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Sensitivity Analysis for Control Parameter Determination for a Nonlinear Cable-Mass System

Katie A. Evans

Abstract— **The MinMax controller results from a differential game approach to solving the H_∞ control problem. As such, the MinMax controller involves a design parameter, which gives a measure of robustness for the controller. There exists no explicit formula for determining the design parameter, and the optimal value must be determined experimentally. Instead of choosing the parameter value experimentally and suffering the computational expense, it would be more efficient if the design parameter could be determined by a prescribed formula based on a mathematically rigorous criterion. In this paper, the author employs continuous sensitivity equation methods to examine the sensitivity of the controlled state with respect to variation of the MinMax control parameter, with the goal being to explore the possibility of determining an efficient assignment of the parameter that is mathematically justified. Numerical simulations are performed on a one-dimensional nonlinear cable-mass system, and the results are presented.**

I. INTRODUCTION

Since its introduction by Zames [1], the H_∞ controller has received much research attention because of its robustness to disturbances and uncertainties. Rhee and Speyer later introduced what has become known as the MinMax controller, which is a differential game approach to solving the H_∞ control problem [2]. As with the standard H_∞ control problem, the mathematical formulation of the MinMax controller involves a design parameter, which gives a measure of robustness for the controller. There is no explicit formula for determining the design parameter, and the optimal value - optimal in the sense of controller robustness - must be determined experimentally, thus adding computational expense to the design of the MinMax controller. Presently, a costly iterative procedure is used to choose the value for θ. Still yet, Chen identifies the problem of finding an optimal value for the control design as an unsolved problem in systems and control theory [3]. Instead of choosing the parameter value experimentally and suffering the computational expense, it would be more efficient if the design parameter could be determined by a prescribed formula based on a mathematically rigorous criterion.

In this paper, the author employs continuous sensitivity equation methods to mathematically examine the sensitivity of the controlled state and the controller itself with respect to variation of the MinMax control parameter, with the goal being to explore the possibility of determining an efficient assignment of the parameter that is mathematically justified. Numerical simulations are performed on a one-dimensional nonlinear cable-mass system [4], [5], [6].

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K. Evans is with the Faculty of Mathematics and Statistics, Louisiana Tech University, P.O. Box 10348, Ruston, LA 71272, USA kevans@latech.edu

The outline of the paper is as follows. The MinMax controller is summarized in Section II. Section III provides a description of the equations governing the cable-mass PDE, along with the variational forms of the PDE equations and state sensitivity. Numerical results are presented in Section IV. Conclusions and directions for future work are given in Section V.

II. MINMAX CONTROL DESIGN

In this section, the author presents a short overview of the MinMax compensator design [2]. Assume the existence of a nonlinear PDE system of the form

$$\begin{aligned} \dot{x}(t) &= \mathscr{A}_0 x(t) + \mathscr{N}(x(t)) + \mathscr{B}u(t) + \mathscr{D}\eta(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t) = x(t,\cdot) \in X$ is the state of the nonlinear system and X is a Hilbert space. Here, $\mathscr{A}_0$ is the system operator defined on $\mathbf{D}(\mathscr{A}_0) \subseteq X$ that, by assumption, generates an exponentially stable C_0 semigroup, $\mathscr{N}$ defined on X is the nonlinearity in the system, $\mathscr{B}$ is the control operator, $\mathscr{D}$ is the disturbance operator, $u(t)$ is the control input, and $\eta(t)$ is the disturbance, with the latter two functions defined on Hilbert space U. It is assumed that knowledge of only part of the system can be obtained through the state measurement y on Hilbert space Y where

$$y(t) = \mathscr{C}x(t). \tag{2}$$

Assume an estimate of the state is used in the control law. To provide this estimate, a compensator is used that has the form

$$\dot{x}_c(t) = \mathscr{A}_c x_c(t) + \mathscr{F}y(t), \quad x_c(0) = x_{c_0} \tag{3}$$

and the feedback control law is written

$$u(t) = -\mathscr{K}x_c(t) \tag{4}$$

where $x_c(t) = x_c(t,\cdot) \in X$ is the state estimate. Designing a controller of this type requires determining $\mathscr{A}_c, \mathscr{F}$, and $\mathscr{K}$.

The MinMax compensator is defined for linear systems, so one must first linearize the system in (1), (2). Doing so yields the linear distributed parameter control system (with state x_ℓ) defined on X

$$\dot{x}_\ell(t) = \mathscr{A}x_\ell(t) + \mathscr{B}u(t) + \mathscr{D}\eta(t), \quad x_\ell(0) = x_{\ell_0} \tag{5}$$

with sensed output

$$y(t) = \mathscr{C}x_\ell(t). \tag{6}$$

By solving the Riccati equations

$$\mathscr{A}^*\Pi+\Pi\mathscr{A}-\Pi(\mathscr{B}R^{-1}\mathscr{B}^*-\theta^2\mathscr{B}\mathscr{B}^*)\Pi+\mathscr{C}^*\mathscr{C}=0, \quad (7)$$

where $R:U\to U$ is a weighting operator for the control of the form $R=cI$, with c a scalar and I the identity operator, and

$$\mathscr{A}P+P\mathscr{A}^*-P(\mathscr{C}^*\mathscr{C}-\theta^2\mathscr{C}^*\mathscr{C})P+\mathscr{B}\mathscr{B}^*=0, \quad (8)$$

one can obtain the operators $\mathscr{K}$, $\mathscr{F}$, and $\mathscr{A}_c$ via

$$\begin{aligned}\mathscr{K} &= R^{-1}\mathscr{B}^*\Pi,\\ \mathscr{F} &= (I-\theta^2P\Pi)^{-1}P\mathscr{C}^*,\\ \mathscr{A}_c &= \mathscr{A}-\mathscr{B}\mathscr{K}-\mathscr{F}\mathscr{C}+\theta^2\mathscr{B}\mathscr{B}^*\Pi. \end{aligned} \quad (9)$$

The resulting feedback control is applied to the original nonlinear system; the closed loop nonlinear system is then defined by

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}x(t)\\ x_c(t)\end{bmatrix} &= \begin{bmatrix}\mathscr{A} & -\mathscr{B}\mathscr{K}\\ \mathscr{F}\mathscr{C} & \mathscr{A}_c\end{bmatrix}\begin{bmatrix}x(t)\\ x_c(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}\mathscr{N}(x(t))\\ \mathscr{N}(x_c(t))\end{bmatrix}+\begin{bmatrix}\mathscr{D}\\ 0\end{bmatrix}\eta(t).\end{aligned} \quad (10)$$

For sufficiently small θ, there are guaranteed minimal solutions Π and P to (7) and (8), respectively, such that $(I-\theta^2P\Pi)$ is positive definite and the linearized closed loop system, i.e. the linearized form of (10), is stable. Note that $\theta=0$ yields the classical Linear Quadratic Gaussian (LQG) compensator design. Since there exist no prescribed formulas for θ, there is an inherent computational expense for this control design in choosing the parameter value. The author seeks to use sensitivity analysis to gain a better understanding of the MinMax controller. The goal is to develop a methodology for choosing θ to satisfy performance and robustness criteria, while justifying that choice based on the analysis. To this end, sensitivity analysis is applied to MinMax controlled distributed parameter systems to examine the sensitivity of the controlled state to θ.

III. A STRUCTURAL VIBRATION PROBLEM

The PDE system of interest in this work is the cable mass distributed parameter system described in [4] and also studied numerically in [5] and [6]. In particular, a wave equation with Kelvin-Voigt damping models the elastic cable, which is fixed at one end and attached to a mass at the other end, for $0<s<\ell$. A Duffing's type equation models the oscillator, which is forced by a sinusoidal disturbance, at $s=\ell$. The equations governing this system are as follows:

$$\rho\frac{\partial^2}{\partial t^2}w(t,s)=\frac{\partial}{\partial s}\left[\tau\frac{\partial}{\partial s}w(t,s)+\gamma\frac{\partial^2}{\partial t\partial s}w(t,s)\right], \quad (11)$$

for $0<s<\ell,\ t>0$, and

$$\begin{aligned}m\frac{\partial^2}{\partial t^2}w(t,\ell) &= -\left[\tau\frac{\partial}{\partial s}w(t,\ell)+\gamma\frac{\partial^2}{\partial t\partial s}w(t,\ell)\right]\\ &\quad-\alpha_1 w(t,\ell)-\alpha_3[w(t,\ell)]^3+\eta(t)+u(t),\end{aligned} \quad (12)$$

with boundary condition

$$w(t,0)=0, \quad (13)$$

where $w(t,s)$ represents the displacement of the cable at time t and position s, $w(t,\ell)$ gives the position of the mass at time t, ρ and m are the densities of the cable and mass respectively, τ is tension in the cable, and γ is the coefficient of the damping term. The spring's stiffening terms have coefficients of α_1 and α_3, with α_3 being associated with the nonlinear effects in the spring. A disturbance enters through $\eta(t)$, and $u(t)$ is a control input. This view of the cable mass system can be seen in Figure 1.

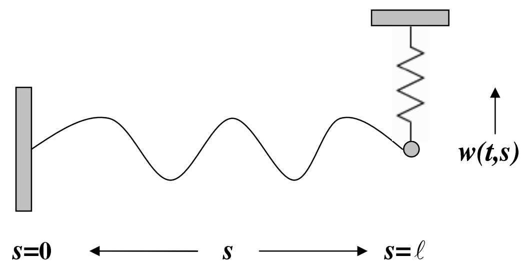

Fig. 1. Cable-mass system.

Sensed information is used to design a feedback controller that attenuates the disturbance $\eta(t)$. It is assumed that the control acts exclusively on the mass, and the only available measured information is the position and velocity of the mass. These two observations take the form

$$y_1(t)=w(t,\ell), \quad y_2(t)=\frac{\partial}{\partial t}w(t,\ell). \quad (14)$$

Identifying operators from (5), (6), $\mathscr{A}$ contains system dynamics, while the control input, disturbance input, and state measurement operators are bounded and defined by

$$\mathscr{B}=\mathscr{D}=\left[0,0,0,\frac{1}{m}\right]^T \text{ and } \mathscr{C}=\begin{bmatrix}0&1&0&0\\0&0&0&1\end{bmatrix}. \quad (15)$$

A. Variational Form and Discretization of Cable Mass System

Now consider the variational form of the cable mass system in order to develop a Galerkin finite element approximation of the problem. One wants to find a $w(s)\in V=\{\varphi=[\varphi_1(\cdot),\varphi_2]^T\in E:\varphi_1(\ell)=\varphi_2\}\subset E=H^1_L(0,\ell)\times\mathbb{R}$ such that for all $\varphi\in V$

$$\begin{aligned}&\int_0^\ell \rho\ddot{w}(t,s)\varphi_1(s)\ ds-\int_0^\ell \tau w''(t,s)\varphi_1(s)\ ds-\\ &\int_0^\ell \gamma\dot{w}''(t,s)\varphi_1(s)\ ds+m\ddot{w}(t,\ell)\varphi_2+\tau w'(t,\ell)\varphi_2+\\ &\gamma\dot{w}'(t,\ell)\varphi_2+\alpha_1 w(t,\ell)\varphi_2+\alpha_3[w(t,\ell)]^3\varphi_2\\ &=\eta(t)\varphi_2+u(t)\varphi_2.\end{aligned} \quad (16)$$

Now choose a basis $\{e_i\}_{i=1}^N$ for the approximating space $V^N\subseteq V$, where N corresponds to the number of gridpoints used in the finite element approximation. In particular, since $V^N \subseteq V \subset E = H_L^1(0,\ell) \times \mathbb{R}$, the state can be approximated by a linear combination of linear B-splines, satisfying $b_i(0) = 0$, of the form

$$e_i = [b_i(s) \quad b_i(\ell)]^T \text{ for } i = 1, \ldots, N. \tag{17}$$

Then the state is approximated as

$$\begin{aligned} \begin{bmatrix} w(t,s) \\ w(t,\ell) \end{bmatrix} &\approx \begin{bmatrix} w^N(t,s) \\ w^N(t,\ell) \end{bmatrix} \\ &= \sum_{i=1}^{N} c_i(t) e_i(s) \\ &= \begin{bmatrix} \sum_{i=1}^{N} c_i(t) b_i(s) \\ c_N(t) b_i(\ell) \end{bmatrix}. \end{aligned} \tag{18}$$

Using the state approximation (18) in (16), we find that (16) can be rewritten as a matrix equation

$$\begin{aligned} M_0 \ddot{c}(t) + D_0 \dot{c}(t) + K_0 c(t) + F_0(c(t)) = \\ B_0 u(t) + G_0 \eta(t), \end{aligned} \tag{19}$$

$$c(0) = c_0, \qquad \dot{c}(0) = c_1,$$

where $c(t) = [c_1(t), \ldots, c_N(t)]^T$, M_0 is the mass matrix, D_0 is the damping matrix, K_0 is the stiffness matrix, $F_0(c(t))$ contains the nonlinear terms, B_0 is the input matrix, G_0 is the disturbance matrix, all defined by the following, for $i, j = 1, \ldots, N$:

$$\begin{aligned} [M_0]_{i,j} &= \int_0^\ell \rho b_i(s) b_j(s)\ ds + m b_i(\ell) b_j(\ell) \\ [D_0]_{i,j} &= \int_0^\ell \gamma b_i'(s) b_j'(s)\ ds \\ [K_0]_{i,j} &= \int_0^\ell \tau b_i'(s) b_j'(s)\ ds + \alpha_1 b_i(\ell) b_j(\ell) \\ F_0(c(t)) &= \alpha_3 [w_N]^3 \\ B_0 = G_0 &= [b_1(\ell), \ldots, b_N(\ell)]^T. \end{aligned} \tag{20}$$

Convert (19) into a first order system by defining $x_1(t) = c(t)$ and $x_2(t) = \dot{x}_1(t) = \dot{c}(t)$, thereby yielding

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -M_0^{-1} K_0 & -M_0^{-1} D_0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ M_0^{-1} B_0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ M_0^{-1} G_0 \end{bmatrix} \eta(t) - \\ &\begin{bmatrix} 0 \\ -M_0^{-1} F_0(w(t)) \end{bmatrix}, \end{aligned} \tag{21}$$

$$x(0) = x_0,$$

where $x = [x_1(t), x_2(t)]^T = \left[x_1(t), \frac{d}{dt} x_1(t) \right]^T$. Note that (21) is a finite-dimensional approximation of the system in (1).

B. Variational Form and Discretization of Sensitivity Equation for Cable Mass System

This framework now provides the basis for implementing control techniques discussed in Section II. Beyond control design, the author is interested in examining the effects of the MinMax control parameter, θ, on the displacement of the cable, displacement of the mass, and the controller itself. The dependence of these quantities on θ is denoted explicitly with the following notation: $w(t,s) = w(t,s;\theta)$, $w(t,\ell) = w(t,\ell;\theta)$, and $u(t) = u(t;\theta)$, respectively. A continuous sensitivity equation method is employed for examining the sensitivities of these quantities to changes in the value of θ used in the MinMax control design. Make the following definitions for the sensitivities: $s_w(t,s;\theta) = \frac{\partial}{\partial \theta} w(t,s;\theta)$ for the sensitivity of cable displacement with respect to θ at time t and spatial location s, $s_w(t,\ell;\theta) = \frac{\partial}{\partial \theta} w(t,\ell;\theta)$ for the sensitivity of mass displacement with respect to θ at time t, and $s_u(t;\theta) = \frac{\partial}{\partial \theta} u(t;\theta)$ for the sensitivity of the controller with respect to θ at time t.

Now derive the variational form of the sensitivity equation by adding (11) and (12) and differentiating with respect to θ. One seeks a $s_w(s;\theta) \in V = \{\varphi = [\varphi_1(\cdot), \varphi_2]^T \in E : \varphi_1(\ell) = \varphi_2\} \subset E = H_L^1(0,\ell) \times \mathbb{R}$ such that for all $\varphi \in V$

$$\begin{aligned} \int_0^\ell \rho \ddot{s}_w(t,s;\theta) \varphi_1(s)\ ds - \int_0^\ell \tau s_w''(t,s;\theta) \varphi_1(s)\ ds \\ - \int_0^\ell \gamma \dot{s}_w''(t,s;\theta) \varphi_1(s)\ ds + m \ddot{s}_w(t,\ell,\theta) \varphi_2 + \\ \tau s_w'(t,\ell;\theta) \varphi_2 + \gamma \dot{s}_w'(t,\ell;\theta) \varphi_2 + \\ \alpha_1 s_w(t,\ell;\theta) \varphi_2 + 3\alpha_3 [w(t,\ell;\theta)]^2 s_w(t,\ell;\theta) \varphi_2 \\ = s_u(t;\theta) \varphi_2. \end{aligned} \tag{22}$$

We choose the same basis $\{e_i\}_{i=1}^N$ for the approximating space $V^N \subseteq V$ as was used in the state approximation. Then the state sensitivity is approximated as

$$\begin{aligned} \begin{bmatrix} s_w(t,s;\theta) \\ s_w(t,\ell;\theta) \end{bmatrix} &\approx \begin{bmatrix} s_w^N(t,s;\theta) \\ s_w^N(t,\ell;\theta) \end{bmatrix} \\ &= \sum_{i=1}^{N} s_{ci}(t) e_i(s) \\ &= \begin{bmatrix} \sum_{i=1}^{N} s_{ci}(t) b_i(s) \\ s_{cN}(t) b_i(\ell) \end{bmatrix}, \end{aligned} \tag{23}$$

and a finite dimensional approximation of (22) can be rewritten as a matrix equation

$$M_0\ddot{s}_c(t)+D_0\dot{s}_c(t)+K_0 s_c(t)+F_1(c(t),s_c(t)) = B_0 s_u(t;\theta), \quad (24)$$

$$s_c(0)=s_{c0}, \qquad \dot{s}_c(0)=s_{c1},$$

where $s_c(t)=[s_{c1}(t),\ldots,s_{cN}(t)]^T$, M_0, D_0, K_0, and B_0 are defined in (20) and

$$F_1(c(t),s_c(t))=3\alpha_3[w_N]^2(s_w)_N. \quad (25)$$

Convert (24) into a first order system by defining $s_{x1}(t)=s_c(t)$ and $s_{x2}(t)=\dot{s}_{x1}(t)=\dot{s}_c(t)$, thereby yielding

$$\begin{bmatrix}\dot{s}_{x1}(t)\\ \dot{s}_{x2}(t)\end{bmatrix}=\begin{bmatrix}0 & I\\ -M_0^{-1}K_0 & -M_0^{-1}D_0\end{bmatrix}\begin{bmatrix}s_{x1}(t)\\ s_{x2}(t)\end{bmatrix} + \begin{bmatrix}0\\ M_0^{-1}B_0\end{bmatrix}s_u(t)+\begin{bmatrix}0\\ -M_0^{-1}F_1(w(t),s_w(t))\end{bmatrix}, \quad (26)$$

$$s(x0)=s_{x0},$$

where $s_x=[s_{x1}(t),s_{x2}(t)]^T=\left[s_{x1}(t),\frac{d}{dt}s_{x1}(t)\right]^T$. Combining (21) and (26) yields the coupled system

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{s}_{x1}(t)\\ \dot{s}_{x2}(t)\end{bmatrix}=\begin{bmatrix}0 & I & 0 & 0\\ H_1 & H_2 & 0 & 0\\ 0 & 0 & 0 & I\\ 0 & 0 & H_1 & H_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ s_{x1}(t)\\ s_{x2}(t)\end{bmatrix} + \begin{bmatrix}0\\ H_3\ u(t)\\ 0\\ H_3\ s_u(t)\end{bmatrix}+\begin{bmatrix}0\\ H_4\ \eta(t)\ +\ H_5\\ 0\\ H_6\end{bmatrix}, \quad (27)$$

where I is the identity operator and

$$\begin{aligned} H_1 &= -M_0^{-1}K_0\\ H_2 &= -M_0^{-1}D_0\\ H_3 &= M_0^{-1}B_0\\ H_4 &= M_0^{-1}G_0\\ H_5 &= -M_0^{-1}F_0(w(t))\\ H_6 &= -M_0^{-1}F_1(w(t),s_w(t)). \end{aligned} \quad (28)$$

Now, (27) is a finite-dimensional approximation to a system similar to the form of (1), where the additional terms appear due to the coupled sensitivity equation. One can replace the control $u(t)$ in (27) by the full state feedback control law

$$u(t,\theta)=-\mathscr{K}x(t,\theta)=-\mathscr{K}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}. \quad (29)$$

Furthermore, one can differentiate (29) with respect to θ to compute $s_u(t,\theta)$ as follows

$$\begin{aligned} s_u(t,\theta) &= \frac{d}{d\theta}u(t,\theta)\\ &= -R^{-1}\mathscr{B}^*\Pi\frac{dx(t,\theta)}{d\theta}-R^{-1}\mathscr{B}^*\frac{d\Pi}{d\theta}x(t,\theta) \quad (30)\\ &= -\mathscr{K}s_w(t,\theta)-R^{-1}\mathscr{B}^*\frac{d\Pi}{d\theta}x(t,\theta), \end{aligned}$$

where the sensitivity of Π with respect to θ, $\frac{d\Pi}{d\theta}$, is computed by differentiating (7) with respect to θ and solving a resulting Lyapunov equation [7], [8].

IV. NUMERICAL RESULTS

To obtain a solution to the system in (27), initial conditions are chosen of the form

$$\begin{bmatrix}x_1(0)\\ x_2(0)\\ s_{x1}(0)\\ s_{x2}(0)\end{bmatrix}=\begin{bmatrix}s\\ -2\\ 0.75*s\\ 0.75*-2\end{bmatrix}. \quad (31)$$

That is, to generate a nonzero state sensitivity, the author chooses the initial conditions for the sensitivity equation to be 15% of the initial conditions for the state equation. A finite element approximation of order $N=80$ for the spatial discretization is employed to simulate (27), and the parameter values for the cable-mass distributed system are provided in the following table. For this discretization

TABLE I
SYSTEM PARAMETERS

ρ	τ	γ	m	ℓ	α_1	α_3
1	1	.005	1.5	2	.01	3

and set of parameter values, it was found that the largest possible MinMax controller parameter θ that will guarantee $(I-\theta^2 P\Pi)$ being positive definite is 0.45. Therefore, all MinMax controllers implemented in this paper use $\theta=0.45$. Still, the reader is reminded of the interest in examining the sensitivity of the state with respect to θ variation.

To this end, approximate state and state sensitivities to θ are computed for several values of the parameter, namely $\theta=0.00$ (LQG compensator), 0.20, 0.40 and 0.45. For reference, the uncontrolled state plot is given in Figure 2.

Figure 3 shows the controlled state plot for $\theta=0.45$. The plot demonstrates how well the MinMax controller is able to regulate the position state to the exponentially stable equilibrium of zero. The controller also satisfactorily regulates the velocity state and mass and midcable positions. Due to space limitations, the controlled state plots for $\theta=0.00$, $\theta=0.20$ and $\theta=0.40$ are not provided in this paper. However, they tell much the same story as Figure 3. The MinMax controllers for $\theta=0.00$, $\theta=0.20$, and $\theta=0.40$ regulate the states to zero quite well, and it is difficult to visually distinguish the controlled states from those obtained with $\theta=0.45$.

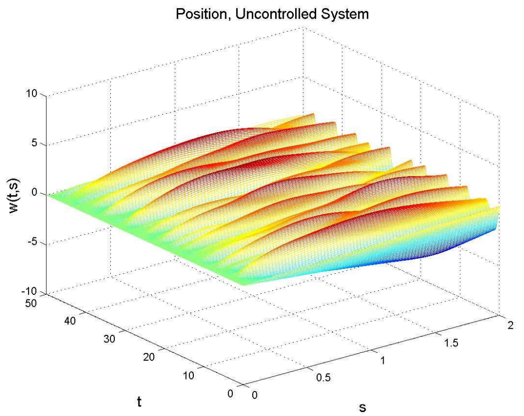

Fig. 2. Uncontrolled Position State

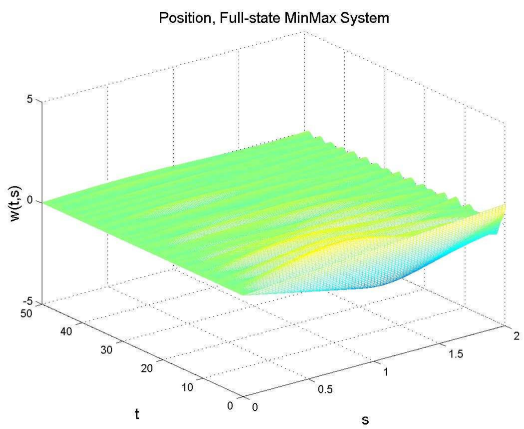

Fig. 3. MinMax Controlled Position State ($\theta = 0.45$)

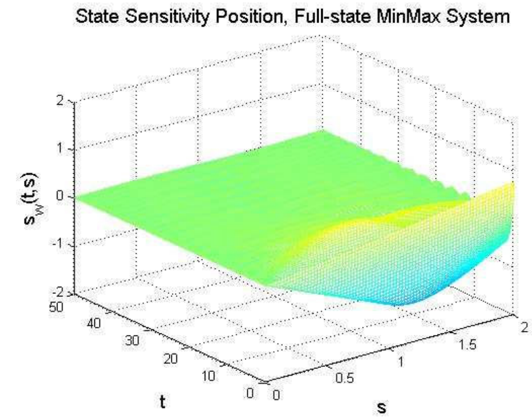

Fig. 4. State Sensitivity ($\theta = 0.00$)

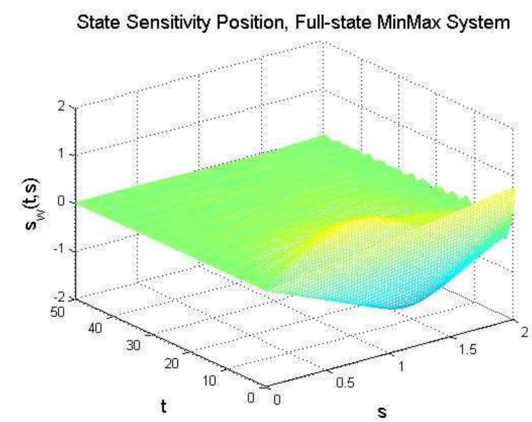

Fig. 5. State Sensitivity ($\theta = 0.20$)

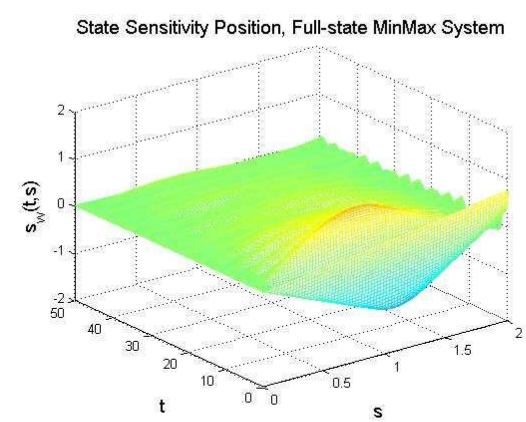

Fig. 6. State Sensitivity ($\theta = 0.40$)

The primary question of interest in this paper is to examine how sensitive the controlled cable and mass displacements are to variation in the MinMax control parameter, θ. Given the similarity of the controlled state plots for a range of θ values between 0.00 and 0.45, the author expects the state sensitivity to be similar for these same parameter values. Figures 4-7 show the approximate sensitivities of the cable and mass displacements with respect to θ for $\theta = 0.00$, 0.20, 0.40 and 0.45, respectively.

The four plots all show increased amplitude near $s = 1$, meaning the sensitivities of the cable and mass displacements are largest near the midcable position for the parameter values considered. The sensitivities increase as the θ parameter increases. This seems reasonable since the MinMax controller with $\theta = 0.45$ does a visibly better job of regulating the cable and mass displacements to the exponentially stable equilibrium of $[0,0]^T$ than the LQG controller where $\theta = 0.00$ [9]. The sensitivities also demonstrate that the fixed end of the cable is not sensitive to small changes in the θ parameter. This seems reasonable given that the control, which is dependent on θ, is only applied at the mass location and not along the cable. It is interesting to note that the sensitivities remain bounded over the time interval

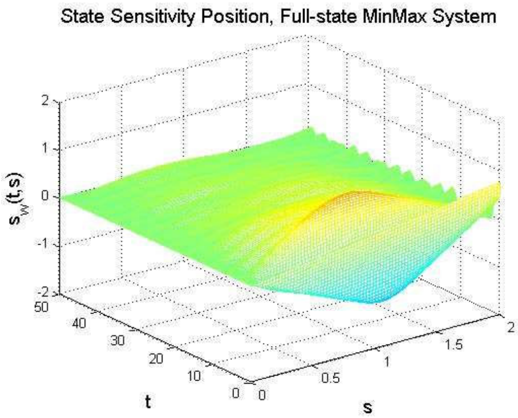

Fig. 7. State Sensitivity ($\theta = 0.45$)

$[0, 50]$ seconds, even at the midcable position. In fact, the sensitivities are oscillatory, and the amplitudes of the waves are comparable to the amplitudes of the controlled cable and mass displacements observed in Figure 3. Additionally, the author examined $s_u(t;\theta)$, the sensitivity of the controller with respect to θ, and these plots are found in Figure 8. Figure 8 shows that the controllers are more sensitive initially and then taper off toward a zero sensitivity. Additionally, as θ increases, increased controller sensitivity is observed.

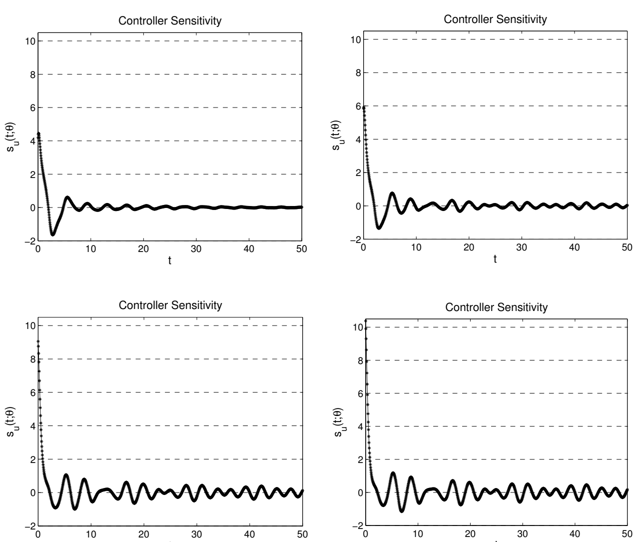

Fig. 8. Control Sensitivities: $\theta = 0.00$ (top left), $\theta = 0.20$ (top right), $\theta = 0.40$ (bottom left), $\theta = 0.45$ (bottom right)

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this paper, a nonlinear cable-mass distributed parameter system is considered, and a continuous sensitivity equation method is applied to derive the sensitivity of the cable and mass displacements with respect to the MinMax control parameter, θ. Numerical results calculated using a Galerkin finite element approximation for the PDEs have been presented. Approximate state sensitivities and controller sensitivities were calculated for varying θ parameter values. Based on the computational results presented here, both the state and controller sensitivities increase gradually in magnitude for increasing values of $\theta \in [0.00, 0.45]$. Increasing the value of θ also leads to improved performance in the regulation of the state to equilibrium. These sensitivity and controller performance results do not suggest a clear indication for choosing the control parameter θ in an alternate way.

B. Future Works

An analysis similar to that presented in this paper will be applied to other hyperbolic and parabolic distributed parameter systems, in hope of determining a strategy for an efficient assignment of the MinMax control parameter that is mathematically justified and still satisfies certain performance and robustness criteria. Other related research directions pertain to sensitivity and conditioning of solutions to Riccati equations and controller robustness [8].

VI. ACKNOWLEDGMENTS

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