2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Reach Control on Simplices by Continuous State Feedback

Mireille E. Broucke

Abstract— **This paper studies a theoretical problem of whether continuous state feedback and affine feedback are equivalent from the point of view of making an affine system defined on a simplex reach a prespecified facet in finite time. We show that the two classes of feedbacks are equivalent. As a byproduct, new necessary and sufficient conditions for solvability based more directly on the problem data are obtained.**

I. Introduction

This paper studies a theoretical problem of whether continuous state feedback and affine feedback are equivalent from the point of view of making an affine system defined on a simplex reach a prespecified facet in finite time. In general, such problems have been overlooked in the literature on reachability problems via feedback control. This contrasts with the situation for stabilization, where it has long been known that linear state feedback is the largest class of feedbacks needed to stabilize a linear system.

The problem studied is for an affine system to reach a prespecified facet of a simplex in finite time and is taken from [8], [13]. Facet reachability problems on simplices and polytopes, with minor variations in assumptions, were first introduced in [6] and further studied in [7], [10], [11]. In [8], [13], two sets of conditions called *invariance conditions* and a *flow condition* were given as necessary and sufficient conditions for existence of an affine feedback to solve the problem of reaching a facet of a simplex in finite time. The invariance conditions can be shown to be necessary for continuous state feedback [7]. The necessity of the flow condition is tied to its direct link to existence of closed-loop equilibria, assuming the closed-loop vector field is convex. Once one relaxes the class of controls to continuous state feedback, convexity is lost, and the necessity of the flow condition becomes problematic to establish.

Therefore, it is the flow condition which is the focus of attention. A key observation is that the flow condition can be bypassed if we triangulate the polytopic state space in a manner adapted to the system dynamics. Since typically triangulations are performed by standalone software libraries that are not taylored to control problems, our requirement for a proper triangulation is no loss of generality.

Our results have implications for the study of piecewise linear and piecewise affine feedbacks to solve more general reachability problems on polytopes and unions of polytopes. See, for example, [5], [1], [4], [15].

Supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

Dept. of Electrical and Computer Engineering, University of Toronto, Toronto ON Canada M5S 3G4 broucke@control.utoronto.ca

Notation 1: For a vector $x \in \mathbb{R}^n$, the notation $x \succ 0$ ($x \succeq 0$) means $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notation $x \prec 0$ ($x \preceq 0$) means $-x \succ 0$ ($-x \succeq 0$). For a matrix $A \in \mathbb{R}^{n\times n}$, the notation $A \succ 0$ ($A \succeq 0$) means $a_{ij} > 0$ ($a_{ij} \geq 0$) for $1 \leq i, j \leq n$.

II. Problem Statement and Background

Consider an n-dimensional simplex $\mathcal{S}$ with vertices $v_0, v_1, \ldots, v_n$ and facets $\mathcal{F}_0, \ldots, \mathcal{F}_n$ such that the index of each facet is determined by the vertex it does not contain. Let h_i, $i = 0, \ldots, n$ be the unit normal vector to each facet $\mathcal{F}_i$ pointing outside of the simplex. Let $\mathcal{F}_0$ be the target set in $\mathcal{S}$.

We consider the following affine control system on $\mathcal{S}$:

$$\dot{x} = Ax + a + Bu =: f(x, u), \quad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n\times m}$ with rank$(B) = m$. Let $\phi_u(t, x_0)$ be the trajectory of (1) under a control u starting from $x_0 \in \mathcal{S}$ and evaluated at time t.

We are interested in studying reachability of the target $\mathcal{F}_0$ from $\mathcal{S}$ by way of feedback control. A number of results on finding feedbacks to solve reachability specifications on simplices have already appeared in the literature. In particular, the following problem was proposed in [8], [13].

Problem 1: Consider system (1) defined on $\mathcal{S}$. Find an affine feedback control $u = Kx + g$ such that for every $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\epsilon > 0$ satisfying:

(i) $\phi_u(t, x_0) \in \mathcal{S}$ for all $t \in [0, T]$;
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$;
(iii) $\phi_u(t, x_0) \notin \mathcal{S}$ for all $t \in (T, T + \epsilon)$.

Condition (iii) is interpreted to mean that the closed-loop dynamics on $\mathcal{S}$ are extended to a neighborhood of $\mathcal{S}$. In this paper, we extend Problem 1 to continuous state feedback. This is termed the *reach control problem*.

The following notation will be used. Define the set of vertices of $\mathcal{S}$ to be V. Define the index sets $I := \{1, \ldots, n\}$ and $I_i := I \setminus \{i\}$. Define the closed, convex cones

$$\begin{aligned} \mathcal{C}_i &:= \left\{ y \in \mathbb{R}^n \ : \ h_j \cdot y \leq 0, \ j \in I_i \right\}. \\ \text{cone}(\mathcal{S}) &:= \mathcal{C}_0 = \text{cone}\{v_1 - v_0, \ldots, v_n - v_0\}. \end{aligned}$$

Definition 1: A point $x_0 \in \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ by continuous state feedback, denoted $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a continuous state feedback $u(x)$ such that properties (i)-(iii) of Problem 1 hold. A set $\mathcal{S}' \subseteq \mathcal{S}$ can reach $\mathcal{F}_0$ with constraint in $\mathcal{S}$ by continuous state feedback, denoted by $\mathcal{S}' \xrightarrow{\mathcal{S}} \mathcal{F}_0$, if there exists a continuous state feedback such that for every $x_0 \in \mathcal{S}'$, $x_0 \xrightarrow{\mathcal{S}} \mathcal{F}_0$.

Let $\mathcal{B}$ denote the m-dimensional subspace spanned by the column vectors of B (namely, $\mathcal{B} = \text{Im}(B)$, the image of B). Define the set

$$\mathcal{O} := \{ \; x \in \mathbb{R}^n \; : \; Ax + a \in \mathcal{B} \; \} \, .$$

It is fairly easy to prove that $\mathcal{O} = \emptyset$ when $\text{Im}(A) \subseteq \mathcal{B}$ and $a \notin \mathcal{B}$; $\mathcal{O} = \mathbb{R}^n$ when $\text{Im}(A) \subseteq \mathcal{B}$ and $a \in \mathcal{B}$; and $\mathcal{O}$ is an affine space, otherwise. Notice that vector field $f(x, u)$ can vanish on $\mathcal{O}$ for an appropriate choice of u, so $\mathcal{O}$ is the set of all possible equilibrium points of the system. Define

$$\mathcal{G} := \mathcal{S} \cap \mathcal{O}.$$

Associated with $\mathcal{G}$ is its vertex index set $I_{\mathcal{G}} := \{i : v_i \in V \cap \mathcal{G}\}$.

Definition 2: The *invariance conditions* require that there exist $u_0, \ldots, u_n \in \mathbb{R}^m$ such that:

$$h_j \cdot (Av_i + a + Bu_i) \leq 0 \, , \quad i \in \{0, \ldots, n\}, \quad j \in I_i \, . \quad (2)$$

For Problem 1 the following necessary and sufficient conditions have been established.

Theorem 1: [8], [13] We have $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback if and only if there exists an affine feedback $u(x) = Kx + g$ with $u_1 = u(v_1), \ldots, u_n = u(v_n)$, such that: (a) The invariance conditions hold; (b) The closed-loop system has no equilibrium in $\mathcal{S}$.

A more computational set of necessary and sufficient conditions are the following.

Theorem 2: [8], [13] We have $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback if and only if there exists an affine feedback $u(x) = Kx + g$, with $u_1 = u(v_1), \ldots, u_n = u(v_n)$, and a vector $\xi \in \mathbb{R}^n$ such that

(a) The invariance conditions hold.
(b) The flow condition holds:
$\xi \cdot (Av_i + a + Bu_i) < 0 \, , \qquad i \in \{0, \ldots, n\} \, .$

The invariance conditions (2) are suitable for affine feedback, but for continuous state feedback, the following stronger conditions must hold.

Definition 3: The *invariance conditions* for state feedback $u(x)$ require that for all $j \in I$ and $x \in \mathcal{F}_j$,

$$h_j \cdot (Ax + Bu(x) + a) \leq 0 \, . \quad (3)$$

The following result is easily proved (see the analogous result in [7] for conditions (2)) and forms the starting point for our investigation of continuous state feedback.

Lemma 3: Solvability of the invariance conditions (3) is necessary to solve the reach control problem $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by continuous state feedback.

III. Existence of Linear Affine Feedback

As we have seen in Theorem 2, the invariance conditions by themselves are generally not enough to establish that the reach control problem is solvable by affine feedback. However, there is one extreme case when the invariance conditions are also sufficient to solve the problem. These depend on combining Theorem 1 with the fact that $\mathcal{O}$ is the only place in the state space where equilibria can appear. See also [13].

Theorem 4: Suppose $\mathcal{G} = \emptyset$. If the invariance conditions are solvable, then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

In general it is difficult to extend results such as Theorem 4. However, if one propitiously chooses a triangulation of the state space which respects the underlying structure of the system, then new necessary and sufficient conditions for solvability of the reach control problem are obtainable and, moreover, the boundary between affine and continuous state feedback can be clarified. We propose the following triangulation.

Assumption 1: Simplex $\mathcal{S}$ and system (1) satisfy the following condition: if $\mathcal{G} \neq \emptyset$, then $\mathcal{G}$ is a κ-dimensional face of $\mathcal{S}$, where $0 \leq \kappa \leq n$.

Remark 1: We have discussed that there are three possibilities for $\mathcal{O}$. If $\mathcal{O} = \emptyset$, then one applies Theorem 4. If $\mathcal{O}$ is the entire state space then we will see in Remark 3 that there are easily derived necessary and sufficient conditions for solvability. The only interesting case is when $\mathcal{O}$ is a κ-dimensional affine subspace with $\kappa < n$. This case arises, for example, when (A, B) is controllable.

Based on the proposed triangulation, we can find several new sufficient conditions for existence of affine feedback. First we require a preliminary lemma which provides a sufficient condition for existence of a flow condition on a polytope.

Lemma 5: Let $\mathcal{P}$ be a polytope. If $\mathcal{O} \cap \mathcal{P} = \emptyset$, then there exists $\beta \in \text{Ker}(B^T)$ such that $\beta^T(Ax + a) < 0$, $\forall x \in \mathcal{P}$.

Theorem 6: Suppose Assumption 1 holds and $\mathcal{G} \neq \emptyset$. Suppose the following conditions hold.

1) The invariance conditions are solvable.
2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq 0$.

Then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

Proof: Let $\mathcal{G} = \text{conv}\{v_{i_1}, \ldots, v_{i_{\kappa+1}}\}$, a κ-dimensional facet of $\mathcal{S}$ where $0 \leq \kappa \leq n$. Thus, $I_{\mathcal{G}} = \{i_1, \ldots, i_{\kappa+1}\}$. Let $b \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, $b \neq 0$, and select control values u_i such that $y(v_i) = Av_i + Bu_i + a = b$ for all $i \in I_{\mathcal{G}}$ (notice this is always achievable for $v_i \in \mathcal{O}$). Clearly, by the assumption that $b \in \text{cone}(\mathcal{S})$, $y(v_i)$ satisfies the invariance conditions for $v_i \in V \cap \mathcal{O}$. We can select the remaining controls u_i for $i \in \{0, \ldots, n\} \setminus I_{\mathcal{G}}$ such that $y(v_i) \neq 0$ (since $v_i \notin \mathcal{O}$) and $y(v_i)$ satisfies the invariance conditions. Finally, using $\{u_0, \ldots, u_n\}$ and the synthesis procedure in [7], construct the affine feedback $u(x) = Kx + g$.

Now let us show that a flow condition holds in $\mathcal{S}$. First, a flow condition trivially holds for the closed loop vector field $y(x) := (A + BK)x + Bg + a$ on $\mathcal{G}$. Let $\beta_1 := -b$. We have $\beta_1^T y(v_i) = -\|b\|^2 < 0$ for all $i \in I_{\mathcal{G}}$. By the convexity of $y(x)$, this implies a flow condition holds on $\mathcal{G}$. Now we claim that a flow condition holds on all of $\mathcal{S}$. Let $\mathcal{P} := \text{conv}\{v_i \mid i \in \{0, \ldots, n\} \setminus I_{\mathcal{G}}\}$. Note that $\mathcal{P} \cap \mathcal{O} = \emptyset$, so according to Lemma 5, there exists $\beta_2 \in \text{Ker}(B^T)$ such that for all $x \in \mathcal{P}$, $\beta_2^T(Ax + a) < 0$. Define $\beta = \alpha\beta_1 + (1-\alpha)\beta_2$ for some $\alpha \in (0, 1)$. Now consider $v_i \in V \cap \mathcal{O}$. Using the fact that $\beta_2^T b = 0$, we have $\beta^T y(v_i) = \beta^T b = -\alpha\|b\|^2 < 0$. Next consider $v_i \in V \setminus \mathcal{O}$. We have $\beta^T(Av_i + Bu_i + a) = \alpha\beta_1^T(Av_i + Bu_i + a) + (1-\alpha)\beta_2^T(Av_i + a)$. The term $\beta_1^T(Av_i + Bu_i + a)$ is a constant of unknown sign, whereas we know $\beta_2^T(Av_i + a) < 0$. Therefore it is possible to select α sufficiently small so that $\beta^T(Av_i + Bu_i + a) < 0$ for all $v_i \in V \setminus \mathcal{O}$. We have shown that for all $v_i \in V$, $\beta^T(Av_i + Bu_i + a) < 0$, so by convexity of the vector field $y(x)$, a flow condition holds on all of $\mathcal{S}$. Therefore, by Theorem 2 with $\xi = \beta$, the control $u(x) = Kx + g$ solves $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. ∎

One can also obtain sufficient conditions for existence of affine feedback even when $\mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$. Of course, this will only be possible if $v_0 \notin \mathcal{G}$ (see Remark 3). This relies on the idea that there are enough degrees of freedom in $\mathcal{B}$ with respect to $\mathcal{G}$. We make the following assumptions.

Assumption 2:

(A1) W.l.o.g. $\mathcal{G} = \text{conv}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \le \kappa < m$.

(A2) $\mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$.

(A3) There exists a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$.

The important new assumption is (A3) which says that $\mathcal{B}$ and $\mathcal{G}$ are arranged with respect to each other so that there are enough degrees of freedom in $\mathcal{B}$ both to span a $\kappa+1$-dimensional subspace of $\mathcal{B}$ and at the same time satisfy all the invariance conditions for the vertices of $\mathcal{G}$. For this to work, it is of course necessary that $\kappa < m$. We now show that under Assumption 2, the linearly independent vectors $\{b_1, \ldots, b_{\kappa+1}\}$ can always be modified to obtain a new set $\{y_1, \ldots, y_{\kappa+1} \mid y_i \in \mathcal{B} \cap \mathcal{C}_i\}$ which permits a flow condition on $\mathcal{G}$. To do so, we introduce the following family of matrices. Let $1 \le p \le q \le \kappa + 1$ and define

$$M_{p,q} := \begin{bmatrix} (h_p \cdot b_p) & (h_p \cdot b_{p+1}) & \cdots & (h_p \cdot b_q) \\ \vdots & \vdots & & \vdots \\ (h_q \cdot b_p) & (h_q \cdot b_{p+1}) & \cdots & (h_q \cdot b_q) \end{bmatrix}.$$

Define the matrices

$$H_{p,q} := [h_p \cdots h_q], \quad Y_{p,q} := [b_p \cdots b_q].$$

Then $M_{p,q} = H_{p,q}^T Y_{p,q}$. We say a matrix M is a $\mathscr{Z}$-matrix if the off-diagonal elements are non-positive; i.e. $m_{ij} \le 0$ for all $i \ne j$ [2]. Since $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$, each $M_{p,q}$ is a $\mathscr{Z}$-matrix. Also under the condition that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$, $M_{p,q}$ adopts further algebraic properties. In particular, we require the notion of an $\mathscr{M}$-matrix. The following theorem characterizes non-singular $\mathscr{M}$-matrices (see [2], Ch. 6).

Theorem 7: Let $M \in \mathbb{R}^{k \times k}$ be a $\mathscr{Z}$-matrix. Then the following are equivalent:

(i) M is a non-singular $\mathscr{M}$-matrix.

(ii) $\Re(\lambda) > 0$ for all eigenvalues λ of M.

(iii) There exists a vector $\xi \succeq 0$ in $\mathbb{R}^k$ such that $M\xi \succ 0$.

(iv) The inequalities $y \succeq 0$ and $My \preceq 0$ have only the trivial solution $y = 0$, and M is non-singular.

(v) M is monotone; that is, $My \succeq 0$ implies $y \succeq 0$ for all $y \in \mathbb{R}^k$.

(vi) M is nonsingular and M^{-1} is a non-negative matrix.

Lemma 8: Suppose $\mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$. Let $1 \le p \le q \le \kappa + 1$ and suppose $\{b_p, \ldots, b_q \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$ are linearly independent. Then $M_{p,q}$ is a non-singular $\mathscr{M}$-matrix.

Proof: First we note that since $\text{rank}(H_{p,q}) = q - p + 1$ and by assumption $\text{rank}(Y_{p,q}) = q - p + 1$, we have that $M_{p,q}$ is non-singular. Next, we claim that $M_{p,q}$ has a positive diagonal; that is, $(M_{p,q})_{ii} > 0$ for $i = 1, \ldots, q - p + 1$. For if not, we would have $h_j \cdot b_{p+i-1} \le 0$ for all $j = 1, \ldots, n$, which implies $0 \ne b_{p+i-1} \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction. Now suppose there exists $c \in \mathbb{R}^{q-p+1}$ with $c \ne 0$ and $c \succeq 0$ such that $M_{p,q}c \preceq 0$. Define the vector $\bar{y} = Y_{p,q}c \in \mathcal{B}$. Note that $\bar{y} \ne 0$ because $\{b_p, \ldots, b_q\}$ are linearly independent. Then $M_{p,q}c = H_{p,q}^T Y_{p,q} c = H_{p,q}^T \bar{y} \preceq 0$ implies $h_j \cdot \bar{y} \le 0$ for $j = p, \ldots, q$. Also, $h_j \cdot \bar{y} = \sum_{i=p}^q c_i (h_j \cdot b_i) \le 0$ for $j \notin \{p, \ldots, q\}$. This implies $0 \ne \bar{y} \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction. Therefore, $M_{p,q}$ has the property that the only solution of the inequalities $c \succeq 0$ and $M_{p,q}c \preceq 0$ is $c = 0$. By Theorem 7 this implies that $M_{p,q}$ is a non-singular $\mathscr{M}$-matrix. ∎

We will construct a set $\{y_1, \ldots, y_{\kappa+1} \mid y_i \in \mathcal{B} \cap \mathcal{C}_i\}$ which permits a flow condition on $\mathcal{G}$ by an inductive procedure. The following lemma establishes the initial step of the induction.

Lemma 9: Suppose Assumption 2 holds. Then w.l.o.g. (by reordering the indices $1, \ldots, \kappa + 1$ and the indices $\kappa + 2, \ldots, n$), $h_{\kappa+2} \cdot b_1 < 0$.

Proof: Suppose not. That is,

$$h_j \cdot b_i = 0, \quad i = 1, \ldots, \kappa + 1, \quad j = \kappa + 2, \ldots, n. \quad (4)$$

By Assumption 2 and Lemma 8, $M_{1,\kappa+1}$ is a non-singular $\mathscr{M}$-matrix, so by Theorem 7(iii), there exists $c \preceq 0$, $c \ne 0$, such that $M_{1,\kappa+1}c =: d \preceq 0$. Let $\bar{y} := Y_{1,\kappa+1}c$. Note $\bar{y} \ne 0$ since $\{b_1, \ldots, b_{\kappa+1}\}$ are linearly independent. Now $\bar{y}$ satisfies $H_{1,\kappa+1}^T \bar{y} = H_{1,\kappa+1}^T Y_{1,\kappa+1} c = M_{1,\kappa+1} c \preceq 0$. Combining with (4) we have $h_j \cdot \bar{y} \le 0$ for $j = 1, \ldots, \kappa + 1$ and $h_j \cdot \bar{y} \le 0$ for $j = \kappa + 2, \ldots, n$. Therefore, $0 \ne \bar{y} \in \mathcal{B} \cap \text{cone}(\mathcal{S})$, a contradiction. ∎

The following proposition shows how one can modify the linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$ to obtain a new set of velocity vectors satisfying both the invariance conditions and also a flow condition on $\mathcal{G}$.

Proposition 10: Suppose Assumption 2 holds. Then there exists an assignment $\{y_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$ and a vector $\beta_1 \in \mathcal{B}$ such that $\beta_1 \cdot y_i < 0$ for all $i \in I_{\mathcal{G}}$.

Proof: In the first step of the proof, we will construct an assignment $\{y_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$ such that for each $i \in I_{\mathcal{G}}$,

$$(\exists p_i \in \{\kappa + 2, \ldots, n\}) \qquad h_{p_i} \cdot y_i < 0. \quad (5)$$

In the second step of the proof, we will show that the sets $\text{conv}\{y_1, \ldots, y_{\kappa+1}\}$ and $\{0\}$ are strongly separated, and this will lead to the desired result.

For the first step, the proof is by induction on an index $l = 0, \ldots, \kappa$. Following Assumption 2, let $\{b_1, \ldots, b_{\kappa+1}\}$ be a linearly independent set satisfying $b_i \in \mathcal{B} \cap \mathcal{C}_i$, $i \in I_{\mathcal{G}}$. Set $l := 0$ and $y_1 := b_1$. Assuming indices have been ordered according to Lemma 9, we have that $\{y_1\}$ satisfies the property (5). Now suppose there exists $\{y_1, \ldots, y_{l+1}\}$ satisfying (5) and there remain $\{b_{l+2}, \ldots, b_{\kappa+1}\}$ which have not been modified. If w.l.o.g. (by reordering indices $l + 2, \ldots, \kappa + 1$) there exists b_{l+2} satisfying property (5), then set $y_{l+2} = b_{l+2}$

and the induction step is done. Suppose instead that no such b_i exists. That is,

$$h_j \cdot b_i = 0\,, \qquad i = l{+}2, \ldots, \kappa{+}1\,, \quad j = \kappa{+}2, \ldots, n\,. \quad (6)$$

Now we claim that w.l.o.g. (by reordering indices $l + 2, \ldots, \kappa + 1$) there exist b_{l+2} and $q \in \{1, \ldots, l+1\}$ such that $h_q \cdot b_{l+2} < 0$.

For suppose not. Then since $b_i \in \mathcal{B} \cap \mathcal{C}_i$,

$$h_j \cdot b_i = 0\,, \qquad i = l{+}2, \ldots, \kappa{+}1\,, \quad j = 1, \ldots, l{+}1\,. \quad (7)$$

Now consider $M_{l+2,\kappa+1}$ formed using the linearly independent vectors $\{b_{l+2}, \ldots, b_{\kappa+1}\}$. By Assumption 2 and Lemma 8, it is a non-singular $\mathscr{M}$-matrix. By Theorem 7(iii), there exists $c \preceq 0$, $c \neq 0$ such that $M_{l+2,\kappa+1} c \prec 0$. Let $\overline{y} := Y_{l+2,\kappa+1} c$. Note $\overline{y} \neq 0$ since $\{b_{l+2}, \ldots, b_{\kappa+1}\}$ are linearly independent. Now $\overline{y}$ satisfies $H^T_{l+2,\kappa+1} \overline{y} = H^T_{l+2,\kappa+1} Y_{l+2,\kappa+1} c = M_{l+2,\kappa+1} c \prec 0$. Combining with (6)-(7) we have $h_j \cdot \overline{y} \leq 0$ for $j = l+2, \ldots, \kappa+1$ and $h_j \cdot \overline{y} = 0$ for $j = 1, \ldots, l+1, \kappa+2, \ldots, n$. Therefore, $0 \neq \overline{y} \in \mathcal{B} \cap \mathrm{cone}(\mathcal{S})$, a contradiction.

Consequently we know that w.l.o.g. (by reordering indices $\{1, \ldots, l+1\}$) for b_{l+2} there exists $q \in \{1, \ldots, l+1\}$ such that $h_q \cdot b_{l+2} < 0$. Now define $y_{l+2} := \alpha y_q + b_{l+2} \in \mathcal{B}$. Even though $h_q \cdot y_q > 0$, we have $h_q \cdot b_{l+2} < 0$, so $\alpha > 0$ can be selected sufficiently small so that $h_q \cdot (\alpha y_q + b_{l+2}) \leq 0$. Also, we know that $h_j \cdot (\alpha y_q + b_{l+2}) \leq 0$ for $j \in I \setminus \{l{+}2, q\}$ since $h_j \cdot y_q \leq 0$ and $h_j \cdot b_{l+2} \leq 0$ for all $j \in I \setminus \{l+2, q\}$. Therefore, $y_{l+2} \neq 0$ satisfies the invariance conditions at v_{l+2}. Also, by assumption of the induction step, there exists $p_q \in \{\kappa{+}2, \ldots, n\}$ such that $h_{p_q} \cdot y_q < 0$. Since $h_{p_q} \cdot b_{l+2} \leq 0$, we obtain $h_{p_q} \cdot (\alpha y_q + b_{l+2}) < 0$. Therefore y_{l+2} satisfies property (5) with $p_{l+2} = p_q$. This completes the induction step.

Now we consider the second step of the proof. Let $\{y_1, \ldots, y_{\kappa+1}\}$ be the (not necessarily linearly independent) assignment of feasible velocity vectors for v_i, $i \in I_{\mathcal{G}}$, constructed in the first step and satisfying property (5). Consider the set $\mathcal{C} := \mathrm{conv}\{y_1, \ldots, y_{\kappa+1}\} \subset \mathcal{B}$. We observe that $0 \notin \mathcal{C}$ because no convex combination of y_i's can sum to zero by property (5). (For consider $\overline{y} = \sum_i c_i y_i$, $c_i \geq 0$, and $\sum_i c_i = 1$. Suppose w.l.o.g. that $c_1 > 0$. By assumption $\exists\, p_1 \in \{\kappa{+}2, \ldots, n\}$ such that $h_{p_1} \cdot y_1 < 0$. Also, $h_{p_1} \cdot y_i \leq 0$ for all $i = 2, \ldots, \kappa+1$, so $h_{p_1} \cdot \overline{y} < 0$ which implies $\overline{y} \neq 0$.) Now applying the Separating Hyperplane Theorem ([12],p.98), there exists a hyperplane $\mathcal{H}$ separating $\mathcal{C}$ and $\{0\}$ strongly in $\mathcal{B}$. That is, there exists $\beta_1 \in \mathcal{B}$ such that for all $y \in \mathcal{C}$, $\beta_1^T y < 0$. ■

The proof of the following theorem is analogous to that of Theorem 6.

Theorem 11: Suppose Assumption 1 holds and $\mathcal{G} = \mathrm{conv}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < m$. Suppose the following conditions hold.

1) The invariance conditions are solvable.
2) There exists a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$.

Then $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.

Remark 2: An interesting aspect of Theorems 6 and 11 is that one is able to show existence of a flow condition on $\mathcal{S}$ without explicitly computing controls for the vertices. In this manner the problem of finding controls to satisfy the invariance conditions and that of satisfying a flow condition are decoupled. This property is achieved due to the method of triangulation of the state space relative to $\mathcal{O}$.

IV. Existence of Equilibria

In this section we explore cases when equilibria appear on $\mathcal{G}$ when an assignment of a continuous state feedback $y(x)$ is made on $\mathcal{S}$, so that the reach control problem is not solvable by continuous state feedback. Particular attention is given to the case when $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = 0$. Let $u(x)$ be a continuous state feedback defined on $\mathcal{S}$. We restrict our attention to such controls which yield unique solutions on $\mathcal{S}$ and which satisfy the invariance conditions (3) on $\mathcal{S}$. Define the closed-loop system

$$\dot{x} = Ax + Bu(x) + a =: y(x)\,. \quad (8)$$

First we consider an obvious necessary condition for the problem to be solvable, which is that one must be able to assign $y(v_i) \neq 0$ at each vertex $v_i \in \mathcal{G}$.

Proposition 12: Suppose Assumption 1 holds and let $u(x)$ be a continuous state feedback such that the closed-loop system has unique solutions and the invariance conditions hold. If at some $i \in I_{\mathcal{G}}$, $\mathcal{B} \cap \mathcal{C}_i = 0$, then the closed-loop system $\dot{x} = Ax + Bu(x) + a$ has an equilibrium point at $v_i \in \mathcal{G}$.

Remark 3: When $v_0 \in \mathcal{G}$, then Proposition 12 immediately implies that a necessary condition for existence of a continuous state feedback is that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq 0$.

From Proposition 12 a necessary condition for a solution is that there exists a set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid b_i \neq 0, i \in I_{\mathcal{G}}\}$. In the special case of $v_0 \in \mathcal{G}$ this completely settles the question of necessary conditions since in that case we require that $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq 0$. More generally, if $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) \neq 0$, the question is settled because of Theorem 6. Therefore, other necessary conditions for a solution are studied in this section under the following assumptions.

Assumption 3:

(E1) W.l.o.g. $\mathcal{G} = \mathrm{conv}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < n$.
(E2) $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = 0$.
(E3) The maximum number of linearly independent vectors in any set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$ is $m^\star$ with $1 \leq m^\star \leq \kappa$.

Asssumption (E3) says there does not exist a full linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}$ as in Assumption 2. This automatically holds true when $\kappa = m$, in which case (E3) could simply be removed. We remark that $m^\star$ is well-defined (for $\dim(\mathrm{sp}\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_{\mathcal{G}}\}) \in \{0, \ldots, \kappa+1\}$ defines a finite set of integers for which the maximum always exists).

Given $1 \leq m^\star \leq \kappa$ as above, w.l.o.g. let

$$\{b_1, \ldots, b_{m^\star} \mid b_i \in \mathcal{B} \cap \mathcal{C}_i\}$$

be one such maximal linearly independent set. By construction, every $b_j \in \mathcal{B} \cap \mathcal{C}_j$ for $j = m^\star + 1, \ldots, \kappa + 1$

satisfies $b_j \in \mathrm{sp}\{b_1, \ldots, b_{m^\star}\}$. Indeed for each $j \in \{m^\star + 1, \ldots, \kappa + 1\}$ there exists $1 \leq \kappa_j \leq m^\star$ such that w.l.o.g. (reordering indices $1, \ldots, m^\star$), $\mathcal{B} \cap \mathcal{C}_j \subset \mathrm{sp}\{b_1, \ldots, b_{\kappa_j}\}$, and $\mathrm{sp}\{b_1, \ldots, b_{\kappa_j}\}$ is the smallest such subspace in $\mathcal{B}$. Therefore we can say $\kappa_j = \dim(\mathcal{B} \cap \mathcal{C}_j)$. Now consider $\mathcal{B} \cap \mathcal{C}_{m^\star+1}$. Following the arguments above, let $\kappa^\star := \dim(\mathcal{B} \cap \mathcal{C}_{m^\star+1})$ and w.l.o.g. (reordering indices $1, \ldots, m^\star$) assume $\mathcal{B} \cap \mathcal{C}_{m^\star+1} \subset \mathrm{sp}\{b_1, \ldots, b_{\kappa^\star}\}$. If $\kappa^\star < m^\star$, swap the indices $m^\star + 1 \iff \kappa^\star + 1$. (The index swap is to make incrementing of indices easier below). Finally select any vectors $\beta_i \in \mathcal{B}$, $i = \kappa^\star + 1, \ldots, m$ such that

$$\mathcal{B} = \mathrm{sp}\{b_1, \ldots, b_{\kappa^\star}, \beta_{\kappa^\star+1}, \ldots, \beta_m\} . \tag{9}$$

With our reordering of indices we have that for all $b_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$, $b_{\kappa^\star+1} = c_1 b_1 + \cdots + c_{\kappa^\star} b_{\kappa^\star}$. Also define

$$\mathcal{G}^\star := \mathrm{conv}\{v_1, \ldots, v_{\kappa^\star+1}\} .$$

The following results will show that there exists an equilibrium in $\mathcal{G}^\star$ for any closed-loop vector field $y(x)$ satisfying the invariance conditions on $\mathcal{S}$. We begin by isolating the defect in available degrees of freedom in $\mathcal{B}$ with respect to $\mathcal{G}^\star$.

Proposition 13: Suppose Assumptions 1 and 3 hold. Suppose that the closed-loop system $\dot{x} = y(x)$ satisfies the invariance conditions. Then for all $x \in \mathcal{G}^\star$,

$$h_j \cdot y(x) = 0 , \qquad j = \kappa^\star + 2, \ldots, n .$$

Proof: W.l.og. let a basis of $\mathcal{B}$ be as in (9) and select $b_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$ such that

$$b_{\kappa^\star+1} = c_1 b_1 + \cdots + c_{\kappa^\star} b_{\kappa^\star} , \qquad c_i \neq 0 .$$

(Such a vector exists by the definition of $\kappa^\star$ and convexity of $\mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$.) Define $c := (c_1, \ldots, c_{\kappa^\star})$. Since $\{b_1, \ldots, b_{\kappa^\star}\}$ are linearly independent and $\mathcal{B} \cap \mathrm{cone}(\mathcal{S}) = 0$, by Lemma 8, $M_{1,\kappa^\star}$ is a non-singular $\mathscr{M}$ matrix. Consider the following invariance conditions

$$H^T_{1,\kappa^\star} b_{\kappa^\star+1} = H^T_{1,\kappa^\star} Y_{1,\kappa^\star} c = M_{1,\kappa^\star} c \preceq 0 .$$

By Theorem 7(v) and the fact that $c_i \neq 0$, we obtain $c \prec 0$. Now consider the invariance conditions

$$h_j \cdot b_{\kappa^\star+1} = h_j \cdot \left(\sum_{i=1}^{\kappa^\star} c_i b_i \right) \leq 0 , \quad j = \kappa^\star + 2, \ldots, n .$$

Every term in the sum is non-negative, since $b_i \in \mathcal{B} \cap \mathcal{C}_i$ and $c_i < 0$, and so we obtain

$$h_j \cdot b_i = 0 , \quad i = 1, \ldots, \kappa^\star + 1, \quad j = \kappa^\star + 2, \ldots, n . \tag{10}$$

Now by Theorem 7(iii) there exists $c' = (c'_1, \ldots, c'_{\kappa^\star})$ such that $c' \preceq 0$ and $M_{1,\kappa^\star} c' \prec 0$. Define $b'_{\kappa^\star+1} := Y_{1,\kappa^\star} c'$. The vector $H^T_{1,n} b'_{\kappa^\star+1} \in \mathbb{R}^n$ has the following sign pattern:

$$(-, \ldots, -, *, 0, \ldots, 0) \tag{11}$$

where the $*$ appears in the $(\kappa^\star+1)$th component. In particular $b'_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$ and the first $\kappa^\star$ invariance conditions are strictly negative. Now suppose we find a non-zero vector $\beta \in \mathrm{sp}\{\beta_{\kappa^\star+1}, \ldots, \beta_m\}$ such that

$$h_j \cdot \beta \leq 0 , \qquad j = \kappa^\star + 2, \ldots, n . \tag{12}$$

Then for $\alpha > 0$ we can form $b''_{\kappa^\star+1} := b'_{\kappa^\star+1} + \alpha\beta$. Using (11) and (12), α can be selected sufficiently small so that $h_j \cdot b''_{\kappa^\star+1} \leq 0$ for all $j = 1, \ldots, \kappa^\star, \kappa^\star + 2, \ldots, n$. That is, $b''_{\kappa^\star+1} \in \mathcal{B} \cap \mathcal{C}_{\kappa^\star+1}$. Moreover, with $\beta \neq 0$, $\{b_1, \ldots, b_{\kappa^\star}, b''_{\kappa^\star+1}\}$ is a linearly independent set. This contradicts that $\mathcal{B} \cap \mathcal{C}_{\kappa^\star+1} \subset \mathrm{sp}\{b_1, \ldots, b_{\kappa^\star}\}$. The conclusion is that there does not exist $\beta \in \mathrm{sp}\{\beta_{\kappa^\star+1}, \ldots, \beta_m\}$, $\beta \neq 0$, satisfying (12).

Now let $y(x)$ be any continuous closed-loop vector field on $\mathcal{S}$ satisfying the invariance conditions (3). Using (9), for $x \in \mathcal{G}$, let

$$y(x) = c_1(x) b_1 + \cdots + c_{\kappa^\star}(x) b_{\kappa^\star} + \beta(x) , \tag{13}$$

where $\beta(x) \in \mathrm{sp}\{\beta_{\kappa^\star+1}, \ldots, \beta_m\}$. From (3) we know that for each $x \in \mathcal{G}^\star$, $h_j \cdot y(x) \leq 0$, for $j = \kappa^\star + 2, \ldots, n$. Using (10) and (13) these conditions become

$$h_j \cdot \beta(x) \leq 0 , \qquad j = \kappa^\star + 2, \ldots, n ,$$

but we have just shown that no such non-zero β exists, so it must be that $\beta(x) = 0$. Therefore for each $x \in \mathcal{G}^\star$, $h_j \cdot y(x) = 0$ for $j = \kappa^\star + 2, \ldots, n$, as desired. ■

Remark 4: Proposition 13 has the following intuitive meaning. For simplicity suppose $v_0 = 0$. We know from the geometry of the simplex that the state space can be decomposed as follows:

$$\mathbb{R}^n = \mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\} \oplus \mathrm{sp}\{h_{\kappa^\star+2}, \ldots, h_n\} . \tag{14}$$

Therefore, Proposition 13 says that

$$\mathrm{sp}\{b_1, \ldots, b_{\kappa^\star}\} \subset \mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\} .$$

Moreover, for all $x \in \mathcal{G}^\star$,

$$y(x) \in \mathrm{sp}\{b_1, \ldots, b_{\kappa^\star}\} .$$

Geometrically, $\mathcal{G}^\star$ lies in $\mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\}$, a $\kappa^\star + 1$ dimensional affine space in $\mathbb{R}^n$, and $\mathcal{B}$ provides to $\mathcal{G}^\star$ only $\kappa^\star$ usable directions (which also lie in $\mathrm{aff}\{v_0, \ldots, v_{\kappa^\star+1}\}$) to resolve all its invariance conditions.

Proposition 13 captures the fundamental geometric structure of the problem which forces the existence of an equilibrium. The proof that an equilibrium exists can now be executed in a number of different ways, including index theory and the Brouwer Fixed Point Theorem. A particularly efficient proof can be obtained based on Sperner's Lemma [14].

Let $\mathbb{T}$ be a triangulation of n-dimensional simplex $\mathcal{S}$. A *proper labeling* of the vertices of $\mathbb{T}$ is as follows: (P1) vertices of the original simplex $\mathcal{S}$ have $n + 1$ distinct labels. (P2) Vertices of $\mathbb{T}$ on a face of $\mathcal{S}$ are labeled using only the labels of the vertices forming the face. Given a properly labeled triangulation of $\mathcal{S}$, we say a simplex in $\mathbb{T}$ is *distinguished* if its vertices have all $n + 1$ labels. Sperner's lemma says that every properly labeled triangulation of $\mathcal{S}$ has an odd number of distinguished simplices.

Theorem 14: Suppose Assumptions 1 and 3 hold. Let $u(x)$ be a continuous state feedback such that the closed-loop system $\dot{x} = Ax + Bu(x) + a = y(x)$ has unique solutions and the invariance conditions (3) hold. Then the closed-loop system has an equilibrium point in $\mathcal{G}$.

Proof: By assumption $\mathcal{G} = \text{conv}\{v_1, \ldots, v_{\kappa+1}\}$. If $\kappa > m$, redefine $\mathcal{G}$ as $\mathcal{G} = \text{conv}\{v_1, \ldots, v_{m+1}\}$. Define the simplex $\mathcal{G}^\star$ using the construction above and let $I^\star := \{1, \ldots, \kappa^\star + 1\}$. Now we show how to obtain a proper labeling of $\mathcal{G}^\star$. We begin by defining the sets:

$$\mathcal{Q}_i^\star := \{x \in \mathcal{G}^\star \mid h_i \cdot y(x) > 0\}, \qquad i \in I^\star.$$

Observe that $v_i \in \mathcal{Q}_i^\star$ and $v_i \notin \mathcal{Q}_j^\star$, $i, j \in I^\star$, $i \neq j$ (for otherwise, we would have $y(v_i) \in \mathcal{B} \cap \text{cone}(\mathcal{S})$ which either contradicts that $\mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$ or implies $y(v_i)$ is an equilibrium). Therefore, we either immediately conclude there is an equilibrium on a vertex of $\mathcal{G}^\star$ or we conclude that inclusion in a set $\mathcal{Q}_i^\star$ provides a distinct label for the vertices $v_i \in \mathcal{G}^\star$. This satisfies (P1) of a proper labeling of $\mathcal{G}^\star$. Next, let $\mathbb{T}$ be any triangulation of $\mathcal{G}^\star$ and consider a vertex v of $\mathbb{T}$ which is not a vertex of $\mathcal{G}^\star$ and lies in $\partial\mathcal{G}^\star$. W.l.o.g. let $v \in \text{conv}\{v_1, \ldots, v_{l+1}\}$ for some $1 \leq l < \kappa^\star$. Then it must be that $v \in \mathcal{Q}_k^\star$ for some $1 \leq k \leq l+1$ (by the same reasoning that otherwise $y(v) \in \mathcal{B} \cap \text{cone}(\mathcal{S})$). Clearly this labeling of v satisfies the second condition (P2) for a proper labeling. Finally, for vertices v of $\mathbb{T}$ in the interior of $\mathcal{G}^\star$, any label $\mathcal{Q}_i^\star$ such that $h_i \cdot y(v) > 0$ can be used (at least one such exists because if all $h_i \cdot y(v) \leq 0$, $i \in I^\star$, it implies $h_i \cdot y(v) \leq 0$ for all $i = 1, \ldots, n$ or $y(v) \in \mathcal{B} \cap \text{cone}(\mathcal{S})$).

Now for each $k > 0$, $k \in \mathbb{Z}$, define a triangulation $\mathbb{T}^k$ of $\mathcal{G}^\star$ such that each simplex of $\mathbb{T}^k$ has diameter $\frac{1}{k}$. Apply Sperner's lemma for each $\mathbb{T}^k$ to obtain a distinguished simplex $\text{conv}\{v_1^k, \ldots, v_{\kappa^\star+1}^k\}$ and its baricenter x^k. $\{x^k\}$ defines a bounded sequence in $\mathcal{G}^\star$ which has a convergent subsequence, again denoted $\{x^k\}$. We have $\lim_{k\to\infty} x^k = \overline{x} \in \mathcal{G}^\star$, since $\mathcal{G}^\star$ is closed. Also, by construction $v_i^k \to \overline{x}$, $i \in I^\star$. By Sperner's lemma we know that $h_i \cdot y(v_i^k) > 0$, $i \in I^\star$, so by continuity of $y(x)$ this implies $h_i \cdot y(\overline{x}) \geq 0$, $i \in I^\star$. Combined with Proposition 13, we obtain that $-y(\overline{x}) \in \mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$, which implies $\overline{x} \in \mathcal{G}^\star$ is an equilibrium of the closed-loop system $\dot{x} = y(x)$. ■

V. Existence of Continuous State Feedback

In this section we collect the previous results to resolve the boundary between continuous state feedback and affine feedback.

Theorem 15: Suppose Assumption 1 holds. Then the following statements are equivalent:

1) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback.
2) $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by continuous state feedback.

Proof: (1) $\Longrightarrow$ (2) is obvious.
(2) $\Longrightarrow$ (1) Suppose there exists a continuous state feedback $u(x)$ such that the closed loop system (8) has a unique solution for each initial condition in $\mathcal{S}$ and Problem 1 is solved using $u(x)$. By Lemma 3 the invariance conditions (2) must be solvable. Suppose $\mathcal{G} = \emptyset$. Then by Theorem 4, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Suppose $\mathcal{G} \neq \emptyset$. Also, suppose $\mathcal{B} \cap \text{cone}(\mathcal{S}) \neq 0$. Then by Theorem 6, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. Instead suppose $\mathcal{G} \neq \emptyset$ and $\mathcal{B} \cap \text{cone}(\mathcal{S}) = 0$. Suppose $v_0 \in \mathcal{G}$. Then by Proposition 12, the closed-loop system has an equilibrium point $x_0 \in \mathcal{S}$, a contradiction. Instead suppose $v_0 \notin \mathcal{G}$ and w.l.o.g. $\mathcal{G} = \text{conv}\{v_1, \ldots, v_{\kappa+1}\}$, with $0 \leq \kappa < n$. Suppose there does not exist a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_\mathcal{G}\}$. Then by Theorem 14 the closed-loop system has an equilibrium point $x_0 \in \mathcal{S}$, a contradiction. Instead suppose there does exist a linearly independent set $\{b_i \in \mathcal{B} \cap \mathcal{C}_i \mid i \in I_\mathcal{G}\}$. Then by Theorem 11, $\mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0$ by affine feedback. ■

References

[1] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*. vol. 35, pp. 407–428, March 1999.

[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York. 1979.

[3] F. Blanchini, A. Megretski. Robust state feedback control of LTV systems: nonlinear is better than linear. *IEEE Trans Automatic Control*. vol 44, issue 4, April 1999, pp. 802–807.

[4] F. Blanchini and F.A. Pellegrino. Relatively optimal control: A static piecewise affine solution. *SIAM Journal on Control and Optimization* vol. 46, issue 2, p585–603, 2007.

[5] P.J. Collins, L.C.G.J.M. Habets, A. Kuut, M. Nool, M. Petreczky, and J.H. van Schuppen. ConPAHS - A software package for control of piecewise-affine hybrid systems. *Proc. 2006 IEEE Conference on Computer Aided Control System Design* (CACSD '06). (Munich, Germany, October 4-6, 2006), IEEE Press, New York, pp. 76–81, 2006.

[6] L.C.G.J.M. Habets and J.H. van Schuppen. Control of piecewise-linear hybrid systems on simplices and rectangles, in: M.D. Di Benedetto and A.L. Sangiovanni-Vincentelli (Eds.) *Hybrid Systems: Computation and Control, Lecture Notes in Computer Science*. Springer Verlag, vol. 2034, pp. 261–274, 2001.

[7] L.C.G.J.M. Habets and J.H. van Schuppen. A control problem for affine dynamical systems on a full-dimensional polytope. *Automatica*. no. 40, pp. 21–35, 2004.

[8] L.C.G.J.M. Habets, P.J. Collins, and J.H. van Schuppen. Reachability and control synthesis for piecewise-affine hybrid systems on simplices. *IEEE Trans. Automatic Control* no. 51, pp. 938–948, 2006.

[9] C. W. Lee. Subdivisions and triangulations of polytopes. *Handbook of Discrete and Computational Geometry*. CRC Press Series Discrete Math. Appl., pp. 271–290, 1997.

[10] Z. Lin and M.E. Broucke. Resolving control to facet problems for affine hypersurface systems on simplices. *IEEE Conference on Decision and Control* (CDC '06). December 2006.

[11] Z. Lin and M.E. Broucke. Reachability and control of affine hypersurface systems on polytopes. *IEEE Conference on Decision and Control* (CDC '07). December 2007.

[12] R.T Rockafellar. *Convex Analysis*. Princeton University,Princeton, New Jersey 1970.

[13] B. Roszak and M. E. Broucke. Necessary and sufficient conditions for reachability on a simplex. *Automatica*. vol. 42, no. 11, pp. 1913–1918, November 2006.

[14] Y.A. Shashkin. *Fixed Points*. American Mathematical Society, Providence, R.I., 1991.

[15] M. L. Tyler and M. Morari. Propositional logic in control and monitoring problems. Automatica, vol. 335, no. 4, pp. 565–582, 1999.

[16] G. M. Zeigler. *Lectures on Polytopes*. Graduate Texts in Mathematics. Springer, 1995.