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# Dynamic Output Feedback Control for Consensus of Multi-Agent Systems: An $H_\infty$ Approach

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*Abstract*— **This paper deals with the consensus control of a multi-agent system with state and measurement disturbances, and proposes a distributed dynamic output feedback protocol. By defining an appropriate controlled output, the consensus control problem is reformulated as an $H_\infty$ control problem. Using $H_\infty$ techniques, a sufficient condition in terms of linear matrix inequalities (LMIs) is given to ensure consensus with the prescribed $H_\infty$ performance, and the undetermined system matrix of the proposed protocol is obtained. Simulation results show that a multi-agent system under the proposed protocol with system matrix solved by the LMI (linear matrix inequality) approach possesses the desired $H_\infty$ consensus performance.**

*Index Terms*— **Consensus, Multi-agent systems, Dynamic $H_\infty$ control, External disturbances**

## I. Introduction

The multi-agent system has received considerable attention due to its applications in many areas, such as formation control of unmanned air and underwater vehicles, flocking of mobile vehicles, distributed optimization of multiple mobile robotic systems, and scheduling of automated highway systems. In the literature, consensus control generally means to design a distributed protocol based on communication networks such that the states of all agents are asymptotically driven to a common value, which is well accepted as one of the most important and fundamental issues in the cooperative control of multi-agent systems.

During the past decade, large numbers of interesting results have been obtained for the consensus of multi-agent systems [1]-[20]. In reality, agents are usually under uncertain environments with external disturbances and subject to communication noises arising from the information interaction. Thus recently, consensus of such a multi-agent system has attracted the attention of some researchers [13]-[20]. In [17], a stochastic model was considered for the average consensus of discrete-time multi-agent systems, in which each new updated value was corrupted by an additive noise with zero mean, and the optimal weighted adjacency matrix was designed to minimize the static mean square consensus error by convex optimization. In [18], Li et al. considered the disturbance rejection problem arising in the coordination control of multi-agent systems subject to external disturbances, and they proved that the disturbance rejection problem of an agent network could be solved by analyzing the $H_\infty$ control problem of a set of independent systems whose dimensions were all equal to that of a single agent. In [19], Lin et al. studied the consensus problem of first-order multi-agent systems with external disturbances and model uncertainties for directed networks with zero and nonzero time delays, and they showed that this problem could be transformed into a robust $H_\infty$ control problem.

In this paper, we study the consensus control problem for a network of autonomous agents with high-dimension linear coupling dynamics and state and measurement disturbances. To this end, a controlled output is firstly defined to measure the state disagreements among agents, and the consensus control problem of multi-agent systems is reformulated as an $H_\infty$ control problem. Then, a distributed dynamic output feedback protocol is proposed with an undetermined system matrix, and a closed-loop system is obtained with an uncontrollable subblock in the state matrix. To solve this uncontrollable problem, we conduct a model transformation by two steps, and derive an equivalent reduced-order system regarding the $H_\infty$ performance, based on which a sufficient condition in terms of LMIs is given to ensure consensus with the desired $H_\infty$ performance, and the system matrix of the proposed protocol is further determined by solving two LMIs. Finally, a simulation example is included to demonstrate the effectiveness of the proposed protocol and the correctness of the theoretical results.

Throughout this paper, $\mathbf{1}$ and $\mathbf{0}$ denote the column vectors with appropriate dimensions whose elements are all ones and all zeros, respectively, while $\mathbf{1}_n$ and $\mathbf{0}_n$ denote the corresponding $n\times 1$ column vectors; $I_n$ and $0_n$ ($0_{n\times m}$) denote the $n\times n$ identity matrix and the $n\times n$ ($n\times m$) null matrix; notations $X<0$ and $X>0$ represent that the symmetric matrix $X$ is negative and positive definite, respectively; in symmetric block matrices, $*$ is used as an ellipsis for terms induced by symmetry; $\mathrm{diag}\{M_1,\cdots,M_n\}$ denotes a block diagonal matrix whose diagonal blocks are given by $M_1,\cdots,M_n$; the notation $\otimes$ denotes the Kronecker product; superscripts "T" and "-1" stand for matrix transposition and matrix inverse, respectively; the space of square-integrable vector functions over $[0,\infty)$ is denoted by $\mathscr{L}_2[0,\infty)$, and for $v(t)\in\mathscr{L}_2[0,\infty)$, its normalized energy is defined by $\|v(t)\|_2=(\int_0^\infty \|v(t)\|^2\mathrm{d}t)^{1/2}$, where $\|v(t)\|^2=v^{\mathrm{T}}(t)v(t)$.

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## II. Preliminaries and Problem Reformulation

### A. Preliminaries

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ be a weighted undirected graph of order $n$ with the set of nodes $\mathscr{V} = \{v_1, \cdots, v_n\}$, the set of undirected edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and a symmetric adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency weights $a_{ij}$. The adjacency elements associated with edges are positive, i.e., $(v_i, v_j)$ or $(v_j, v_i) \in \mathscr{E} \Leftrightarrow a_{ij} = a_{ji} > 0$. In graph $\mathscr{G}$, node $v_i$ represents the $i$th agent, and edge $(v_i, v_j)$ represents that information flows between agents $i$ and $j$. Then the set of neighbors of $v_i$ is denoted by $\mathscr{N}_i = \{v_j \in \mathscr{V} : (v_i, v_j) \in \mathscr{E}\}$. The Laplacian of a weighted graph $\mathscr{G}$ is defined as $L = D - A$, where diagonal matrix $D = \mathrm{diag}\{d_1, \cdots, d_n\}$ is named the degree matrix of $\mathscr{G}$, whose diagonal elements are $d_i = \sum_{j=1}^{n} a_{ij}$. An undirected path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2})$, $(v_{i_2}, v_{i_3})$, $\cdots$, $(v_{i_{l-1}}, v_{i_l})$ in an undirected graph, where $v_{i_j} \in \mathscr{V}$, $j = 1, \cdots, l$. If there is a path from every node to every other node, the undirected graph is said to be connected.

*Lemma 1:* [2] Let $L$ be the Laplacian of an undirected graph $\mathscr{G}$. Then $L$ has at least one zero eigenvalue and all of the nonzero eigenvalues are positive. Furthermore, matrix $L$ has exactly one zero eigenvalue if and only if (iff) the undirected graph $\mathscr{G}$ is connected, and the eigenvector associated with zero is $\mathbf{1}$.

*Lemma 2:* [11] Let $L_c = [L_{c_{ij}}]$ be a symmetric matrix with

$$L_{c_{ij}} = \begin{cases} \frac{n-1}{n}, & i = j \\ -\frac{1}{n}, & i \neq j. \end{cases} \tag{1}$$

Then the following statements hold:

(1) The eigenvalues of $L_c$ are 1 with multiplicity $n-1$ and 0 with multiplicity 1. The vectors $\mathbf{1}_n^{\mathrm{T}}$ and $\mathbf{1}_n$ are the left and the right eigenvectors of $L_c$ associated with the zero eigenvalue, respectively.

(2) There exists an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ such that

$$U^{\mathrm{T}} L_c U = \begin{bmatrix} I_{n-1} & \mathbf{0}_{n-1} \\ * & 0 \end{bmatrix},$$

and the last column of $U$ is $\mathbf{1}_n/\sqrt{n}$. Furthermore, let $L \in \mathbb{R}^{n \times n}$ be the Laplacian of any undirected graph, then

$$U^{\mathrm{T}} L U = \begin{bmatrix} L_1 & \mathbf{0}_{n-1} \\ * & 0 \end{bmatrix},$$

where $L_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ is positive definite iff the graph is connected.

### B. Problem statement and reformulation

Consider the multi-agent system consisting of $n$ identical agents with the $i$th one modeled by the following linear dynamic system with external disturbances:

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_1\omega_i(t) + B_2u_i(t) \\ y_i(t) &= Cx_i(t) + D\omega_i(t), \quad i = 1, \cdots, n, \end{aligned} \tag{2}$$

where $x_i(t) \in \mathbb{R}^m$, $u_i(t) \in \mathbb{R}^{m_2}$ and $y_i(t) \in \mathbb{R}^p$ are the state, the protocol and the measured output of agent $i$, respectively, and $\omega_i(t) \in \mathbb{R}^{m_1}$ is the external disturbance that belongs to $\mathscr{L}_2[0, \infty)$. It is assumed that $(A, B_2)$ is stabilizable, and without loss of generality, $B_2$ is of full column rank. The objective is to design a distributed output feedback protocol such that the multi-agent system (2) asymptotically reaches consensus on the states that is expressed by

$$\lim_{t\to\infty}(x_i(t) - x_j(t)) = \mathbf{0}, \ \ \forall i, j \in \{1, \cdots, n\} \triangleq \mathscr{N}. \tag{3}$$

Under the influence of external disturbances, the accurate consensus may be hard to achieve. In view of this, we attempt to design a protocol to attenuate the interference of external disturbances to the consensus performance. In order to quantitatively analyze the effect of disturbances to the consensus, define a controlled output function

$$z_i(t) = x_i(t) - \frac{1}{n}\sum_{j=1}^{n} x_j(t), \ \ i = 1, \cdots, n \tag{4}$$

to measure the disagreement of $x_i(t)$ to the average state of all agents. Note that if $z_i(t) = \mathbf{0}$ for all $i \in \mathscr{N}$, then $x_i(t) = x_j(t)$ for $\forall i, j \in \mathscr{N}$, that is, the consensus is achieved. Denote $x(t) = [x_1^{\mathrm{T}}(t) \ \cdots \ x_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{mn}$, $\omega(t) = [\omega_1^{\mathrm{T}}(t) \ \cdots \ \omega_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{m_1 n}$, $u(t) = [u_1^{\mathrm{T}}(t) \ \cdots \ u_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{m_2 n}$, $y(t) = [y_1^{\mathrm{T}}(t) \ \cdots \ y_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pn}$ and $z(t) = [z_1^{\mathrm{T}}(t) \ \cdots \ z_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{mn}$. Then combining the dynamic equation (2) with the controlled output (4) yields the following system written in matrix form

$$\begin{aligned} \dot{x}(t) &= (I_n \otimes A)x(t) + (I_n \otimes B_1)\omega(t) + (I_n \otimes B_2)u(t) \\ y(t) &= (I_n \otimes C)x(t) + (I_n \otimes D)\omega(t) \\ z(t) &= (L_c \otimes I_m)x(t), \end{aligned} \tag{5}$$

where $L_c$ is defined in (1).

Since $z(t) = \mathbf{0}$ implies $x_i(t) = x_j(t)$ for $\forall i, j \in \mathscr{N}$, the attenuating ability of the multi-agent system on consensus against external disturbances can be quantitatively measured by the $H_\infty$ norm of the closed-loop transfer function matrix $T_{z\omega}(s)$ from the external disturbance $\omega(t)$ to the controlled output $z(t)$ that is defined by

$$\|T_{z\omega}(s)\|_\infty = \sup_{\nu\in\mathbb{R}} \bar{\sigma}(T_{z\omega}(\mathrm{j}\nu)) = \sup_{\mathbf{0}\neq\omega(t)\in\mathscr{L}_2[0,\infty)} \frac{\|z(t)\|_2}{\|\omega(t)\|_2}, \tag{6}$$

where $\bar{\sigma}$ denotes the largest singular value. Thus, the objective changes into designing an output feedback protocol $u_i(t)$ $(i \in \mathscr{N})$ such that $\|T_{z\omega}(s)\|_\infty < \gamma$, or equivalently, the closed-loop system satisfies the dissipation inequality

$$\int_0^\infty \|z(t)\|^2 \mathrm{d}t < \gamma^2 \int_0^\infty \|\omega(t)\|^2 \mathrm{d}t, \quad \forall \omega \in \mathscr{L}_2[0, \infty),$$

where $\gamma > 0$ is a given $H_\infty$ index. In this way, the consensus control problem of the multi-agent system with external disturbances is reformulated as the above $H_\infty$ control problem.

## III. $H_\infty$ Consensus Control

### A. Protocol design and model transformation

Using the neighbors' measured outputs, a distributed dynamic output feedback protocol is designed as

$$\begin{aligned} \dot{v}_i(t) &= A_K v_i(t) + B_K \sum_{j\in\mathscr{N}_i} a_{ij}(y_i(t) - y_j(t)) \\ u_i(t) &= C_K v_i(t) + D_K \sum_{j\in\mathscr{N}_i} a_{ij}(y_i(t) - y_j(t)), \ i = 1, \cdots, n, \end{aligned} \tag{7}$$

where $v_i(t) \in \mathbb{R}^{m_K}$ ($m_K$ is a preassigned dimension) is the state of the dynamic output feedback controller, and $a_{ij}$ are the adjacency weights of the interaction graph $\mathscr{G}$. Define the system matrix of protocol (7) as

$$K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}. \tag{8}$$

Substituting protocol (7) into the system (5) results in the following closed-loop system

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} =& \begin{bmatrix} I_n \otimes A + L \otimes B_2 D_K C & I_n \otimes B_2 C_K \\ L \otimes B_K C & I_n \otimes A_K \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \\
&+ \begin{bmatrix} I_n \otimes B_1 + L \otimes B_2 D_K D \\ L \otimes B_K D \end{bmatrix} \omega(t) \\
z(t) =& \begin{bmatrix} L_c \otimes I_m & 0_n \otimes 0_{m \times m_K} \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix},
\end{aligned} \tag{9}$$

where $v(t) = [v_1^{\mathrm{T}}(t) \cdots v_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{m_K n}$, and $L$ is the Laplacian of graph $\mathscr{G}$.

Since the symmetric matrix $L$ is singular, it can be proved that the state matrix of the system (9) is unstable if the given matrix $A$ is unstable, which means that the state matrix of the closed-loop system is uncontrollable. To solve this problem, we conduct a model transformation by two steps.

**Step 1**: Let

$$\begin{aligned}
\bar{x}(t) &= x(t) - \frac{\mathbf{1}_n}{n} \otimes \Big(\sum_{j=1}^{n} x_j(t)\Big) = (L_c \otimes I_m) x(t) \\
\bar{v}(t) &= v(t) - \frac{\mathbf{1}_n}{n} \otimes \Big(\sum_{j=1}^{n} v_j(t)\Big) = (L_c \otimes I_{m_K}) v(t).
\end{aligned}$$

By (9), it is derived that

$$\begin{aligned}
&\dot{\bar{x}}(t) \\
=&(L_c \otimes I_m)\, \dot{x}(t) \\
=&(L_c \otimes I_m)[(I_n \otimes A + L \otimes B_2 D_K C) x(t) + (I_n \otimes B_2 C_K) v(t) \\
&+ (I_n \otimes B_1 + L \otimes B_2 D_K D)\omega(t)] \\
=&(L_c \otimes A + L_c L \otimes B_2 D_K C) x(t) + (L_c \otimes B_2 C_K) v(t) \\
&+ (L_c \otimes B_1 + L_c L \otimes B_2 D_K D)\omega(t) \\
=&(L_c \otimes A + L_c L \otimes B_2 D_K C)[\bar{x}(t) + \frac{\mathbf{1}_n}{n} \otimes (\textstyle\sum_{j=1}^{n} x_j(t))] \\
&+ (L_c \otimes B_2 C_K)[\bar{v}(t) + \frac{\mathbf{1}_n}{n} \otimes (\textstyle\sum_{j=1}^{n} v_j(t))] \\
&+ (L_c \otimes B_1 + L_c L \otimes B_2 D_K D)\omega(t) \\
=&(L_c \otimes A + L_c L \otimes B_2 D_K C)\bar{x}(t) + (L_c \otimes B_2 C_K)\bar{v}(t) \\
&+ (L_c \otimes B_1 + L_c L \otimes B_2 D_K D)\omega(t),
\end{aligned} \tag{10}$$

where the facts $L_c \mathbf{1}_n = \mathbf{0}_n$ and $L \mathbf{1}_n = \mathbf{0}_n$ have been used. Similarly, we have

$$\dot{\bar{v}}(t) = (L_c L \otimes B_K C)\bar{x}(t) + (L_c \otimes A_K)\bar{v}(t) + (L_c L \otimes B_K D)\omega(t) \tag{11}$$

and

$$z(t) = (L_c \otimes I_m)\bar{x}(t) + (0_n \otimes 0_{m \times m_K})\bar{v}(t). \tag{12}$$

Combining equations (10) (11) and (12), we obatin the following system written in matrix form

$$\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}(t) \\ \dot{\bar{v}}(t) \end{bmatrix} =& \begin{bmatrix} L_c \otimes A + L_c L \otimes B_2 D_K C & L_c \otimes B_2 C_K \\ L_c L \otimes B_K C & L_c \otimes A_K \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{v}(t) \end{bmatrix} \\
&+ \begin{bmatrix} L_c \otimes B_1 + L_c L \otimes B_2 D_K D \\ L_c L \otimes B_K D \end{bmatrix} \omega(t) \\
\triangleq& \bar{A} \begin{bmatrix} \bar{x}(t) \\ \bar{v}(t) \end{bmatrix} + \bar{B}\omega(t) \\
z(t) =& \begin{bmatrix} L_c \otimes I_m & 0_n \otimes 0_{m \times m_K} \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{v}(t) \end{bmatrix} \\
\triangleq& \bar{C} \begin{bmatrix} \bar{x}(t) \\ \bar{v}(t) \end{bmatrix}.
\end{aligned} \tag{13}$$

**Step 2**: By Lemma 2, there exists an orthogonal matrix $U = [U_1\ U_2] \in \mathbb{R}^{n \times n}$ with $U_2 = \mathbf{1}_n / \sqrt{n}$ such that

$$U^{\mathrm{T}} L_c U = \begin{bmatrix} I_{n-1} & \mathbf{0}_{n-1} \\ * & 0 \end{bmatrix} \triangleq \bar{L}_c$$

and

$$U^{\mathrm{T}} L U = \begin{bmatrix} L_1 & \mathbf{0}_{n-1} \\ * & 0 \end{bmatrix} \triangleq \bar{L},$$

where $L_1 \in \mathbb{R}^{(n-1) \times (n-1)}$ is positive definite iff the graph $\mathscr{G}$ is connected. Perform the orthogonal transformation:

$$\begin{aligned}
\hat{x}(t) &= (U^{\mathrm{T}} \otimes I_m)\bar{x}(t),\ \hat{v}(t) = (U^{\mathrm{T}} \otimes I_{m_K})\bar{v}(t) \\
\hat{\omega}(t) &= (U^{\mathrm{T}} \otimes I_{m_1})\omega(t),\ \hat{z}(t) = (U^{\mathrm{T}} \otimes I_m) z(t).
\end{aligned}$$

Then according to (13), we have

$$\begin{aligned}
\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{v}}(t) \end{bmatrix} =& \begin{bmatrix} \bar{L}_c \otimes A + \bar{L}_c \bar{L} \otimes B_2 D_K C & \bar{L}_c \otimes B_2 C_K \\ \bar{L}_c \bar{L} \otimes B_K C & \bar{L}_c \otimes A_K \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{v}(t) \end{bmatrix} \\
&+ \begin{bmatrix} \bar{L}_c \otimes B_1 + \bar{L}_c \bar{L} \otimes B_2 D_K D \\ \bar{L}_c \bar{L} \otimes B_K D \end{bmatrix} \hat{\omega}(t) \\
\triangleq& \hat{A} \begin{bmatrix} \hat{x}(t) \\ \hat{v}(t) \end{bmatrix} + \hat{B}\hat{\omega}(t) \\
\hat{z}(t) =& \begin{bmatrix} \bar{L}_c \otimes I_m & 0_n \otimes 0_{m \times m_K} \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{v}(t) \end{bmatrix} \\
\triangleq& \hat{C} \begin{bmatrix} \hat{x}(t) \\ \hat{v}(t) \end{bmatrix}.
\end{aligned} \tag{14}$$

Note that the last rows of symmetric matrices $\bar{L}_c$ and $\bar{L}_c \bar{L}$ are both zeros. Then by (14), we derive an reduced-order system

$$\begin{aligned}
\begin{bmatrix} \dot{\hat{x}}^1(t) \\ \dot{\hat{v}}^1(t) \end{bmatrix} =& \begin{bmatrix} I_{n-1} \otimes A + L_1 \otimes B_2 D_K C & I_{n-1} \otimes B_2 C_K \\ L_1 \otimes B_K C & I_{n-1} \otimes A_K \end{bmatrix} \begin{bmatrix} \hat{x}^1(t) \\ \hat{v}^1(t) \end{bmatrix} \\
&+ \begin{bmatrix} I_{n-1} \otimes B_1 + L_1 \otimes B_2 D_K D \\ L_1 \otimes B_K D \end{bmatrix} \hat{\omega}^1(t) \\
\triangleq& \hat{A}^1 \begin{bmatrix} \hat{x}^1(t) \\ \hat{v}^1(t) \end{bmatrix} + \hat{B}^1 \hat{\omega}^1(t) \\
\hat{z}^1(t) =& \begin{bmatrix} I_{n-1} \otimes I_m & 0_{n-1} \otimes 0_{m \times m_K} \end{bmatrix} \begin{bmatrix} \hat{x}^1(t) \\ \hat{v}^1(t) \end{bmatrix} \\
\triangleq& \hat{C}^1 \begin{bmatrix} \hat{x}^1(t) \\ \hat{v}^1(t) \end{bmatrix}
\end{aligned} \tag{15}$$

that is equivalent to (14) considering the $H_\infty$ performance from the external disturbance to the controlled output, where

$\hat{x}^1(t) = (U_1^{\mathrm{T}} \otimes I_m)\bar{x}(t) \in \mathbb{R}^{m(n-1)}$, $\hat{v}^1(t) = (U_1^{\mathrm{T}} \otimes I_{m_K})\bar{v}(t) \in \mathbb{R}^{m_K(n-1)}$, $\hat{\omega}^1(t) = (U_1^{\mathrm{T}} \otimes I_{m_1})\bar{\omega}(t) \in \mathbb{R}^{m_1(n-1)}$ and $\hat{z}^1(t) = (U_1^{\mathrm{T}} \otimes I_m)\bar{z}(t) \in \mathbb{R}^{m(n-1)}$.

From (13) and (14), we get $T_{z\omega}(s) = \bar{C}(sI-\bar{A})^{-1}\bar{B}$ and $T_{\hat{z}\hat{\omega}}(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B}$. It can be easily verified that

$$(U^{\mathrm{T}} \otimes I_m)[\bar{C}(sI-\bar{A})^{-1}\bar{B}](U \otimes I_{m_1}) = \hat{C}(sI-\hat{A})^{-1}\hat{B},$$

which leads to $\|T_{z\omega}(s)\|_\infty = \|T_{\hat{z}\hat{\omega}}(s)\|_\infty = \|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty$ by the definition of $H_\infty$ norm as given in (6). On the other hand, if $\hat{x}^1(t) = \mathbf{0}$, then $\bar{x}(t) = \mathbf{1}$ yields from $\hat{x}^1(t) = (U_1^{\mathrm{T}} \otimes I_m)\bar{x}(t)$, which implies $x(t) = c\mathbf{1}$ ($c$ is a constant), i.e., the consensus on $x_i(t)$ is achieved. Thus, by designing a distributed protocol such that the reduced-order system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty < \gamma$, we can ensure consensus of the multi-agent system (2) with the $H_\infty$ performance index $\gamma$.

*B. Conditions on $H_\infty$ consensus problem*

*Lemma 3:* [24] Let $G(s) = C(sI-A)^{-1}B$. Then $A$ is a stable matrix and $\|G(s)\|_\infty < \gamma$, iff there exists a positive definite matrix $P$ to the following Riccati inequality

$$A^{\mathrm{T}}P + PA + \gamma^{-2}PBB^{\mathrm{T}}P + C^{\mathrm{T}}C < 0.$$

*Lemma 4:* (Schur Complement) For a given symmetric matrix $S$ of the form $S = [S_{ij}]$, $S_{11} \in \mathbb{R}^{r\times r}$, $S_{12} \in \mathbb{R}^{r\times(n-r)}$, $S_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, then $S < 0$ iff

$$S_{11} < 0, \quad S_{22} - S_{21}S_{11}^{-1}S_{12} < 0$$

or equivalently

$$S_{22} < 0, \quad S_{11} - S_{12}S_{22}^{-1}S_{21} < 0.$$

*Theorem 1:* Consider the network with an undirected interaction graph $\mathscr{G}$ that is connected. For a given index $\gamma > 0$, the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty < \gamma$, if there exist a dynamic output feedback $u(t)$ with the system matrix $K$ defined in (8) and positive definite matrices $P_i \in \mathbb{R}^{(m+m_K)\times(m+m_K)}$ $(i = 1,\cdots,n-1)$, such that the following matrix inequalities are satisfied for $i = 1,\cdots,n-1$:

$$A_i^{1\mathrm{T}}P_i + P_iA_i^1 + \gamma^{-2}P_iB_i^1B_i^{1\mathrm{T}}P_i + C_0^{1\mathrm{T}}C_0^1 < 0, \tag{16}$$

where

$$\begin{aligned} A_i^1 &= \begin{bmatrix} A+\lambda_iB_2D_KC & B_2C_K \\ \lambda_iB_KC & A_K \end{bmatrix} \\ B_i^1 &= \begin{bmatrix} B_1+\lambda_iB_2D_KD \\ \lambda_iB_KD \end{bmatrix} \\ C_0^1 &= \begin{bmatrix} I_m & 0_{m\times m_K} \end{bmatrix}, \end{aligned} \tag{17}$$

and $\lambda_i > 0$ is the $i$th eigenvalue of the symmetric matrix $L_1$.

*Proof*: Since the interaction graph $\mathscr{G}$ is connected, the matrix $L_1$ is positive definite by Lemma 2, and there exists an orthogonal matrix $F \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$F^{\mathrm{T}}L_1F = \mathrm{diag}\{\lambda_1,\cdots,\lambda_{n-1}\} \triangleq \Delta,$$

where $0 < \lambda_1 \le \cdots \le \lambda_{n-1}$.

Let $\tilde{x}^1(t) = (F^{\mathrm{T}} \otimes I_m)\hat{x}^1(t)$, $\tilde{v}^1(t) = (F^{\mathrm{T}} \otimes I_{m_K})\hat{v}^1(t)$, $\tilde{\omega}^1(t) = (F^{\mathrm{T}} \otimes I_{m_1})\hat{\omega}^1(t)$ and $\tilde{z}^1(t) = (F^{\mathrm{T}} \otimes I_m)\hat{z}^1(t)$. Then the system (15) can be restated in terms of $\tilde{x}^1(t)$, $\tilde{v}^1(t)$, $\tilde{\omega}^1(t)$ and $\tilde{z}^1(t)$ as follows

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}^1(t) \\ \dot{\tilde{v}}^1(t) \end{bmatrix} &= \begin{bmatrix} I_{n-1}\otimes A+\Delta\otimes B_2D_KC & I_{n-1}\otimes B_2C_K \\ \Delta\otimes B_KC & I_{n-1}\otimes A_K \end{bmatrix} \begin{bmatrix} \tilde{x}^1(t) \\ \tilde{v}^1(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} I_{n-1}\otimes B_1+\Delta\otimes B_2D_KD \\ \Delta\otimes B_KD \end{bmatrix} \tilde{\omega}^1(t) \\ &\triangleq \tilde{A}^1 \begin{bmatrix} \tilde{x}^1(t) \\ \tilde{v}^1(t) \end{bmatrix} + \tilde{B}^1\tilde{\omega}^1(t) \\ \tilde{z}^1(t) &= \begin{bmatrix} I_{n-1}\otimes I_m & 0_{n-1}\otimes 0_{m\times m_K} \end{bmatrix} \begin{bmatrix} \tilde{x}^1(t) \\ \tilde{v}^1(t) \end{bmatrix} \\ &\triangleq \tilde{C}^1 \begin{bmatrix} \tilde{x}^1(t) \\ \tilde{v}^1(t) \end{bmatrix}. \end{aligned} \tag{18}$$

By the property of orthogonal transformations, $\hat{x}^1(t) = \mathbf{0}$ iff $\tilde{x}^1(t) = \mathbf{0}$, and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty = \|T_{\tilde{z}^1\tilde{\omega}^1}(s)\|_\infty$ holds. Thus, combining with Lemma 3, we have that the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty < \gamma$, iff there exists a matrix $\tilde{P} > 0$ such that

$$\tilde{A}^{1\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A}^1 + \gamma^{-2}\tilde{P}\tilde{B}^1\tilde{B}^{1\mathrm{T}}\tilde{P} + \tilde{C}^{1\mathrm{T}}\tilde{C}^1 < 0. \tag{19}$$

On the other hand, by rearranging elements of the state vector in the system (18), we can obtain an equivalent system of (18) with the state matrix $A^1$, the input matrix $B^1$ and the output matrix $C^1$ as

$$\begin{aligned} A^1 &= \mathrm{diag}\{A_1^1,\cdots,A_{n-1}^1\} \\ B^1 &= \mathrm{diag}\{B_1^1,\cdots,B_{n-1}^1\} \\ C^1 &= \mathrm{diag}\{C_0^1,\cdots,C_0^1\}, \end{aligned}$$

where $A_i^1$, $B_i^1$ and $C_0^1$ are defined in (17), $i = 1,\cdots,n-1$. Then there exists a matrix $\tilde{P} > 0$ such that (19) holds, iff there is a matrix $P > 0$ to satisfy

$$A^{1\mathrm{T}}P + PA^1 + \gamma^{-2}PB^1B^{1\mathrm{T}}P + C^{1\mathrm{T}}C^1 < 0. \tag{20}$$

Considering the diagonal block structure of matrices $A^1$, $B^1$ and $C^1$, let $P$ be a diagonal block definite matrix that is denoted by $P = \mathrm{diag}\{P_1,\cdots,P_{n-1}\}$, where $0 < P_i \in \mathbb{R}^{(m+m_K)\times(m+m_K)}$, $i = 1,\cdots,n-1$. In this case, if there exist positive definite matrices $P_i$ to satisfy (16) for $i = 1,\cdots,n-1$, then the matrix inequality (20) holds, from which it is derived that the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty < \gamma$. This completes the proof. □

To proceed, the system matrix of the dynamic output feedback protocol $u(t)$ is further solved in Theorem 2.

*Theorem 2:* Consider the network with an undirected interaction graph $\mathscr{G}$ that is connected. For a given index $\gamma > 0$, the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty < \gamma$, if there exist a positive definite matrix

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & 0_{(m_2+m_K)\times(m-m_2)} \\ * & \bar{P}_2 \end{bmatrix} \in \mathbb{R}^{(m+m_K)\times(m+m_K)} \tag{21}$$

and a matrix

$$\bar{Q} = \begin{bmatrix} \bar{Q}_1 \\ 0_{(m-m_2)\times(p+m_K)} \end{bmatrix} \in \mathbb{R}^{(m+m_K)\times(p+m_K)} \tag{22}$$

such that LMIs

$$\begin{bmatrix} \bar{A}_{(1)}^{1\mathrm{T}}\bar{P}+\bar{P}\bar{A}_{(1)}^{1}+\bar{A}_{i(2)}^{1\mathrm{T}}\bar{Q}^{\mathrm{T}}+\bar{Q}\bar{A}_{i(2)}^{1}+\bar{C}_0^{1\mathrm{T}}\bar{C}_0^{1} & \bar{P}\bar{B}_{(1)}^{1}+\bar{Q}B_{i(2)}^{1} \\ * & -\gamma^2 I_{m_1} \end{bmatrix} < 0, \quad (23)$$

are simultaneously satisfied for $i=1$ and $n-1$, where

$$\bar{A}_{(1)}^{1}=VA_{(1)}^{1}V^{-1}, \bar{A}_{i(2)}^{1}=A_{i(2)}^{1}V^{-1}, \bar{C}_0^{1}=C_0^{1}V^{-1}, \bar{B}_{(1)}^{1}=VB_{(1)}^{1},$$

$$A_{(1)}^{1}=\begin{bmatrix} A & 0_{m\times m_K} \\ * & 0_{m_K} \end{bmatrix},\ A_{i(2)}^{1}=\begin{bmatrix} 0_{m_K\times m} & I_{m_K} \\ \lambda_i C & 0_{p\times m_K} \end{bmatrix},$$

$$B_{(1)}^{1}=\begin{bmatrix} B_1 \\ 0_{m_K\times m_1} \end{bmatrix}, B_{i(2)}^{1}=\begin{bmatrix} 0_{m_K\times m_1} \\ \lambda_i D \end{bmatrix}, C_0^{1}=\begin{bmatrix} I_m \\ 0_{m_K\times m} \end{bmatrix}^{\mathrm{T}},$$

and $V\in\mathbb{R}^{(m+m_K)\times(m+m_K)}$ is a nonsingular matrix such that

$$\bar{E}=VE=\begin{bmatrix} I_{m_2+m_K} \\ 0_{(m-m_2)\times(m_2+m_K)} \end{bmatrix} \quad (24)$$

with

$$E=\begin{bmatrix} 0_{m\times m_K} & B_2 \\ I_{m_K} & 0_{m_K\times m_2} \end{bmatrix}.$$

Further, if (23) hold for $i=1$ and $n-1$, then the system matrix of the distributed protocol is given by

$$K=\bar{P}_1^{-1}\bar{Q}_1.$$

*Proof*: Note that the system matrix $K$ of $u(t)$ can be decompounded from matrices $A_i^1$ and $B_i^1$ as follows

$$\begin{aligned} A_i^1 &= \begin{bmatrix} A & 0_{m\times m_K} \\ * & 0_{m_K} \end{bmatrix} + \begin{bmatrix} 0_{m\times m_K} & B_2 \\ I_{m_K} & 0_{m_K\times m_2} \end{bmatrix} K \begin{bmatrix} 0_{m_K\times m} & I_{m_K} \\ \lambda_i C & 0_{p\times m_K} \end{bmatrix} \\ &= A_{(1)}^1 + EKA_{i(2)}^1 \\ B_i^1 &= \begin{bmatrix} B_1 \\ 0_{m_K\times m_1} \end{bmatrix} + \begin{bmatrix} 0_{m\times m_K} & B_2 \\ I_{m_K} & 0_{m_K\times m_2} \end{bmatrix} K \begin{bmatrix} 0_{m_K\times m_1} \\ \lambda_i D \end{bmatrix} \\ &= B_{(1)}^1 + EKB_{i(2)}^1. \end{aligned} \quad (25)$$

To solve the system matrix $K$, we impose the qualification $P_i=P_0$ $(i=1,\cdots,n-1)$ on the sufficient condition of Theorem 1. Then combining with Lemma 4, we know that the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty<\gamma$ if there exists a common matrix $P_0>0$ such that

$$\begin{bmatrix} A_i^{1\mathrm{T}}P_0+P_0A_i^1+C_0^{1\mathrm{T}}C_0^1 & P_0B_i^1 \\ * & -\gamma^2 I_{m_1} \end{bmatrix} < 0$$

hold for $i=1,\cdots,n-1$. Substituting (25) into the above inequality leads to

$$\begin{aligned} &\begin{bmatrix} \Theta_i & P_0B_{(1)}^1+P_0EKB_{i(2)}^1 \\ * & -\gamma^2 I_{m_1} \end{bmatrix} < 0, \\ &\Theta_i = A_{(1)}^{1\mathrm{T}}P_0+P_0A_{(1)}^1+A_{i(2)}^{1\mathrm{T}}K^{\mathrm{T}}E^{\mathrm{T}}P_0+P_0EKA_{i(2)}^1+C_0^{1\mathrm{T}}C_0^1. \end{aligned} \quad (26)$$

Since $B_2$ is of full column rank, there exists an nonsingular matrix $V$ such that (24) holds. Pre- and post-multiplying the inequality (26) with $\bar{V}=\mathrm{diag}\{V^{-\mathrm{T}},I_{m_1}\}$ and $\bar{V}^{\mathrm{T}}$ yields the matrix inequality (23) with

$$\bar{P}=V^{-\mathrm{T}}P_0V^{-1},\ \bar{Q}=\bar{P}\bar{E}K.$$

Correspondingly, the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty<\gamma$, if there exist a positive definite matrix $\bar{P}\in\mathbb{R}^{(m+m_K)\times(m+m_K)}$ and a matrix $\bar{Q}\in\mathbb{R}^{(m+m_K)\times(p+m_K)}$ such that matrix inequalities (23) are simultaneously satisfied for $i=1,\cdots,n-1$.

The above condition can be further predigested. Note that for a fixed subscript $i$, (23) is an LMI with respect to variables $\bar{P}$ and $\bar{Q}$, thus has the convex property. Therefore, only two LMIs in (23) associated with the largest eigenvalue $\lambda_{n-1}$ and the smallest eigenvalue $\lambda_1$ need to be verified, as stated in the theorem.

Furthermore, if (23) hold for $i=1$ and $n-1$, the undetermined system matrix of the proposed protocol can be solved. Substituting (21) (22) and (24) into $\bar{Q}=\bar{P}\bar{E}K$ yields $K=\bar{P}_1^{-1}\bar{Q}_1$. This completes the proof. □

*Corollary 1:* If the input-space dimension is equal to the state-space dimension, i.e., $m_2=m$, then the input matrix $B_2$ is reversible from the condition that $B_2$ is of full column rank. In this case, the system (15) is asymptotically stable and $\|T_{\hat{z}^1\hat{\omega}^1}(s)\|_\infty<\gamma$ if there exist $0<P\in\mathbb{R}^{(m+m_K)\times(m+m_K)}$ and $Q\in\mathbb{R}^{(m+m_K)\times(p+m_K)}$ such that the following LMIs are satisfied for $i=1$ and $n-1$:

$$\begin{bmatrix} A_{(1)}^{1\mathrm{T}}P+PA_{(1)}^1+A_{i(2)}^{1\mathrm{T}}Q^{\mathrm{T}}+QA_{i(2)}^1+C_0^{1\mathrm{T}}C_0^1 & PB_{(1)}^1+QB_{i(2)}^1 \\ * & -\gamma^2 I_{m_1} \end{bmatrix} < 0.$$

Further, the system matrix $K$ is obtained by $K=E^{-1}P^{-1}Q$.

## IV. Simulation Example

We conduct the numerical simulation on a network of four agents with each one modeled by

$$\begin{aligned} \dot{x}_i(t) &= \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega_i(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_i(t) \\ y_i(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega_i(t) \qquad i=1,2,3,4. \end{aligned}$$

Then the controlled output function $z_i(t)$ is defined as

$$z_i(t)=x_i(t)-\frac{1}{4}\sum_{j=1}^{4}x_j(t) \qquad i=1,2,3,4.$$

The undirected interaction graph is given in Fig. 1, and the $H_\infty$ index $\gamma$ is chosen as 1. The external disturbance is $\omega(t)=[\omega_1(t)\ \omega_2(t)\ \omega_3(t)\ \omega_4(t)]^{\mathrm{T}}=[-0.5w(t)\ 0.8w(t)\ w(t)\ -0.6w(t)]^{\mathrm{T}}$, where $w(t)$ is the energy-limited white noise that is supposed to disturb the system only during the initial two seconds for simplicity of simulation.

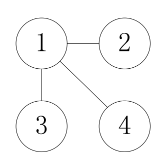


Fig. 1. Undirected interaction graph $\mathscr{G}$.

Under the assumption that all nonzero weighting factors $a_{ij}$ are 1, the Laplacian of the graph $\mathscr{G}$ is

$$L=\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

whose largest and smallest nonzero eigenvalues are $\lambda_3 = 4$ and $\lambda_1 = 1$, respectively. Then by solving two LMIs related to $\lambda_3$ and $\lambda_1$, we obtain the dynamic output feedback protocol (7) with $v_i(t) \in \mathbb{R}^2$ and the system matrix

$$K = \begin{bmatrix} -0.5000 & 0 & 0 & 0 \\ 0 & -0.5000 & 0 & 0 \\ 0 & 0 & -3.5181 & 0.1907 \\ 0 & 0 & 5.9000 & -1.1708 \end{bmatrix}.$$

With the above protocol, Figs. 2 and 3 depict the state error trajectories and the corresponding energy under the zero-valued initial condition, respectively. Obviously, it can be seen that the consensus is asymptotically achieved with $\|T_{z\omega}(s)\|_\infty < 1$, which validates the effectiveness of the proposed output feedback protocol and demonstrates the correctness of the theoretical results.

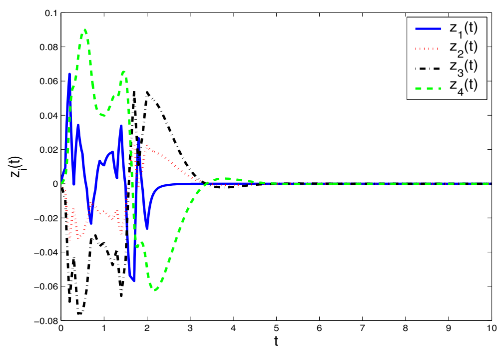


Fig. 2. State error trajectories of the four agents.

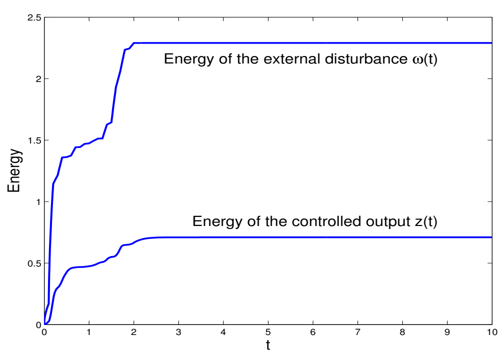


Fig. 3. Energy trajectories of $\omega(t)$ and $z(t)$.

## V. Conclusions

In this paper, the consensus control problem has been addressed for networks of autonomous agents with high-dimension linear coupling dynamics and external disturbances. By transforming the original problem into an $H_\infty$ control problem, a distributed dynamic output feedback protocol is proposed, and conditions in terms of LMIs are derived to ensure consensus with the prescribed $H_\infty$ performance. Meanwhile, the system matrix of the proposed protocol is determined by solving two LMIs. It deserves pointing out that the method introduced in this paper can be applied to networks of agents with switching topology.

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