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Lyapunov stability for Quantum Markov Processes

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Abstract— **In this paper we prove some Lyapunov stability results for Quantum systems. The evolution of open quantum systems can be described using a one parameter semigroup of completely positive operators with which we can associate a minimal quantum Markov dilation. Analogous to Lyapunov stability theorems of classical Markov processes, we develop Lyapunov stability theorems for minimal Markov dilations of quantum systems. This theory depends on a quantum version of Dynkin's formula.**

Index Terms— **Lyapunov stability, Quantum Control, Quantum Markov Processes.**

I. INTRODUCTION

Lyapunov stability theory has been used extensively in the design of controllers for classical nonlinear systems. In this paper, we prove some Lyapunov type stability results for quantum systems whose evolution can be described using quantum Markov processes. One can describe an open quantum system using a quantum stochastic differential equation (QSDE). Hudson and Parthasarathy [1] prove that the solution of a QSDE leads to a one parameter semigroup of completely positive operators that can be used to describe the system evolution. Given such a semigroup one can associate a minimal Markov dilation for the group (see Section III). We develop Lyapunov stability theory for quantum systems whose evolution is described using a one parameter semigroup of completely positive operators.

This paper is organised as follows: In the next section, we review classical Markov processes and Lyapunov stability of classical systems. We state two simple Lyapunov stability results that we wish to generalise to the quantum case. In Section III we review quantum Markov processes and state a quantum version of Dynkin's formula. Finally, in Section IV we prove a quantum Lyapunov stability result. We also discuss a simple example.

II. CLASSICAL MARKOV PROCESS AND LYAPUNOV STABILITY

We recall some Lyapunov stability results of classical stochastic systems [2].

A. Classical Markov Processes

Suppose $(x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x)$ is a Markov process in the sense of Dynkin [3]. Here, $x_t : \Omega \to X$ is defined on some sample space Ω and X is a metric space, ζ is an $\mathbb{R}^+$ valued random variable defined on Ω, $\mathscr{M}_t$ is a σ-algebra on the space $\Omega_t = \{\omega : \zeta(\omega) > t\}$ and $\mathbf{P}_x$ is a probability measure on some σ-algebra $\mathscr{M}^0$ such that $\mathscr{M}_t \subset \mathscr{M}^0$ for all $t \geq 0$.

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We can think of ζ as the terminal time of the process. For a fixed ω, the function $x_t(\omega)$ defines in the space X, the trajectory corresponding to the sample path ω. The σ-algebra $\mathscr{M}_t$ can be visualised as the totality of events which are observed in the time-interval $[0, t]$. Finally, $\mathbf{P}_x(A)$ gives the probability of event A given the initial condition $x_0 = x$.

The probability measure $\mathbf{P}_x\{\cdot\}$ satisfies the Markov property:

$$\mathbf{P}_x\{x_{t+h} \in \Gamma | \mathscr{M}_t\} = \mathbf{P}_{x_t}\{x_h \in \Gamma\} \tag{1}$$

The transition function corresponding to this Markov process is given by

$$\tilde{P}(t, x, \Gamma) = \mathbf{P}_x\{x_t \in \Gamma\},$$

where, $t \geq 0$, $x, x_t \in X$ and $\Gamma \in \mathscr{B} = \mathscr{B}(X)$, the Borel σ-algebra of X. Now let $B = B(X, \mathscr{B})$ denote the $*$-algebra of all complex-valued, bounded and measurable functions defined on $(X, \mathscr{B})$. The transition function determines a unique positive semigroup of unital operators T_t on the space B as follows

$$(T_t f)(x) = \int_X \tilde{P}(x, t, dy) f(y).$$

The weak infinitesimal operator, corresponding to this semigroup of operators, is defined

$$\tilde{A} f = \mathrm{wlim}_{h \downarrow 0} \frac{T_h f - f}{h}.$$

Here, $\mathrm{wlim}_{h\downarrow 0}$ denotes the weak limit as h decreases to zero. Dynkin's formula [3] plays a crucial rule in proving several stochastic stability results and is a stochastic version of the second fundamental theorem of calculus. Suppose $f \in \mathrm{Dom}(\tilde{A})$, the domain of $\tilde{A}$. Then, Dynkin's formula states

$$\mathbf{E}_x f(x_\tau) - f(x) = \mathbf{E}_x \int_0^\tau \tilde{A} f(x_s) ds. \tag{2}$$

Here, $\mathbf{E}_x$ is the expectation value with respect to the measure $\mathbf{P}_x$. In the following, for ease of notation, we denote by x_t^z the Markov process $(x_t, \zeta, \mathscr{M}_t, \mathbf{P}_z)$. The superscript z will be dropped if it causes no confusion. Also, if τ_m is any Markov time, then the stopped process $x_{t\wedge\tau_m}$ is defined as

$$x_{t\wedge\tau_m} = \begin{cases} x_t & \text{if } t \leq \tau_m, \\ x_{\tau_m} & \text{otherwise.} \end{cases}$$

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B. Stochastic Stability

Definition 2.1: The process x_t^z is said to be *stable with respect to the triple* (Q, P, ϵ), where $Q, P \subset X$ and $\epsilon > 0$ if $z \in Q$ implies

$$\mathbf{P}_z \left\{ x_t^z \in P, \ \forall\ t < \infty \right\} \geq \epsilon.$$

We collect some assumptions to be used in later theorems.

A1: For any constant $c > 0$ let Q_c denote the set $\{x : V(x) < c\}$, and suppose $V : X \to \mathbb{R}^+$ is continuous in the open set Q_m for some $m > 0$.

A2: x_t is a right continuous Markov process defined until at least some time $\tau' > \tau_m = \inf\{t : x_t \notin Q_m\}$ with probability 1.

A3: Let $\tilde{A}_m = \tilde{A}_{Q_m}$, the weak infinitesimal operator of $x_{t\wedge\tau_m}$.

A4: $V(x)$ is in the domain of $\tilde{A}_m$.

Let $B_m = \{\omega : x_t \in Q_m, \ \forall\ t < \infty\}$. The following lemma and theorem are from Kushner [2, ch. 2].

Lemma 2.1: Assume (A1) to (A4). Let $\tilde{A}_m V(x) \leq 0$. Then, $V(x_{t\wedge\tau_m})$ is a supermartingale and for $\lambda \leq m$, and initial condition $x_0 = z \in Q_m$,

$$\mathbf{P}_z \left\{ \sup_{\infty > t \geq 0} V(x_{t\wedge\tau_m}) \geq \lambda \right\} \leq \frac{V(z)}{\lambda}.$$

Also, there is a random variable $c(\omega)$, $0 \leq c(\omega) \leq m$, such that with probability 1 relative to B_m, $V(x_t) \to c(\omega)$ as $t \to \infty$ and $\mathbf{P}_z\{B_m\} \geq 1 - \frac{V(z)}{m}$.

This lemma is a direct consequence of Dynkin's formula and in order to generalise Lyapunov theory to quantum Markov processes, we need a quantum version of Dynkin's formula.

Theorem 2.2 (Stability): Assume (A1) to (A4) for some $m > 0$ and suppose $\tilde{A}_m V(x) \leq 0$. Let $V(0) = 0$ and $z \in Q_m$. Then the system is stable relative to $(Q_r, Q_m, 1 - \frac{r}{m})$ for any $r = V(x_0) = V(z) \leq m$. Also, for almost all $\omega \in B_m$, $V(x_{t\wedge\tau_m}) \to c(\omega) \leq m$. If $V(x) > 0$ for $x \neq 0$ and $z \in Q_m$ then the origin is stable with probability 1.

III. Quantum Markov Processes

A. Reformulation of Classical Markov Processes

In order to motivate the definition of quantum Markov processes, we reformulate the classical Markov process in a different mathematical terminology. Consider the Markov process[1] $(x_t, \mathscr{M}_t, \mathbf{P}_x)$. Let $\mathcal{H}$ be the probability space $\mathcal{L}^2(\mathbf{P}_x)$, the space of all square integrable functions defined on Ω. Also, let $\mathcal{H}_t$ denote the subspace of $\mathcal{H}$ of functions measurable with respect to $\mathscr{M}_t$ and let F_t be the projection onto $\mathcal{H}_t$. Then F_t is an increasing sequence of projections in $\mathcal{H}$. For any $g \in B = B(X, \mathscr{B})$, we can define the operator $j_t(g)$ in $\mathcal{B}(\mathcal{H})$, the set of bounded operators on $\mathcal{H}$, by

$$(j_t(g)\phi)(\omega) = g(x_t)(F_t\phi)(\omega), \ \forall\ \phi \in \mathcal{H}.$$

Then j_t is a $*$-homomorphism from B to $\mathcal{B}(\mathcal{H})$. The Markov property (1) is encapsulated in the operator relations

$$\begin{aligned} j_t(\mathbf{1}) &= F_t \\ F_t j_s(g) F_t &= j_t(T_{s-t} g), \ g \in B, s \geq t. \end{aligned}$$

[1]We assume that $\zeta(\omega) = \infty$ for almost all ω for the Markov process $(x_t, \zeta, \mathscr{M}_t, \mathbf{P}_x)$.

Here, $\mathbf{1}$ is the identity operator. Dynkin's formula (2) can be written as

$$F_0 j_\tau(f) F_0 = j_0(f) + F_0 \int_0^\infty \mathbf{1}_{\tau > s} j_s(\tilde{A}(f)) ds F_0.$$

Here $\mathbf{1}_{\tau>s}$ is the indicator function of the event $\tau > s$.

B. Quantum Markov Dilations

Let $\mathscr{A}_t$, $t \geq 0$ be a unital C^*-algebra of operators in the Hilbert space $\mathcal{K}_t$, and let $T(s,t) : \mathscr{A}_t \to \mathscr{A}_s$, $s < t$ be a stochastic operator. That is,

1) $T(s,t)\mathbf{1} = \mathbf{1}$.
2) $T(s,t)$ is completely positive: i.e., for all $n = 1, 2, \ldots$ and all $X_1, \ldots, X_n \in \mathscr{A}_t$, $Y_1, \ldots, Y_n \in \mathscr{A}_s$, we have
$$\sum_{1 \leq i,j \leq n} Y_i^* T(s,t)(X_i^* X_j) Y_j \geq 0.$$
3) If $Y_n \in \mathscr{A}_t$ for $n = 1, 2, \ldots$ and $Y_n \to Y \in \mathscr{A}_t$ weakly, then $T(s,t)(Y_n) \to T(s,t)Y$ weakly in $\mathscr{A}_s$.

Also suppose that the Chapman-Kolgomorov equation holds:

$$T(r,s)T(s,t) = T(r,t) \ \forall\ r < s < t.$$

Let $T(t,t)$ be the identity operator on $\mathscr{A}_t$.

Bhat and Parthasarathy [4,5] prove the following theorem (see e.g. [6, Prop 9.8] and the appendix).

Theorem 3.1: Suppose $\{\mathscr{A}_t\}$ is a family of unital C^*-algebras of operators on Hilbert space $\mathcal{K}_t$, $t \geq 0$ and let $\{T(s,t) : \mathscr{A}_t \to \mathscr{A}_s, s < t\}$ be a family of stochastic operators obeying the Chapman-Kolgomorov conditions. Then there exists a triple $(\mathcal{H}, F_t, j_t)$ such that

1) $\mathcal{H}$ is a Hilbert space and F_t is a projection in $\mathcal{H}$ satisfying $F_t \uparrow \mathbf{1}$ as $t \uparrow \infty$.
2) $j_t : \mathscr{A}_t \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism and $j_t(\mathbf{1}) = F_t$.
3) $F_s j_t(X) F_s = j_s(T(s,t)(X))$ for $s < t$, $X \in \mathscr{A}_t$.
4) $\{j_{t_1}(X_1) \ldots j_{t_n}(X_n)u, t_1 > \ldots > t_n, X_{t_i} \in \mathscr{A}_{t_i}, u \in \mathcal{K}_0\}$ is total in $\mathcal{H}$.
5) The subspace $\mathcal{H}_t$, the range of F_t is spanned by the set $\{j_{t_1}(X_1) \ldots j_{t_n}(X_n)u, t \geq t_1 > \ldots > t_n, X_{t_i} \in \mathscr{A}_{t_i}, u \in \mathcal{K}_0\}$.
6) The triple $(\mathcal{H}, F_t, j_t)$ is unique up to an isometric transformation.

Definition 3.1: The triple $(\mathcal{H}, F_t, j_t)$ is called the minimal Markov dilation for the family $\{T(s,t)\}$ of stochastic operators.

In the above theorem, Statement 3) is the quantum version of the classical Markov property (1).

C. Markov stop times and the Strong Markov property

In this subsection, we consider a one parameter semigroup of stochastic operators (i.e. $T(s,t) = T(t-s)$). Let $\mathcal{K}_t = \mathcal{K}$ be a Hilbert space and let $\mathscr{A}_t = \mathscr{A} \subset \mathcal{B}(\mathcal{K})$ be a unital C^*-algebra. Let $\{T_t : \mathscr{A}_{t+t_0} \to \mathscr{A}_{t_0}, t > 0, t_0 \geq 0\}$ be a one parameter semigroup of stochastic operators. Suppose $(\mathcal{H}, F_t, j_t)$ is a Markov dilation associated with the stochastic operator T_t.

Definition 3.2: [6, p. 111] A *stoptime (or Markov time)* τ for the flow $(\mathcal{H}, F_t, j_t)$ is a spectral measure on $[0, \infty]$ with values in orthogonal projections on $\mathcal{H}$ satisfying the condition

$$[\tau([0,s]), j_t(X)] = 0, \ \forall \ s \le t \text{ and } X \in \mathscr{A}.$$

Here $[\cdot, \cdot]$ denotes the commutator of two operators.
The projection $\tau([0,t])$ is to be interpreted as the event of, stopping the Markov process, has occurred at or before time t. We denote by $\mathbf{1}_E$ the event $\tau(E)$ for all Borel subsets E of $[0, \infty]$. For any two stoptimes τ_1, τ_2 that commute (i.e. $[\tau_1([0,a]), \tau_2([0,b])] = 0$) we can define the minimum $\tau_1 \wedge \tau_2$ and maximum $\tau_1 \vee \tau_2$ stoptimes, of τ_1 and τ_2 as

$$\begin{aligned} \mathbf{1}_{\tau_1 \wedge \tau_2 \le t} &= \mathbf{1}_{\tau_1 \le t} + \mathbf{1}_{\tau_2 \le t} - \mathbf{1}_{\tau_1 \le t}\mathbf{1}_{\tau_2 \le t}, \\ \mathbf{1}_{\tau_1 \vee \tau_2 \le t} &= \mathbf{1}_{\tau_1 \le t}\mathbf{1}_{\tau_2 \le t}. \end{aligned}$$

Also, if $t \ge 0$, then denote the 'deterministic' stoptime τ defined as $\tau(\{t\}) = \mathbf{1}$ by t.

Now suppose τ is a simple stoptime (i.e. the support of τ is a finite set). Then define the operators

$$\begin{aligned} j_\tau(X) &= \sum_s \tau(\{s\}) j_s(X), \ X \in \mathscr{A}, && (3) \\ F_\tau &= j_\tau(\mathbf{1}). && (4) \end{aligned}$$

Here, the summation is over the support of τ. For any stoptime τ and any set $E = \{t_1 < t_2 < \ldots < t_n\} \subset (0, \infty)$, define the stoptime τ_E as

$$\tau_E([0,s]) = \begin{cases} \tau(\{0\}) & \text{if } s < t_1, \\ \tau([0, t_{i-1}]) & \text{if } t_{i-1} \le s < t_i, i = 1, \ldots, n \\ \tau([0, t_n]) & \text{if } t_n \le s < \infty, \\ \mathbf{1} & \text{if } s = \infty. \end{cases} \quad (5)$$

Now, if τ is any stoptime then $\{\tau_E : E = \{t_1 < t_2 < \ldots < t_n\} \subset (0, \infty)\}$ is a monotone decreasing net of projections in $\mathcal{H}$ which converges strongly to a projection [6, Proposition 13.1, Corollary 12.4]. We define

$$F_\tau = \mathrm{slim}_E F_{\tau_E}.$$

Here, slim stands for the strong limit. Attal and Parthasarathy [7] prove the following theorem, provided the infinitesimal generator of the stochastic operator T_t satisfies a technical condition, called condition S[2] [6, p. 123].

Theorem 3.2: [6, Theorem 16.5] Let $\{T_t\}$ be a strongly continuous semigroup of stochastic operators on a C^*-algebra $\mathscr{A}$ of operators on a Hilbert space $\mathcal{K}$, whose infinitesimal generator satisfies condition S. Let $(\mathcal{H}, F(t), j_t), \ t \ge 0$ be its minimal Markov dilation and let τ be any stoptime. Then there exists a $*$-homomorphism $j_\tau : \mathscr{A} \to \mathcal{B}(\mathcal{H})$ satisfying the following conditions

1) $j_\tau(X) = \mathrm{slim}_{t \to \infty} j_{\tau \wedge t}(X) \tau([0, \infty])$ for all $X \in \mathscr{A}$.
2) $j_\tau(\mathbf{1}) = F_\tau \tau([0, \infty])$.
3) $F_\tau j_{\tau + t}(X) F_\tau = j_\tau(T_t X)$ for all $t > 0$ and $X \in \mathscr{A}$.

The final property in the above theorem encapsulates the strong Markov property of the minimal Markov dilation.

[2] If $\mathcal{L}$, with domain $\mathrm{Dom}(\mathcal{L})$, is the generator of T_t, then Condition S is satisfied if there exists a dense $*$-subalgebra $\mathcal{A}_0 \subset \mathrm{Dom}(\mathcal{L})$ such that

1) $\mathcal{A}_0$ is invariant under the action of $\{T_t\}$.
2) The map $t \mapsto \mathcal{L}(X_1 T_t(Y) X_2)$ is locally bounded for every $X_1, X_2, Y \in \mathcal{A}_0$.

D. Dynkin's Formula

Recall that if T_t is a family of stochastic operators on a C^* algebra of operators $\mathscr{A}$ then the differential generator of T_t is defined to be the operator $\mathcal{L}$ with domain consisting of the set of all $X \in \mathscr{A}$ such that the limit

$$\lim_{h \to 0} \frac{T_h X - X}{h}$$

is well-defined. The above limit is defined to be $\mathcal{L}(X)$.

Note that if τ is a non-negative real-valued spectral measure on $\mathcal{H}$ then we can define a self-adjoint operator $\hat{\tau} = \int_0^\infty s P^\tau(ds)$. Here $P^\tau(ds)$ is the projection corresponding to ds under the spectral measure τ. We can define the square root of $\hat{\tau}$ as $\hat{\tau}^{1/2} = \int_0^\infty s^{1/2} P^\tau(ds)$.

Theorem 3.3: [7, Theorem 9.3] Suppose $\{T_t\}$ is a family of stochastic operators with differential generator $\mathcal{L}$ and associated minimal Markov dilation $(\mathcal{H}, F_t, j_t)$. Also let τ be a finite stoptime and suppose $\psi \in \mathcal{H}$ is such that $F_0 \psi \in \mathrm{Dom}(\hat{\tau}^{1/2})$. Then for all $X \in \mathrm{Dom}(\mathcal{L})$,

$$\begin{aligned} F_0 j_\tau(X) F_0 \psi &= j_0(X) F_0 \psi \\ &\quad + F_0 \int_0^\infty \mathbf{1}_{\tau > s} j_s(\mathcal{L}(X)) ds F_0 \psi. \quad (6) \end{aligned}$$

Because the space $\mathcal{K}$ is isomorphic to $\mathcal{H}_0$ and we can think of $\mathcal{H}_0$ as $\mathcal{K}$. Therefore, Dynkin's formula (6) holds for all $u \in \mathcal{K}$ with $F_0 \psi$ replaced by u in the above equation.

E. Open quantum systems and Markov dilations

Open quantum systems can be described using quantum stochastic differential equations [8]–[10]. Such a system can be described in a compact manner using its Ito generator matrix (see e.g. [11] for a detailed description)

$$\mathbf{G} = \begin{bmatrix} -\frac{1}{2}\mathbf{L}^\dagger \mathbf{L} - iH & -\mathbf{L}^\dagger \mathbf{S} \\ \mathbf{L} & \mathbf{S} - \mathbf{I} \end{bmatrix}. \quad (7)$$

Here, the entries in $\mathbf{G}$ are bounded linear operators defined on the system Hilbert space $\mathcal{K}$. If we use $\mathscr{A}$ to denote the algebra of bounded operators defined on $\mathcal{K}$, then $\mathbf{L}$ is a vector of length n with entries in $\mathscr{A}$, $H \in \mathscr{A}$ is Hermitian and $\mathbf{S}$ is an $n \times n$ matrix with entries in $\mathscr{A}$, satisfying $\mathbf{S}^\dagger \mathbf{S} = \mathbf{I}$. Here, $(\cdot)^\dagger$ denotes the conjugate transpose of a matrix.

Physically, $\mathbf{G}$ represents a quantum system with Hamiltonian H coupled to a multichannel quantum noise field $\hat{\mathbf{A}}$ through interaction operators $\mathbf{L}$ and scattering matrix $\mathbf{S}$. Here, $\hat{\mathbf{A}}$ is the quantum noise defined on a suitable Boson Fock space and is given by

$$\hat{\mathbf{A}} = \begin{bmatrix} t & \mathbf{A}^T(t) \\ \mathbf{A}^*(t) & \Lambda(t) \end{bmatrix}.$$

$\mathbf{A}$ is a vector (of length n) of annihilation operators and Λ is an $n \times n$ matrix of scattering operators.

Hudson and Parthasarathy [1][3] proved that the evolution of the system operators $X \in \mathscr{A}$ can be described via a one parameter semigroup of operators whose generator can be written as

$$\mathcal{L}(X) = \frac{1}{2}\mathbf{L}^\dagger[X, \mathbf{L}] + \frac{1}{2}[\mathbf{L}^\dagger, X]\mathbf{L} - i[X, H].$$

Following Gough and James [11] we simply use $\mathbf{G} = (\mathbf{S}, \mathbf{L}, H)$ to describe the system with generator matrix (7).

Finally, Gough and James [11] consider the network description of several open quantum systems that are connected together either in parallel (concatenation) or series configurations. Given two systems G_1 and G_2, the concatenation, $G_1 \boxplus G_2$ and series products $G_1 \lhd G_2$ are defined by

$$\begin{aligned} G_1 \boxplus G_2 &= \left(\begin{pmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{pmatrix}, H_1 + H_2 \right) \\ G_1 \lhd G_2 &= (\mathbf{S}_2\mathbf{S}_1, \mathbf{L}_2 + \mathbf{S}_2\mathbf{L}_1, H_1 + H_2 + \\ &\quad \frac{1}{2i}(\mathbf{L}_2^\dagger\mathbf{S}_2\mathbf{L}_1 - \mathbf{L}_1^\dagger\mathbf{S}_2\mathbf{L}_2)) \end{aligned}$$

The series product is well defined only if G_1 and G_2 have the same number of field channels. It describes the open quantum system formed by connecting the field output of the first system to the field input of the second system and can be used to model field mediated interactions.

In analyzing the stability of open quantum system we assume that the plant is described by its generator $\mathbf{G}_p$ and is controlled using a controller with generator $\mathbf{G}_c$. The controller and plant network is described using series and/or concatenation products and is denoted by $\mathbf{G}_p \wedge \mathbf{G}_c$. We look for controllers that stabilize the plant in the sense of Lyapunov as described in the following section

IV. Lyapunov stability for Quantum systems

The main results of this paper are discussed in this section. We define a notion of Lyapunov stability for quantum systems and prove two stability theorems that are similar to the classical results discussed in Section II. We first introduce the notion of a quantum state.

A. State of a Quantum system

Consider a quantum system $\mathscr{S}$ whose states are described in a Hilbert space $\mathcal{K}$. Let ρ be a density matrix on $\mathcal{H}$ so that $\langle\cdot\rangle = \mathrm{trace}\{\cdot\rho\}$ denotes the expectation value of an operator. Suppose $(\Omega, \mathscr{B})$ is some measure space with σ-algebra $\mathscr{B}$. Then, an Ω-valued observable $X : \mathscr{B} \to \mathcal{P}(\mathcal{K})$ is a spectral valued measure that takes values in $\mathcal{P}(\mathcal{K})$, the set of projections in $\mathcal{K}$. We can define a probability distribution on Ω (see [8, Theorem 9.18])

$$\mathrm{Prob}\{X \in A\} = \mu(A) = \mathrm{trace}\{\rho X(A)\}, \ A \in \mathscr{B}. \quad (8)$$

This probability distribution may be interpreted as the probability that a measurement of the observable X takes a value in the set A[4].

[3]Also see [8,11].

[4]Note that if we wish to find a probability distribution of several observables simultaneously, then we can set Ω to be $\mathbb{R}^n$. However, this definition will only make sense if all the n observables commute.

Suppose $\{T_t\}$ is a family of stochastic operators defined on a C^*-algebra $\mathscr{A} \subset \mathcal{B}(\mathcal{K})$ and let $(\mathcal{H}, F_t, j_t)$ be the associated minimal Markov dilation. Suppose $\mathbf{X} = (X_1, \ldots, X_n),\ X_i \in \mathcal{B}(\mathcal{K})$ are system observables and let $j_t(\mathbf{X}) = (j_t(X_1), \ldots, j_t(X_n))$. We call $(\rho, j_t(\mathbf{X}))$ the state of the system at time t.

Remark 4.1: Note that the observables X_i are a set of relevant system observables. In the classical case, there is a notion of minimum set of system state variables and the system stability may be guaranteed provided this set of state variables satisfies certain properties. However, as far as we are aware, there is no similar notion of minimal observables for the quantum system and this is an interesting topic for future research.

In the remaining part of this paper, whenever we say that the system is stable we mean that the system is stable with respect to the selected set of relevant observables.

B. Definitions

There are two notions of Lyapunov stability that we can consider.

Definition 4.1: Suppose $Q, P \subset \mathcal{B}(\mathcal{H})^n$. We say that the system is *stable with respect to* (Q, P) if $j_0(\mathbf{X}) \in Q$ implies

$$F_0 j_t(\mathbf{X}) F_0 \in P, \ \forall t < \infty.$$

This definition may be interpreted as follows: if the initial state of the system is in some set Q, then the conditional expectation of the system state is in some set P for all time. This definition is slightly different to the classical Definition 2.1, wherein, the stability of the system is defined in terms of the probability that the state of the system remains in some set P for all time. However, the proof of Lemma 2.1 uses the following argument to bound the probability that the state of the system remains in some set P for all time:

1) Prove that the expectation value of the state is in some set P'.
2) Use Chebyshev inequality:
$$\mathrm{Prob}\{X > \lambda\} \leq \frac{\mathbf{E}\{X\}}{\lambda}$$
for any positive-valued random variable, to bound the probability that the state of the system remains in some set P for all time.

Therefore, though Definition 4.1 gives a bound on the expectation value of the observables $\mathbf{X}$, in the classical case this is equivalent to giving a bound on the probability that the state of the system remains in some set P for all time.

We may use the density matrix ρ to give a bound on the probability that the measured values of the state are in some set P for all time. Suppose $(\rho, j_t(\mathbf{X}))$ is the state of the system and $\mathbf{X}$ is an $\mathbb{R}^n$ valued observable.

Definition 4.2: Suppose $Q \subset \mathcal{B}(\mathcal{H})^n$, $P \in \mathbb{R}^m$, $f : \mathcal{B}(\mathcal{H})^n \to \mathcal{B}(\mathcal{H})^m$ is some function of the states $\mathbf{X}$ of the system and $\epsilon > 0$. We say that the system is *stable with respect to*[5] (Q, P, ϵ, f) if $j_0(\mathbf{X}) \in Q$ implies

$$\mathrm{Prob}\{F_0 j_t(f(\mathbf{X}))F_0 \in P, \ \forall t < \infty\} \geq \epsilon$$

[5]c.f. Definition 2.1.

Here, Prob is defined in Equation (8)
This definition may be interpreted as follows: if the initial state of the system is in some set Q, the probability that the measured value of the observable $f(\mathbf{X})$ is in some set P is greater than ϵ.

Remark 4.2: Note that for Definition 4.2 to make sense, we need the observables $f_1(\mathbf{X}), f_2(\mathbf{X}), \ldots, f_m(\mathbf{X})$ to commute with each other. Here, $f_i(\mathbf{X})$, the i^{th} component of f is defined in the obvious way.

Example 1: Consider a quantum system with relevant states being the angular momentum operators L_i, L_j and L_k in the three orthogonal directions in $\mathbb{R}^3$. The states satisfy the commutation relations

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k,$$

where ϵ_{ijk} denotes the Levi-Civita symbol. For the Lyapunov function $f = [f_1 \ f_2]^T$, where

$$\begin{aligned} f_1(L_i, L_j, L_k) &= L_i^2 + L_j^2 + L_k^2 = \mathbf{L}^2, \\ f_2(L_i, L_j, L_k) &= L_i^2. \end{aligned}$$

the two components of f commute with each other.

C. Stability results

We collect the following assumptions together for future reference.

A1 Let $\mathcal{K}$ be a Hilbert space and suppose T_t is a stochastic operator on a C^*-algebra $\mathscr{A} \subset \mathcal{B}(\mathcal{K})$ and $(\mathcal{H}, F_t, j_t)$ is the associated minimum Markov dilation.

A2 Let $V : \mathcal{B}(\mathcal{H})^n \to \mathcal{B}(\mathcal{H})$ be of the form

$$\mathbf{X} \mapsto \sum_{i=1}^{p} Y_1 Y_2 \ldots Y_{q_i}.$$

Here p and q_i, $i = 1, \ldots, p$ are finite integers and $Y_1, \ldots, Y_{q_i} \in \{X_1, \ldots, X_n, X_1^*, \ldots X_n^*\}$, $i = 1, \ldots, p$. Suppose $V(\mathbf{X})$ is non-negative (i.e. for all $\psi \in \mathcal{H}, \mathbf{X} \in \mathcal{B}(\mathcal{H})^n$, $\langle \psi, V(\mathbf{X})\psi \rangle \geq 0$) and continuous in the set $Q_m = \{\mathbf{X} \in \mathcal{B}(\mathcal{H})^n : V(\mathbf{X}) \leq m\mathbf{1}\}$. Let $t_0 = \inf\{t : j_t(\mathbf{X}) \notin Q_m\}$ and suppose τ_m determined by $\tau_m(\{t_0\}) = \mathbf{1}$ is a stoptime on $(\mathcal{H}, F_t, j_t)$.

A3 $V(\mathbf{X})$ is in the domain of $\mathcal{L}$, the infinitesimal generator of T_t for all $\mathbf{X} \in Q_m$.

The following lemma is analogous to the classical result in Lemma 2.1.

Lemma 4.1: Suppose (A1)-(A3) are satisfied and let $j_{\tau\wedge t}(\mathcal{L}(V(X)) \leq 0$. Then, $V(j_{\tau\wedge t}(\mathbf{X}))$ is a nonnegative supermartingale[6] in the sense that $F_0 V(j_{\tau\wedge t}(\mathbf{X}))F_0 \leq V(j_{\tau\wedge 0}(\mathbf{X}))$ and for $\lambda \leq m$, initial density matrix ρ and $j_0(\mathbf{X}) \in Q_m$, we have

$$\mathrm{Prob}\{\sup_{\infty > t \geq 0} V(j_{\tau\wedge 0}(\mathbf{X})) \geq \lambda\} \leq \frac{\|V(j_{\tau\wedge 0}(\mathbf{X}))\|}{\lambda} \quad (9)$$

[6]One can think of inequality $F_0 V(j_{\tau\wedge t}(\mathbf{X}))F_0 \leq V(j_{\tau\wedge 0}(\mathbf{X}))$ as the defining inequality of a quantum supermartingale. As far as we are aware, no such definition is known for quantum supermartingales.

Proof: Firstly, note that for arbitrary stoptimes τ

$$\begin{aligned} j_\tau(X_1 X_2) &= j_\tau(X_1) j_\tau(X_2) \\ j_\tau(X_1 + X_2) &= j_\tau(X_1) + j_\tau(X_2) \end{aligned}$$

for all $X_1, X_2 \in \mathscr{A}$. The first equation above follows from the fact that j_τ is a $*$-homomorphism (see [7, p. 21]). The second equation follows directly from the relation $j_t(X) = U^* X U$, for some unitary matrix U and the definition of j_τ. Therefore, we have

$$V(j_\tau(X)) = j_\tau(V(X)).$$

From Dynkin's formula (6) we have

$$\begin{aligned} & F_0 V(j_{\tau\wedge t}(X))F_0\psi \\ =\ & F_0 j_{\tau\wedge t}(V(X))F_0\psi \\ =\ & j_0(V(X))F_0\psi \\ & + F_0 \int_0^\infty \mathbf{1}_{\tau\wedge t > s} j_s(\mathcal{L}(V(X))) ds F_0\psi. \end{aligned}$$

The result now follows from the negativity of $j_{\tau\wedge s}(\mathcal{L}(V(X))$. ■

Note that if $V \geq 0$ then we can write $V = X^*X$ for some operator X. Therefore $j_t(V) = j_t(X^*X) = j_t(X)(j_t(X))^* \geq 0$. Therefore $j_t(\mathcal{L}(V)) \leq 0$ if and only if $\mathcal{L}(V) \leq 0$.

The following theorem is the main result of our paper.

Theorem 4.2: Suppose (A1)-(A3) are satisfied for some $m > 0$ and let $j_{\tau\wedge t}(\mathcal{L}(V(X)) \leq 0$. Then the system is stable relative to (Q_r, Q_m) for any $r \leq m$ (c.f Definition 4.1).

Proof: This theorem is a direct consequence of Lemma 4.1. ■

Now suppose the system is in state $(\rho, \mathbf{X})$ and V is some Lyapunov function. Then $\langle V(j_t(X))\rangle$ is a real-valued stochastic process with probability measure $\mathbf{P}$ on some sample space Ω determined by the conditional expectation $\mathbf{E}_0\cdot$ and density matrix ρ. We make the following two additional assumptions

A2' Let V and $\mathbf{X}$ be as defined in (A2) and let $Q'_m = \{\mathbf{X} \in \mathcal{B}(\mathcal{H})^n : \langle V(\mathbf{X})\rangle \leq m\}$. Also suppose τ'_m, defined similar to τ_m with Q_m replaced with Q'_m, is a stoptime.

A4 T_t is strongly continuous. Therefore, the maps $t \mapsto j_t(X)\psi$ and $t \mapsto F_t\psi$ are continuous for all $X \in \mathcal{K}$ and $\psi \in \mathcal{H}$ [7, Proposition 5.3].

Also let $B_m = \{\omega \in \Omega : j_t(\mathbf{X}) \in Q'_m \ \forall t < \infty\}$.

Corollary 2.1: Suppose (A1),(A2') and (A3) are satisfied for some $m > 0$ and let $j_{\tau\wedge t}(\mathcal{L}(V(\mathbf{X})) \leq 0$. Then the system is stable relative to $(Q'_r, Q'_m, 1 - \frac{r}{m}, V)$ for any $r = \langle V(\mathbf{X})\rangle \leq m$. Also, $\langle V(j_{t\wedge\tau_m}(\mathbf{X}))\rangle$ is a non-negative supermartingale and for almost all $\omega \in B_m$, $\langle V(j_{t\wedge\tau_m}(\mathbf{X}))\rangle \to c(\omega) \leq m$ where c is some random variable.

Proof: From the inequality $F_0 V(j_{\tau\wedge t}(\mathbf{X}))F_0 \leq V(j_{\tau\wedge 0}(\mathbf{X}))$ we have

$$\mathbf{E}\{\langle V(j_{\tau\wedge t}(\mathbf{X}))\rangle\} \leq \langle V(j_{\tau\wedge 0}(\mathbf{X}))\rangle.$$

Also, because T_t is strongly continuous by (A4) we have $\mathbf{E}\{\langle V(j_{\tau\wedge t}(\mathbf{X}))\rangle\} \to \langle V(j_{\tau\wedge 0}(\mathbf{X}))\rangle$ as $t \to 0$. These two facts and the non-negativity of V implies the fact that $\langle V(j_{\tau\wedge t}(\mathbf{X}))\rangle$ is a non-negative supermartingale [3, Theorem 12.6]. The existence of the random variable c follows from the supermartingale convergence theorem. The stability with respect to the triple $(Q'_r, Q'_m, 1-\frac{r}{m})$ follows from Equation (9). ■

D. Example

We consider a two level atom with two input field channels (see example 4.4 in [12]). The generator of the plant is

$$\mathbf{G}_p = (1, \sqrt{\gamma_1}\sigma_-, \frac{1}{2}\omega\sigma_z) \boxplus (1, \sqrt{\gamma_2}\sigma_-, 0).$$

Here σ_x, σ_y and σ_z are the Pauli matrices (see e.g. [13]) and $\sigma_\pm = \sigma_x \pm i\sigma_y$. Consider the Lyapunov function $V = \sigma_1 = \frac{1}{2}(I + \sigma_z) \geq 0$. With vacuum inputs, the mean excited state energy of the two-level atom, $\langle\sigma_1\rangle$ goes to zero due to decoherence modeled by the vacuum noise. Gough and James [12] suggest using a simple plant controller network

$$\mathbf{G}_p \wedge \mathbf{G}_c = (1, \sqrt{\gamma_1}\sigma_-, \frac{1}{2}\omega\sigma_z) \triangleleft \mathbf{G}_c \triangleleft (1, \sqrt{\gamma_2}\sigma_-, 0)$$

with controller $\mathbf{G}_c = (-1, 0, 0)$ to minimize the decoherence.

A simple calculation shows that the generator of V is

$$\mathcal{L}(V) = -(\gamma - 2\sqrt{\gamma_1\gamma_2})V$$

Here $\gamma = \gamma_1 + \gamma_2$. Therefore $\mathcal{L}(V) \leq 0$. From Theorem 4.2 we see that the system is stable with respect to $(Q_r, Q_m, 1-\frac{r}{m}, V)$. That is, if the operator σ_1 is less than $r\mathbf{1}$ for some r then is less than $m\mathbf{1}$ for all time with probability $1-\frac{r}{m}$. Also, if the initial mean energy of the excited state is less than r initially, then it is less than m for all time with probability $1-\frac{r}{m}$.

V. Conclusion

In this paper we generalised Lyapunov stability theory to quantum Markov dilations. Open quantum systems can be described using one parameter semigroups of completely positive operators and an associated minimal Markov dilation. One can define Lyapunov functions for such Markov process to be positive operators that depend on the system observables. The expectation value of a Lyapunov function can be expressed in terms of the integral of its Lindbald generator using a quantum version of Dynkin's formula. The negativity of the generator ensures stability in the sense of Definition 4.1 (c.f. [2]). Several Lyapunov type stability results such as asymptotic stability can be generalised to quantum systems and is a subject of future research.

Appendix

In this appendix, we show how one can evaluate the minimal Markov dilation given a family of stochastic operators $\{T_t\}$. Let $\mathscr{D}$ denote the set of all ordered triples $(\mathbf{t}, \mathbf{X}, u)$, where $\mathbf{t} = \{t_1 > t_2 > \ldots, t_n\}$, $\mathbf{X} = X_1, \ldots X_n$, $X_i \in \mathscr{A}_{t_i}$, $u \in \mathcal{H}_0$ and $n = 1, 2, \ldots$. We can define a positive definite kernel on $\mathscr{D}$ as follows [6, p. 97].

$$L((\mathbf{t},\mathbf{X},u),(\mathbf{t},\mathbf{Y},v)) = \langle u, T(0,t_n)(X_n^* \ldots \\ \ldots \{T(t_3,t_2)(X_2^*\{T(t_2,t_1)(X_1^*Y_1)\}Y_2)\} \ldots Y_n)v\rangle$$

For arbitrary $(\mathbf{s}, \mathbf{X}, u)$ and $(\mathbf{t}, \mathbf{Y}, v)$, let $\mathbf{r} = \mathbf{s} \cup \mathbf{t}$ be ordered as a decreasing sequence and let

$$\tilde{X}_j = \begin{cases} X_k & \text{if } r_j = s_k \text{ for some k,} \\ \mathbf{1} & \text{otherwise.} \end{cases}$$
$$\tilde{Y}_j = \begin{cases} Y_k & \text{if } r_j = t_k \text{ for some k,} \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Now define

$$L((\mathbf{s},\mathbf{X},u)(\mathbf{t},\mathbf{Y},v)) = L((\mathbf{r},\tilde{\mathbf{X}},u)(\mathbf{r},\tilde{\mathbf{Y}},v)).$$

Because, L is a positive definite kernel on $\mathscr{D}$, by the GNS principle[7] there exists a Gelfand pair $(\mathcal{H}, \lambda)$ such that

1) $\mathcal{H}$ is a Hilbert space and $\lambda : \mathscr{D} \to \mathcal{H}$.
2) $\langle\lambda((\mathbf{s},\mathbf{X},u)), \lambda((\mathbf{t},\mathbf{Y},v))\rangle = L((\mathbf{s},\mathbf{X},u),(\mathbf{s},\mathbf{X},u))$.
3) $\{\lambda((\mathbf{s},\mathbf{X},u)) : (\mathbf{s},\mathbf{X},u) \in \mathscr{D}\}$ spans $\mathcal{H}$.

Let $\mathcal{H}_t$ denote the closed linear span of $\{\lambda((\mathbf{s},\mathbf{X},u)) : (\mathbf{s},\mathbf{X},u) \in \mathscr{D}$ and $s_1 \leq t\}$ and let F_t be the projection from $\mathcal{H}$ onto its subspace $\mathcal{H}_t$. Define the representation $j_t^0 : \mathscr{A}_t \to \mathcal{B}(\mathcal{H}_t)$ such that

$$j_t^0(X)\lambda((t,s_1,\ldots,s_n),(Y_0,Y_1,\ldots,Y_n),u) = \lambda((t,s_1,\ldots,s_n),(XY_0,Y_1,\ldots,Y_n),u)$$

and let $j_t : \mathscr{A}_t \to \mathcal{B}(\mathcal{H})$ be defined as

$$j_t(X) = j_t^0(X)F_t. \tag{10}$$

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[7] See e.g. [8, ch. 15]