}$, the *prefix* of σ with length $k-1$ is given by $\sigma_{k-1} = t_{i1} t_{i2} \ldots t_{i(k-1)}$.

Definition 6 Given two markings $M \in (Z_0^+)^n$ and $M' \in (Z_0^+)^n$, we say M' is *comparable* with M if either $M' \leq M$, $M' = M$, or $M' \geq M$.

Example 1 Consider the λ-free labeled Petri net structure shown in Fig. 1 with places $P = \{p_1, p_2, p_3, p_4\}$; transitions $T = \{t_1, t_2, t_3, t_4\}$; labels $\Sigma = \{a, b, c\}$; labeling function defined as $L(t_1) = L(t_2) = a$, $L(t_3) = b$, $L(t_4) = c$; and unknown initial marking. Given the label sequence $\omega = aabc$ as our observation, we see that there are four valid transition firing sequences $\{t_1t_1t_3t_4, t_1t_2t_3t_4, t_2t_1t_3t_4, t_2t_2t_3t_4\}$ corresponding to ω. The minimal initial marking estimate with respect to each of these four transition firing sequences can be found by adding the tokens that are strictly necessary to enable this transition firing sequence (this is essentially the approach of [3], [4]). Thus, it is not difficult to argue that the corresponding (individually) minimal initial marking estimates (each corresponding to one of the transition firing sequences above) are given by $[2\ 0\ 1\ 0]^T, [1\ 0\ 0\ 0]^T, [1\ 1\ 0\ 0]^T, [0\ 2\ 0\ 0]^T$ respectively; therefore, the set of minimal initial marking estimates with respect to ω is given by $Z_{minimal}(\omega) = \{[1\ 0\ 0\ 0]^T, [0\ 2\ 0\ 0]^T\}$. Clearly, the set of minimum initial marking estimates with respect to ω is $Z_{minimum}(\omega) = \{[1\ 0\ 0\ 0]^T\}$ since marking $[1\ 0\ 0\ 0]^T$ has the smallest total number of tokens. □

From Example 1, it is not difficult to see that, given a transition firing sequence $\sigma = t_{i1} t_{i2} \ldots t_{ik}$ (where $t_{ij} \in T$ for $j \in \{1,2,\ldots,k\}$), there is a unique minimal initial marking estimate corresponding to it; this is also essentially established by the analysis in [3], [4]. In particular, the authors of [3], [4] show that this unique minimal initial marking estimate can be obtained recursively via

$$M_0^j = \max\{M_0^{j-1} + B \cdot y^{j-1},\ B^-(:, t_{ij})\} - B \cdot y^{j-1} \quad (2)$$

for $j \in \{1,2,\ldots,k\}$, where max is taken element-wise, $M_0^0 = \vec{\mathbf{0}}_n$, and y^{j-1} is the firing vector of transition firing sequence $t_{i1} t_{i2} \ldots t_{i(j-1)}$ (initially, $y^0 = \vec{\mathbf{0}}_m$ is taken to be an m-dimensional vector of zeros). Note that M_0^{j-1} is the initial marking estimate before transition t_{ij} fires and M_0^j is the initial marking estimate after transition t_{ij} fires. Intuitively, this estimate is obtained by adding (each time a transition in σ fires) the smallest number of tokens to places, such that the prefix of σ seen so far is enabled (i.e., the initial marking estimate M_0^j has just enough tokens so as to enable the prefix of σ of length j).

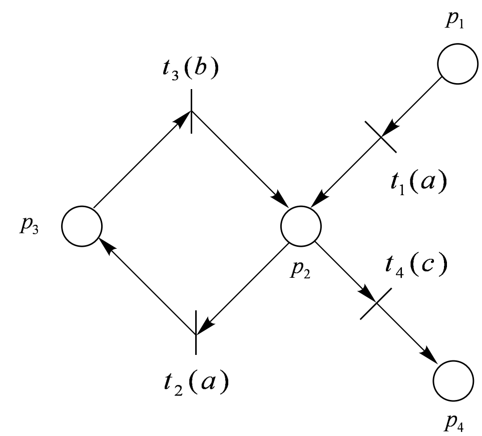

Fig. 1. Labeled Petri net structure for Example 1.

The difficulty in our setup arises due to the fact that an observed sequence of labels may correspond to a set of transition firing sequences. Therefore, given an observed sequence of labels, we have a set of (possibly minimal) initial marking estimates (instead of a single estimate). In order to solve the problem in (1), a straightforward approach would be to enumerate each possible sequence of length k, evaluate for each sequence σ whether it satisfies $L(\sigma) = \omega$, and if so compute its corresponding minimal initial marking estimate. Then, one would have to choose, among all these (individually minimal) initial marking estimates, the one(s) that has (have) the minimum total number of tokens.

The problem with the above approach is that, in the worst case, the number of sequences considered is exponential in the length k of the observed sequence of labels (more precisely, the number of sequences corresponding to the sequence of labels $\omega = l_1 l_2 \ldots l_k$ is given by $\prod_{i=1}^{k} |T_{l_i}|$). Instead of enumerating all such transition firing sequences, we will employ a trellis diagram and use a dynamic programming approach [12] to estimate the set of minimum initial markings more efficiently in a recursive manner. This approach will take advantage of the fact that several of these transition firing sequences correspond to identical firing vectors[2] and satisfy the following property (which we formally establish later): when identical firing vectors are reached with comparable minimal initial marking estimates at identical points in time, the corresponding initial marking

[2]As shown in [7], [8], the number of different such firing vectors is polynomial in the length of the observed label sequence.

estimates need not be explored separately. In fact, among these multiple initial marking estimates we only need to retain the minimal one(s).

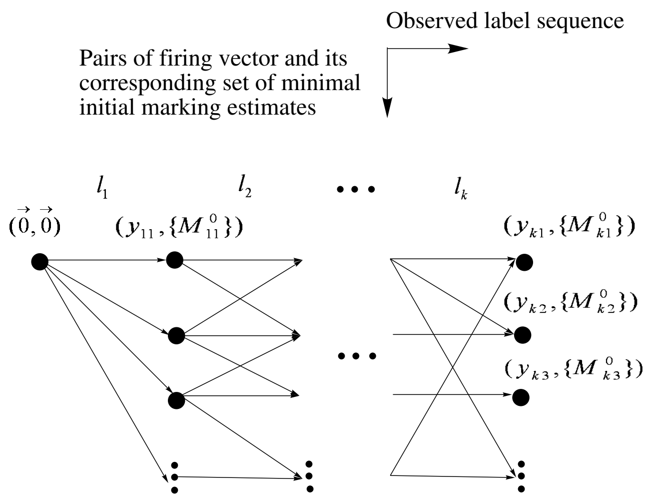

Fig. 2. Trellis diagram capturing the evolution of minimal initial marking estimates.

We start with an overview of our approach in this section and describe its crucial steps in more detail in the next section. In Fig. 2, we illustrate how a trellis diagram can be used to capture the evolution of minimal initial marking estimates as the number of observed labels increases. In particular, $\omega = l_1 l_2 \ldots l_k$ denotes the observed sequence of labels, with time epochs (stages) $\{1,2,\ldots,k\}$ corresponding to the instants each label is observed. Each node in the trellis diagram (drawn as a big black dot) denotes a pair in the form of $(y_{ji},\{M^0_{ji}\})$ such that: for $j \in \{1,2,\ldots,k\}$, y_{ji} is a firing vector that is consistent with $l_1 l_2 \ldots l_j$ and $\{M^0_{ji}\}$ is the set of minimal initial marking estimates that correspond to transition firing sequences $\sigma_j = t_{i1} t_{i2} \ldots t_{ij}$ that satisfy $L(t_{i1}) = l_1,\ L(t_{i2}) = l_2,\ \ldots,\ L(t_{ij}) = l_j$ and $\overline{\sigma_j} = y_{ji}$ (non-minimal markings are removed from $\{M^0_{ji}\}$). We refer to $\{M^0_{ji}\}$ as the set of minimal initial marking estimates for firing vector y_{ji}. Recall that each transition firing sequence has a unique minimal initial marking estimate associated with it; therefore, given a firing vector, we need to capture a *set* of minimal initial marking estimates $\{M^0_{ji}\}$ in our trellis diagram (because a particular firing vector will in general correspond to multiple transition firing sequences). Arcs between nodes in the trellis diagram in Fig. 2 represent transitions whose firings will lead from one firing vector to another. Note that from each minimal initial marking estimate in the trellis diagram, we can easily compute the corresponding current marking estimate (obtained through the firing of the corresponding sequence of transitions) because we capture the firing vector that is associated with these transition firings in the algorithm that we formally describe in the next section.

IV. Obtaining Minimum Initial Marking(s)

A. Algorithm Description

In this section, we describe a recursive algorithm to estimate the set of minimum initial markings based on an observed label sequence $\omega = l_1 l_2 \ldots l_k$ of length k. We use a data structure $\mathscr{C} = (y,\{M^0_{initial}\})$ to capture the information we need to store for each node in the trellis diagram. More specifically, at time epoch j, each node in the trellis diagram captures: (i) the firing vector y associated with sequences of transitions that are consistent with both the sequence of labels observed so far and the net structure (and have firing vector y); and (ii) the set of minimal initial marking estimates $\{M^0_{initial}\}$ associated with y (i.e., associated with at least one consistent firing sequence with firing vector y).

We describe the algorithm in detail below.

Algorithm 1

Input: A labeled Petri net structure $N_L = (P,T,A,W,L,\Sigma)$ and an observed label sequence $\omega = l_1 l_2 \ldots l_k$ of length k.

1. $\mathscr{C}(0) = \{(\vec{\mathbf{0}}_m,\ \vec{\mathbf{0}}_n)\}$.
2. Let $j = 1$.
3. Consider the label ω_j.
4. Set $\mathscr{C}(j) = \emptyset$.
5. For all $R \in \mathscr{C}(j-1)$ do

    For all t such that $L(t) = l_j$ and all markings $M^0 \in R.M^0_{initial}$
        compute $M'^0 = \max\{M^0 + B \cdot R.y\ ,\ B^-(:,t)\} - B \cdot R.y$
        compute $y' = R.y + \bar{t}$
        If y' has not appeared in $\mathscr{C}(j)$
            $\mathscr{C}(j) = \mathscr{C}(j) \bigcup (y',\{M'^0\})$
        Else
            y' has appeared in $R' \in \mathscr{C}(j)$
            Set $Flag$ = True
            For all markings $M^0_{ex} \in R'.M^0_{initial}$
                If $M^0_{ex} \leq M'^0$
                    $Flag$ = False; exit for loop
                Else If $M'^0 \leq M^0_{ex}$ and $M'^0 \neq M^0_{ex}$
                    $R' = (y',\{\{R'.M^0_{initial}\} - \{M^0_{ex}\}\})$
                End If
            End For
            If $Flag$ == True
                $R'.M^0_{initial} = R'.M^0_{initial} \bigcup \{M'^0\}$
            End If
        End If
    End For
    End For

6. $j = j + 1$.
7. If $j = k + 1$, Goto 8; else Goto 3.
8. For all $R \in \mathscr{C}(k)$, search among all minimal initial marking estimates stored in $\{R.M^0_{initial}\}$ and output the one(s) that has (have) the minimum number of tokens over all places. □

Remark 2 At Step 5 of Algorithm 1, after we obtain $(y,\{M^0_{initial}\})$ at each time epoch, we can compute the current marking estimate corresponding to each initial marking estimate $M^0 \in \{M^0_{initial}\}$ (that is required to update the minimal initial marking estimate at next time epoch) as $M^0 + B \cdot y$.

Remark 3 At Step 5 of Algorithm 1, when the same firing vector y' is obtained and is associated with multiple minimal initial marking estimates, we only retain for further consideration those estimate(s) that is (are) minimal among that set of estimates. The $Flag$ is used to keep track of whether the new initial marking estimate is "covered" by a "smaller" existing minimal initial marking estimate or not.

B. Algorithm Analysis

Note that at Step 5 of Algorithm 1, when multiple minimal initial marking estimates are computed (by considering different consistent transition firing sequences that are nevertheless associated with the same firing vector), Algorithm 1 retains for further consideration only those estimates that are *minimal* (among all estimates). To guarantee the optimality of the algorithm, we need to ensure that the minimal initial marking estimates that are retained will always result in smaller initial marking estimates at later stages; this is established in the lemma below.

Lemma 1 Given two initial marking estimates $M_{0,j}$ and $M'_{0,j}$ that satisfy $M_{0,j} \leq M'_{0,j}$ and are associated with the same firing vector y at the j^{th} ($1 \leq j \leq k$) stage of the trellis diagram, the initial marking estimates $M_{0,j'}$ and $M'_{0,j'}$ obtained from markings $M_{0,j}$ and $M'_{0,j}$ respectively via identical transition firing sequences (and thus identical firing vectors) satisfy $M_{0,j'} \leq M'_{0,j'}$ for all stages $j' \in \{j+1, j+2, \ldots, k\}$.

Proof: Without loss of generality, assume transition t is associated with the label observed at $(j+1)^{st}$ stage. Using Eq. (2), we can calculate the associated minimal initial marking estimates $M_{0,j+1}$ and $M'_{0,j+1}$ at the $(j+1)^{st}$ stage as

$$M_{0,j+1} = \max\{M_{0,j} + By, B^-(:,t)\} - By,$$
$$M'_{0,j+1} = \max\{M'_{0,j} + By, B^-(:,t)\} - By.$$

Clearly, since $M_{0,j} \leq M'_{0,j}$, we have

$$M_{0,j} + By \leq M'_{0,j} + By.$$

For places p such that $M'_{0,j}(p) + (By)(p) \leq B^-(p,t)$, we have

$$M'_{0,j+1}(p) = B^-(p,t) - (By)(p) = M_{0,j+1}(p);$$

for places p such that $M'_{0,j}(p) + (By)(p) > B^-(p,t)$, we have

$$\begin{aligned} M'_{0,j+1}(p) &= M'_{0,j}(p) + (By)(p) - (By)(p) \\ &= M'_{0,j}(p) \\ &\geq \max\{M_{0,j}(p) + (By)(p), B^-(p,t)\} - (By)(p) \\ &= M_{0,j+1}(p) \end{aligned} \tag{3}$$

since $M'_{0,j} + By \geq M_{0,j} + By$ and $M'_{0,j}(p) + (By)(p) > B^-(p,t)$.

Therefore, we have $M_{0,j+1} \leq M'_{0,j+1}$. Using induction on the number of stages (and employing similar analysis as above), the result follows. ■

Example 2 Recall the labeled Petri net structure shown in Fig. 1. If the observed label sequence is given by $\omega = aabc$, the corresponding trellis diagram after running Algorithm 1 is shown in Fig. 3. Each node is associated with a pair of a firing vector and its corresponding set of minimal initial marking estimates. In particular, after considering all four labels the possible firing vectors (associated with the firing sequences from the initial marking estimate) are $[2\ 0\ 1\ 1]^T$, $[1\ 1\ 1\ 1]^T$, and $[0\ 2\ 1\ 1]^T$, with corresponding minimal initial marking estimates $\{[2\ 0\ 1\ 0]^T\}$, $\{[1\ 0\ 0\ 0]^T\}$, and $\{[0\ 2\ 0\ 0]^T\}$ respectively. Clearly, the minimum initial marking estimate is given by $[1\ 0\ 0\ 0]^T$ since it has the minimum total number of tokens over all places. Note that at stage 2 of the trellis diagram, after considering both transition firing sequences t_1t_2 and t_2t_1, we have the common firing vector $[1\ 1\ 0\ 0]^T$ that is associated with two (individually) minimal initial marking estimates $[1\ 0\ 0\ 0]^T$ and $[1\ 1\ 0\ 0]^T$. We only retain marking $[1\ 0\ 0\ 0]^T$ because it is minimal (i.e., we discard marking $[1\ 1\ 0\ 0]^T$ from further consideration). □

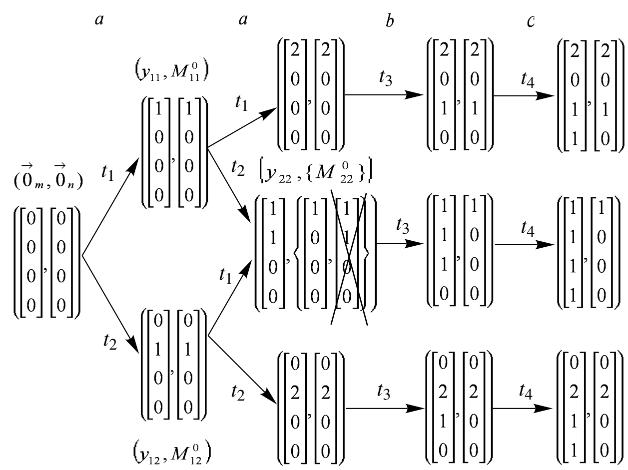

Fig. 3. Trellis diagram of minimal initial marking estimates after running Algorithm 1 for Example 1.

C. Complexity Analysis

To analyze the complexity of Algorithm 1, we use the results established in [7], [8] for state estimation in a labeled Petri net. Specifically, we use the fact that given an observed sequence of labels ω of length k, the total number of possible firing vectors at each time epoch is upper bounded by a polynomial function in k (i.e., $O(k^b)$ where b is a parameter that depends on the structure and the labeling function of the net[3]). This implies that the number of nodes at the j^{th} stage of the trellis diagram is upper bounded by $O(j^b)$. Note, however, that a node (i.e., a firing vector) may be associated with multiple minimal initial marking estimates (depending on the order of transitions in different firing sequences that are consistent with the observed sequence of labels and share the same firing vector, as seen in Example 2). We now establish that for a given labeled Petri net structure with

[3]More precisely, in [7] it is argued that $b = c(d-1)$ where c is the number of nondeterministic labels (i.e., labels $l \in \Sigma$ such that $|T_l| \geq 2$) in the net and d is the maximum number of transitions corresponding to a label in the net (i.e., $d = \max_{l \in \Sigma}\{|T_l|\}$).

n places and m transitions, and an observed sequence of labels of length k, the number of minimal initial marking estimates we retain for each firing vector is upper bounded by a polynomial function in k.

Definition 7 We say a marking $M = [M(p_1)\ M(p_2)\ \dots\ M(p_n)]^T \in (Z_0^+)^n$ is $K-bounded$ if $M(p_i) \leq K$ for all $i \in \{1,2,\dots,n\}$.

Denote the maximum entry in the input incident matrix B^- by c (i.e., $c = \max\ B_{ij}^-$ for $i \in \{1,2,\dots n\}$ and $j \in \{1,2,\dots,m\}$). In Algorithm 1, after each label is observed, the number of tokens we need to add for each place in order to enable any transition is clearly upper bounded by c (also refer to Eq. (2)). Therefore, given an observed sequence of labels of length k, the maximum number of tokens we need to add in each place is upper bounded by ck. Therefore, each minimal initial marking estimate is $ck-bounded$.

Lemma 2 Given a $ck-bounded$ minimal initial marking estimate $M = [M(p_1)\ M(p_2)\ \dots\ M(p_n)]^T \in (Z_0^+)^n$ where $0 \leq M(p_i) \leq ck$ for $i \in \{1,2,\dots,n\}$, the total number of minimal initial marking estimates that are not comparable with M is given by $(ck+1)^n - (\prod_{i=1}^n (M(p_i)+1) + \prod_{i=1}^n (ck - M(p_i) + 1)) + 1$.

Proof: Clearly, the total number of $ck-bounded$ markings is $(ck+1)^n$. The total number of $ck-bounded$ markings M' such that $M' \leq M$ is $\prod_{i=1}^n (M(p_i)+1)$ whereas the total number of $ck-bounded$ markings M' such that $M' \geq M$ is $\prod_{i=1}^n (ck - M(p_i)+1)$. Since marking M is counted in both cases, it follows that the total number of $ck-bounded$ markings that are not comparable with M is given by $(ck+1)^n - (\prod_{i=1}^n (M(p_i)+1) + \prod_{i=1}^n (ck - M(p_i)+1)) + 1$. ∎

It follows from Lemma 2 that the number of minimal marking estimates we might consider for each node (i.e., each firing vector) at stage j of the trellis diagram is upper bounded by a polynomial function in j, i.e., $O(j^n)$. With this observation in hand, the complexity of Algorithm 1 can be obtained as follows. First, regarding space complexity, the storage needed is proportional to the product of the number of firing vectors and the number of corresponding minimal initial marking estimates. Since the number of minimal initial marking estimates at the j^{th} stage of the trellis diagram is upper bounded by $O(j^n \cdot j^b)$, the total space needed to store all marking estimates is $\sum_{j=1}^k O(j^{n+b})$ which can be simplified as $O(k \cdot k^{n+b}) = O(k^{n+b+1})$, i.e., the storage required is polynomial in the length k of the observed sequence of labels.

Using an analysis similar to the above for the case of space complexity, we can establish that the computational complexity of Algorithm 1 is $O(k^{n+2b+1})$, which is polynomial in the length k of the observed sequence of labels.

V. Conclusions and Future Work

In this paper, we considered the problem of estimating the set of minimum initial markings that can explain a given observed sequence of labels in a labeled Petri net. Specifically, given the observation of a sequence of labels (and assuming complete knowledge of the net structure), we aimed at estimating the set of minimum initial markings of the net that (i) allow for the firing of at least one sequence of transitions that is consistent with both the observed sequence of labels and the net structure; (ii) have the least total number of tokens summed over all places. We developed a recursive algorithm that is able to find the set of minimum initial markings with complexity that is polynomial in the length of the observed label sequence. This algorithm can be used for minimum resource allocation in manufacturing systems that are modeled as labeled Petri nets and for online estimation and supervision.

One interesting direction for future work is to investigate subclasses of net structures to further reduce the complexity of the algorithm. Another possible extension is to consider scenarios where unobservable transitions may be present in the net.

References

[1] T. Murata, "Petri nets: properties, analysis and applications," *Proceedings of the IEEE,* vol. 77, no. 4, pp. 541–580, April 1989.

[2] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems,* Springer, 1999.

[3] A. Giua, "Petri net state estimators based on event observation," in *Proc. 36th IEEE Int. Conf. Decision and Control*, pp. 4086–4091, December 1997.

[4] A. Giua and C. Seatzu, "Observability of place/transition nets," *IEEE Trans. on Automatic Control*, vol. 47, no. 9, pp. 1424–1437, September 2002.

[5] A. Giua, J. Julvez, and C. Seatzu, "Marking estimation of Petri nets based on partial observation," in *Proc. American Control Conf.*, pp. 326–331, June 2003.

[6] A. Giua, C. Seatzu, and D. Corona, "Marking estimation of Petri nets with silent transitions," *IEEE Trans. on Automatic Control*, vol. 52, no. 9, pp. 1695–1699, September 2007.

[7] Y. Ru and C. N. Hadjicostis, "State estimation in discrete event systems modeled by labeled Petri nets," in *Proc. 45th IEEE Int. Conf. Decision and Control*, pp. 6022–6027, December 2006.

[8] Y. Ru and C. N. Hadjicostis, "Bounds on the number of markings consistent with label observations in Petri nets." To appear in *IEEE Trans. on Automation Science and Engineering*.

[9] D. Kiritsis, K.-P. Neuendorf, and P. Xirouchakis, "Petri net techniques for process planning cost estimation," *Advances in Engineering Software,* vol. 30, no. 6, pp. 375–387, June 1999.

[10] L. Li and C. N. Hadjicostis, "Least-cost planning sequence estimation in labeled Petri nets." To appear in *Transactions of the Institute of Measurement and Control, Special Issue on "Planning and Scheduling in Complex Systems."*

[11] S. Lin, T. Kasami, T. Fujiwara, and M. Fossorier, *Trellises and Trellis-Based Decoding Algorithms for Linear Block Codes,* Kluwer Academic Publishers, 1998.

[12] R. Bellman, *Dynamic Programming,* Princeton University Press, 1957.