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WLS-based Partially Decentralized Adaptive Control for Coupled ARMAX Multi-agent Dynamical System

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Abstract— **In this paper, partially decentralized adaptive control for a multi-agent dynamic system is studied. There are many agents in the system, and each agent's state evolves like an ARMAX model with unknown parameters while being intervened by its neighborhood agents, in form of unknown linear interactions. Each agent adopts recursive WLS (weighted least-square) algorithm to estimate its local unknown parameters and designs its local adaptive controller by "certainty equivalence" principle. Generally speaking, it is a question whether such local adaptive controllers can guarantee stability of whole system. In this paper, we not only give the affirmative answer to this question, but also rigorously prove the optimality of decentralized WLS adaptive controller.**

Keywords: decentralized adaptive control, multi-agent dynamic system, coupled ARMAX model, weighted least-square algorithm

I. Introduction

In recent years, complex systems ([1], [2], [3], [4], [5]) have attracted many researchers' interests mainly because of their wide range of background and applications. Their typical features, such as nonlinearity, multi-hierarchy and uncertainty, can be roughly classified into architecture complexity and information uncertainty, which may bring many difficulties in theoretical analysis of the whole systems. Hence, adaptive control of complex systems is a challenging research direction and few efforts are devoted in this area.

In the area of conventional adaptive control, which has been developed for several decades, many fruitful results have been obtained and systematic theories have been established ([6], [7], [8], [9], [10]); however, these well-known theories mainly focused on schemes of centralized control, which may not meet increasing demands for the control of complex systems. Recognizing the demands for the control of complex or large-scale systems, the approach of robust control has been applied in some examples of decentralized control (see e.g. [11], [12], [13], [14], [15]), where uncertainties are dealt with in sense of worst-case analysis. To use this approach in practice, we need to know the *a priori* information on the nominal model and make sure that the uncertainty is in a small range. Comparing with the approach of robust control, the approach of adaptive control can usually deal with a large range of uncertainties rather than a small range of uncertainties, therefore it is necessary to develop theory of decentralized adaptive control, which has also attracted many researchers in the past two decades [16], [17], [18], [19], [20], [21]. Among these existing work, most of them were devoted to analyze the stabilization problem of continuous-time systems with uncertain interconnections via Lyapunov methods, while a few of them [19] focused on decentralized adaptive control of discrete-time systems. We shall remark that, in some work (e.g. [19]), only weak coupling uncertainties can be dealt with by the designed decentralized controller.

Noticing the above background and potential wide applications of adaptive control for complex systems, the authors proposed a theoretical problem framework in [23], [24] to study the discrete-time adaptive control of complex systems, and as two concrete examples in this framework, two discrete-time multi-agent adaptive control problems, adaptive control problem for a coupled ARMAX multi-agent system using decentralized ELS (extended-least-squares) algorithm in [24], and adaptive synchronization problem in [22], have been investigated, respectively.

In this paper, in the same framework as the above, we shall study a multi-agent dynamic complex control system, where each agent has connections with its neighborhood agents by unknown linear interactions. The dynamics of each agent is modeled by an ARMAX model with unknown parameters, which is a widely-used frequency-domain representation of linear systems, and each agent can design its local control by using its history information and its neighborhood agents' state information. In this paper, each agent is supposed to use the recursive WLS (weighted-least-squares) algorithm to estimate the unknown parameters and unknown interactions, and it designs its own local adaptive control with all available local information by using the so-called "certainty equivalence" principle. Generally speaking, it is still unclear whether the local WLS-based adaptive controller can guarantee the stability of whole system. We will answer this question in this paper; furthermore, we shall rigorously prove that the decentralized WLS-based adaptive control is optimal in dealing with the external noise.

We shall remark that the scheme of adaptive control discussed in this paper is only partially decentralized in sense that each local agent requires also measurement of its neighbour agents to design the local controller. However, we need to point out that, while totally decentralized control schemes for continuous-time systems may be relatively easy to find by means of high-gain feedback design, totally decentralized schemes for discrete-time systems (even linear systems) may not always exist since generally each agent cannot "overwhelm" the possible strong interactions by the so-called high-gain feedback without using any information on its neighbor agents. This is why we consider only partially decentralized adaptive control scheme based on WLS algorihtm for the discrete-time model investigated here. And in this contribution, we term the considered coupled ARMAX model as a *multi-agent system* rather than a *large-scale system* simply because the latter emphasizes only the scale of the system and its effects, while the former emphasizes that each agent can take active actions (e.g. WLS-based learning and control) in the evolution of the whole system.

The main contributions of this paper are summarized as follows: (i) The considered decentralized adaptive control scheme is based on the recursive WLS (weighted-least-squares) estimation algorithm, which was not examined in previous literature on decentralized adaptive control. (ii) With the decentralized WLS-based adaptive control scheme, various uncertainties (including unknown internal parameters, unknown strength of interactions, unobservable colored noise) can be easily dealt with. Note that in general the connections need not be very weak as required in some existing study, that is to say, the strength of interactions can be arbitrarily large. (iii) Comparing with the decentralized adaptive control based on LS (least-squares) algorithm, the scheme studied in this paper can provide some extra benefits under very weak conditions. Especially, the high freqency gains of agents are not very necessary to be known *a priori* like in decentralized LS algorithm [24] so as to avoid the so-called singularity problem since the parameter estimates obtained by the WLS algorithms are convergent almost surely.

The rest of this paper is organized as follows. In Section II, we shall introduce the complex system model, i.e. coupled ARMAX model, and the decentralized WLS estimation algorithm, and consequently the local adaptive controllers will be designed based on given decentralized WLS algorithm. Then, in Section III, we shall present preliminary assumptions and lemmas, which will be used in later proofs. Next, in Section IV, we shall establish global stability and optimality of the whole closed-loop system and the rigorous theoretical proof of the main theorem will be given. Simulation study will be given in Section V, where simulation examples can verify our theoretical results. Finally, we shall finish this paper with some concluding remarks in Section VI.

II. Problem Formulation

In this paper, we consider the system with N agents, and each of them evolves as coupled ARMAX model:

$$\begin{aligned} A_i(z)x_i(t+1) &= B_i(z)u_i(t)+C_i(z)w_i(t+1) \\ &\quad + \sum_{j\in\mathcal{N}_i} g_{ij}x_j(t) \end{aligned} \tag{1}$$

where $x_i(t)$, $u_i(t)$ and $w_i(t)$ are the state, input and noise of Agent i respectively; $x_j(t)$ is the state of agent j;

$$\mathcal{N}_i = \{n_{i1}, n_{i2}, \cdots, n_{i,m_i}\} \subseteq \{1,2,\cdots,N\} \tag{2}$$

is the set of Agent i's neighborhood agents;

$$\begin{aligned} A_i(z) &= 1+a_{i1}z+a_{i2}z^2+\cdots+a_{i,p_i}z^{p_i} \\ B_i(z) &= b_{i1}+b_{i2}z+\cdots+b_{i,q_i}z^{q_i-1} \\ C_i(z) &= 1+c_{i1}z+c_{i2}z^2+\cdots+c_{i,l_i}z^{l_i} \end{aligned} \tag{3}$$

and g_{ij} is the *intensity of influence* of agent j towards Agent i.

For convenience, we can rewrite the above equation as the following regression model

$$x_i(t+1) = \boldsymbol{\theta}_i^\tau \boldsymbol{\phi}_i^0(t) + w_i(t+1) \tag{4}$$

with

$$\begin{aligned} \boldsymbol{\phi}_i^0(t) &= [x_i(t),\cdots,x_i(t-p_i+1), \\ &\quad u_i(t),\cdots,u_i(t-q_i+1), \\ &\quad w_i(t),\cdots,w_i(t-l_i+1), \bar{\boldsymbol{X}}_i^\tau(t)]^\tau \\ \boldsymbol{\theta}_i &= [-\boldsymbol{a}_i^\tau, \boldsymbol{b}_i^\tau, \boldsymbol{c}_i^\tau, \boldsymbol{g}_i^\tau]^\tau \end{aligned} \tag{5}$$

where

$$\begin{aligned} \boldsymbol{a}_i^\tau &= [a_{i1},\cdots,a_{i,p_i}] \\ \boldsymbol{b}_i^\tau &= [b_{i1},\cdots,b_{i,q_i}] \\ \boldsymbol{c}_i^\tau &= [c_{i1},\cdots,c_{i,l_i}] \\ \boldsymbol{g}_i^\tau &= [g_{i,n_{i1}}, g_{i,n_{i2}},\cdots,g_{i,n_{i,m_i}}] \\ \bar{\boldsymbol{X}}_i^\tau(t) &= [x_{n_{i1}}(t), x_{n_{i2}}(t),\cdots,x_{n_{i,m_i}}(t)]. \end{aligned} \tag{6}$$

As parameters $\boldsymbol{\theta}_i$ are unknown, we need to estimate them. Here we use the following recursive WLS algorithm:

$$\begin{aligned} \hat{\theta}_i(t+1) &= \hat{\theta}_i(t) + a_i(t)P_i(t)\boldsymbol{\phi}_i(t)\times && (7)\\ &\quad (x_i(t+1)-\hat{\theta}_i^\tau(t)\boldsymbol{\phi}_i(t)) && (8)\\ P_i(t+1) &= P_i(t) - a_i(t)P_i(t)\boldsymbol{\phi}_i(t)\boldsymbol{\phi}_i^\tau(t)P_i(t) && (9)\\ a_i(t) &= (\alpha_i^{-1}(t)+\boldsymbol{\phi}_i^\tau(t)P_i(t)\boldsymbol{\phi}_i(t))^{-1} && (10)\\ \hat{w}_i(t+1) &= x_i(t+1)-\hat{\theta}_i^\tau(t+1)\boldsymbol{\phi}_i(t) && (11) \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\phi}_i(t) &= [x_i(t),\cdots,x_i(t-p_i+1), \\ &\quad u_i(t),\cdots,u_i(t-q_i+1), \\ &\quad \hat{w}_i(t),\cdots,\hat{w}_i(t-l_i+1), \\ &\quad \bar{\boldsymbol{X}}_i^\tau(t)]^\tau \end{aligned} \tag{12}$$

and the initial values $\hat{\theta}_i(0)$ and $P_i(0) > 0$ can be chosen arbitrarily; $\{\alpha_i(t)\}$ is the weighting sequence defined by

$$\alpha_i(t) = \frac{1}{h(r_i(t))}, \quad r_i(t) = ||P_i^{-1}(0)|| + \sum_{k=0}^{t} ||\boldsymbol{\phi}_i(k)||^2 \tag{13}$$

with $h(x) = \log^{1+\gamma} x$, $(\gamma > 0)$ or more generally, see [25]. To make sure $r_i(t) > 1$, we need only take $P_i(0)$ such that $0 < P_i(0) < e^{-1}I$, where e is the base of natual logarithm.

In this case, the control objective of each agent is, for Agent i ($i = 1,2,\cdots,N$), at any time instant t, to design a local feedback control $u_i(t)$ based on the past measurements $\{x_i(0),\cdots,x_i(t),u_i(0),\cdots,u_i(t-1)\}$ and its neighbors' states $\{x_j(t), j\in N_i\}$ so that the average tracking error

$$J_i(t) \triangleq \frac{1}{t}\sum_{i=1}^{t} |x_i(t)-x_i^*(t)|^2$$

is asymptotically minimized, where the deterministic signal $\{x_i^*(t)\}$ is the local tracking goal of Agent i.

Then, by using the "certainty equivalence" principle, Agent i can choose its local control $u_i(t)$ such that

$$\boldsymbol{\phi}_i^\tau(t)\hat{\theta}_i(t) = x_i^*(t+1) \tag{14}$$

where the estimate $\hat{\theta}_i(t)$ is given by the above WLS algorithm. From Eq. (14), we can obtain that

$$\begin{array}{rcl} u_i(t) & = & \frac{1}{b_{i1}(t)}\{x_i^*(t+1) \\ & & +[a_{i1}(t)x_i(t)+\cdots+a_{i,p_i}(t)x_i(t-p_i+1)] \\ & & -[b_{i2}(t)u_i(t-1)+\cdots+b_{i,q_i}(t)u_i(t-q_i+1)] \\ & & -[c_{i1}(t)\hat{w}_i(t)+\cdots+c_{i,l_i}(t)\hat{w}_i(t-l_i+1)] \\ & & -\boldsymbol{g}_i^\tau(t)\bar{\boldsymbol{X}}_i(t)\}. \end{array} \tag{15}$$

Note that the above local control laws only utilize local information rather than global information of all the agents. Intuitively, each agent has all the necessary information available to design the local controller, however, we shall remark that the decentralized update laws Eqs. (8)—(12) cannot be obtained from the centralized WLS estimation algorithm for the whole system because we cannot directly get any knowledge on the bounds of $\boldsymbol{X}_i(t)$ and $\boldsymbol{g}_i^\tau(t)$. Therefore, it is not possible to directly use the established results of centralized WLS-based adaptive control [25] to yield the closed-loop stability of the whole system. Due to this reason, we give rigorous mathematical proof for the stability and optimality of the WLS-based decentralized adaptive controller, which is the key merit of this contribution.

III. Preliminary Assumptions And Lemmas

In order to analyze the decentralized WLS-based adaptive controller, here we introduce the following standard conditions:

Assumption 3.1: (noise condition) $\{w_i(t), \mathcal{F}_t\}$ is a martingale difference sequence, with $\{\mathcal{F}_t\}$ being a sequence of nondecreasing σ-algebras, such that

$$\sup_{t\geq 0} E[|w_i(t+1)|^\beta|\mathcal{F}_t] < \infty, a.s. \tag{16}$$

for some $\beta > 2$ and

$$\lim_{t\to\infty} \frac{1}{t}\sum_{k=1}^{t} |w_i(k)|^2 = \sigma_i > 0, a.s. \tag{17}$$

Assumption 3.2: (minimum phase condition) $B_i(z) \neq 0, \forall z \in \mathcal{C} : |z| \leq 1$.

Assumption 3.3: (reference signal) $\{x_i^*(t)\}$ is a bounded deterministic signal.

Assumption 3.4: (strict-positive-real condition) $C_i^{-1}(z) - \frac{1}{2}$ is strictly positive real, i.e.,

$$|C_i(z) - 1| < 1, \forall z \in \mathcal{C} : |z| = 1. \tag{18}$$

Remark 3.1: Assumption 3.1 is very weak, for example, i.i.d. Gaussian noise or noise with other commonly-seen probability distribution all satisfy this assumption. Assumption 3.2 is one standard structure condtion for controlling linear system. And Assumption 3.4 is a standard technical condition to deal with colored noise.

Furthermore, assume that $\{d_i(t)\}$ is a positive nondecreasing deterministic sequence such that

$$w_i^2(t) = O(d_i(t)) \quad a.s. \quad d_i(t+1) = O(d_i(t)) \tag{19}$$

It can be proved that under condition (A.1), $d_i(t)$ can be taken as

$$d_i(t) = t^\delta, \quad \forall \delta \in (\frac{2}{\beta}, 1) \tag{20}$$

To make rigorous analysis, we introduce the following notations firstly:

$$\begin{array}{rcl} \tilde{\theta}_i(t) & \triangleq & \boldsymbol{\theta}_i - \hat{\theta}_i(t) \\ \delta_i(t) & \triangleq & \mathrm{tr}(P_i(t) - P_i(t+1)) \\ \beta_i(t) & \triangleq & \frac{(\boldsymbol{\phi}_i^\tau(t)\tilde{\theta}_i(t))^2}{\alpha_i^{-1}(t)+\boldsymbol{\phi}_i^\tau(t)P_i(t)\boldsymbol{\phi}_i(t)} \end{array} \tag{21}$$

The following properties of WLS algorithm are to be applied in later proof of global stability and optimality:

Lemma 3.1: If Assumptions 3.1 and 3.4 hold, then the recursive WLS algorithm defined by Eqs. (8)—(10) has the following properties almost surely:

$$||P_i^{-\frac{1}{2}}(t+1)\tilde{\theta}_i(t+1)||^2 = O(1) \tag{22}$$

$$\sum_{t=1}^{\infty} \alpha_i(t)(\boldsymbol{\phi}_i^\tau(t)\tilde{\theta}_i(t+1))^2 < \infty \tag{23}$$

$$\sum_{t=1}^{\infty} \alpha_i(t)(\hat{w}_i(t) - w_i(t))^2 < \infty \tag{24}$$

$$\sum_{t=1}^{\infty} \frac{(\boldsymbol{\phi}_i^\tau(t)\tilde{\theta}_i(t))^2}{\alpha_i^{-1}(t) + \boldsymbol{\phi}_i^\tau(t)P_i(t)\boldsymbol{\phi}_i(t)} < \infty \tag{25}$$

And $\hat{\theta}_i(t)$ converges almost surely to a finite random vector $\bar{\theta}_i$ (not necessarily equal to θ_i).

Proof: See [25, Lemma 1] for proof of properties (22)—(25) and [25, Theorem 1] for proof of the last property. □

Remark 3.2: The last property is called self-convergence of the WLS algorithm, which is not available in recursive least-square (LS) algorithm and provides significant benefits in theoretical analysis of WLS-based adaptive controller. And because of this property, in this paper, it is not very necessary to impose the assumption on high-frequency gains, "b_{i1} ($i = 1, 2, \cdots, N$) are known *a priori*", so as to avoid the so-called singularity problem. A detailed comparison between WLS algorithm and LS algorithm can be found in [25].

IV. Stability And Optimality

The closed-loop system of the decentralized adaptive controller based on local WLS estimators has the following stability and optimality:

Theorem 4.1: Under Assumptions 3.1—3.4, the closed-loop system is stable in sense of

$$\limsup_{t\to\infty} \frac{1}{t}[\sum_{k=1}^{t} x_k^2(t) + u_k^2(t)] < \infty, \quad a.s. \quad \forall i = 1, 2, \cdots, N$$

and optimal in sense of

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{k=0}^{t} |x_i(k+1) - x_i^*(k+1) - w_i(k+1)|^2 = 0, \quad a.s. \quad \forall i = 1, 2, \cdots, N \tag{26}$$

And furthermore, the following order estimates

$$R_i(t) = O(h(t) + \bar{d}(t)) \tag{27}$$

hold almost surely for $i = 1, 2, \cdots, N$, where

$$\begin{array}{rcl} R_i(t) & \triangleq & \sum_{k=0}^{t} |x_i(k+1) - x_i^*(k+1) - w_i(k+1)|^2 \\ \bar{d}(t) & = & \max(d_1(t), \cdots, d_N(t)) \end{array} \tag{28}$$

Proof: First, we need to emphasize that both stability and optimality are implied by Eq. (27). In fact, by definitions of $h(\cdot)$ and $\bar{d}(t)$, if Eq. (27) holds, then obviously we have $R_i(t) = O(h(t) + \bar{d}(t)) = o(t)$, which means the optimality of closed-loop system holds. Then, by the optimality and Assumption 3.1, we can obtain that

$$\sum_{k=0}^{t}(x_i(k+1))^2 = O(t);$$

then, by Assumption 3.2, i.e. minimum phase condition, we can obtain that

$$\sum_{k=0}^{t}(u_i(k))^2 = O(t).$$

Thus, the stability

$$\limsup_{t\to\infty} \frac{1}{t}[\sum_{k=1}^{t} x_k^2(t) + u_k^2(t)] < \infty$$

also holds if Eq. (27) is true.

By the discussions above, we need only to prove Eq. (27). We divide the whole proof into 4 steps. In Step 1, we try to analyze each agent individually, and give rough estimates of $x_i^2(t+1)$ and $u_i(t)$, $i = 1, 2, \cdots, N$. Since the results in Step 1 involve coupling agents, we need to make a global analysis for all the agents, which establishes further $||X(t)|| \leq L_t$, where L_t satisfies an linear inequality. In Step 3, based on results in Step 2, we shall give order estimate of L_t, which is critical in the whole proof. In the last step, we shall analyze each agent again so as to give estimates of $r_i(t)$ ($i = 1, 2, \cdots, N$) in terms of $\bar{r}(t)$, which in turn establishes a key relationship on order of $\bar{r}(t)$ and then accurate estimate $\bar{r}(t) = O(t)$ can be obtained, from which validity of Eq. (27) can be deduced finally.

Step 1. For Agent i, $i = 1, 2, \cdots, N$, by putting the control law Eq. (14) into Eq. (1), we have

$$\begin{aligned} x_i(t+1) &= \boldsymbol{\phi}_i^\tau(t)\boldsymbol{\theta}_i + (\boldsymbol{\phi}_i^0(t) - \boldsymbol{\phi}_i(t))^\tau\boldsymbol{\theta}_i + w_i(t+1) \\ &= x_i^*(t+1) - \boldsymbol{\phi}_i^\tau(t)\hat{\theta}_i(t) + \boldsymbol{\phi}_i^\tau(t)\boldsymbol{\theta}_i \\ &\quad +(\boldsymbol{\phi}_i^0(t) - \boldsymbol{\phi}_i(t))^\tau\boldsymbol{\theta}_i + w_i(t+1) \\ &= x_i^*(t+1) + \boldsymbol{\phi}_i^\tau(t)\tilde{\theta}_i(t) \\ &\quad +(\boldsymbol{\phi}_i^0(t) - \boldsymbol{\phi}_i(t))^\tau\boldsymbol{\theta}_i + w_i(t+1) \end{aligned} \tag{29}$$

Then, by Lemma 3.1 and Eq. (13),

$$\begin{aligned} &(\boldsymbol{\phi}_i^\tau(t)\tilde{\theta}_i(t))^2 \\ &= \beta_i(t)[\alpha_i^{-1}(t) + \boldsymbol{\phi}_i^\tau(t)P_i(t)\boldsymbol{\phi}_i(t)] \\ &= \beta_i(t)[\alpha_i^{-1}(t) + \boldsymbol{\phi}_i^\tau(t)P_i(t+1)\boldsymbol{\phi}_i(t) \\ &\quad +\boldsymbol{\phi}_i^\tau(t)(P_i(t) - P_i(t+1))\boldsymbol{\phi}_i(t)] \\ &\leq \beta_i(t)[2\alpha_i^{-1}(t) + \delta_i(t)||\boldsymbol{\phi}_i(t)||^2]; \end{aligned} \tag{30}$$

$$\begin{aligned} &||\boldsymbol{\phi}_i^0(t) - \boldsymbol{\phi}_i(t)||^2 \\ &= O(\sum_{k=1}^{t} |\hat{w}_i(k) - w_i(k)|^2) \\ &= O(\sum_{k=1}^{t} \alpha_i(k)|\hat{w}_i(k) - w_i(k)|^2\alpha_i^{-1}(k)) \\ &= O(h(r_i(t))); \end{aligned} \tag{31}$$

And by the definition of $\beta_i(t)$ and Lemma 3.1, we have $\sum_{k=1}^{\infty} \beta_i(k) < \infty$, hence $\beta_i(t) \to 0$ as $t \to \infty$. Then

$$\begin{aligned} |x_i(t+1)|^2 &\leq 2\beta_i(t)\delta_i(t)||\boldsymbol{\phi}_i(t)||^2 \\ &\quad +O(h(r_i(t))) + O(d_i(t)). \end{aligned} \tag{32}$$

Then, by the minimum phase condition, there exists $\lambda_i \in (0, 1)$ such that

$$\begin{aligned} |u_i(t)|^2 &= O(\sum_{k=0}^{t+1} \lambda_i^{t+1-k}(|x_i(k)|^2 + ||\bar{\boldsymbol{X}}_i(k)||^2 \\ &\quad +|w_i(k+1)|^2)). \end{aligned} \tag{33}$$

This fact holds for all $i = 1, 2, \cdots, N$. Note that in this step, we cannot obtain order estimate of $|u_i(t)|^2$ directly because Eq. (33) involves $\{x_j(k), j \in \mathcal{N}_i\}$ in $\bar{\boldsymbol{X}}_i(k)$, whose order estimates are not available yet.

Step 2. To estimate $|u_i(t)|^2$, here we apply similar idea as in [23], [24]. By the definition of $\boldsymbol{X}(k)$, obviously we can get

$$|x_i(k)|^2 = O(||\boldsymbol{X}(k)||^2), ||\bar{\boldsymbol{X}}_i(k)||^2 = O(||\boldsymbol{X}(k)||^2). \tag{34}$$

Take $\rho = \max\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$. Now define

$$L_t \triangleq \sum_{k=0}^{t} \rho^{t-k}||\boldsymbol{X}(k)||^2 \tag{35}$$

Then we can obtain that

$$\begin{aligned} |u_i(t)|^2 &= O(L_{t+1}) + O(\sum_{k=0}^{t+1} \rho^{t+1-k}\bar{d}(k)) \\ &= O(L_{t+1}) + O(\bar{d}(t+1)). \end{aligned} \tag{36}$$

Also by the definition of $\boldsymbol{\phi}_i(t)$, together with

$$\hat{w}_i(k) = (\hat{w}_i(k) - w_i(k)) + w_i(k) \tag{37}$$

we can obtain that

$$\begin{aligned} ||\boldsymbol{\phi}_i(t)||^2 &= O(||\boldsymbol{X}(t)||^2) + O(L_t) + O(\bar{d}(t)) \\ &\quad +O(h(r_i(t)) + d_i(t)) \\ &= O(L_t + h(\bar{r}(t)) + \bar{d}(t)) \end{aligned} \tag{38}$$

where

$$\bar{r}(t) \triangleq \max(r_1(t), r_2(t), \cdots, r_N(t)). \tag{39}$$

Hence from the above, by Eq. (32), for Agent i, there exists a constant $C_i > 0$ such that

$$\begin{aligned} |x_i(t+1)|^2 &\leq C_i\beta_i(t)\delta_i(t)L_t \\ &\quad +O(\beta_i(t)\delta_i(t)[h(\bar{r}(t)) + \bar{d}(t)]) \\ &\quad +O(d_i(t) + h(r_i(t))). \end{aligned} \tag{40}$$

Then, by the definition of $\delta_i(t)$, noting that

$$\sum_{k=1}^{\infty} \delta_i(k) = \sum_{k=1}^{\infty} \mathrm{tr}(P_i(t) - P_i(t+1)) < \infty$$

we can get $\delta_i(t) \to 0$ as $t \to \infty$. Since $\beta_i(t)\delta_i(t) \to 0$ as $t \to \infty$, we have

$$|x_i(t+1)|^2 \leq C_i\beta_i(t)\delta_i(t)L_t + O(h(\bar{r}(t)) + \bar{d}(t)) \tag{41}$$

for $i = 1, 2, \cdots, N$.

Thus

$$\begin{aligned} ||\boldsymbol{X}(t+1)||^2 &= \sum_{i=1}^{N} |x_i(t+1)|^2 \\ &\leq [\sum_{i=1}^{N} C_i\beta_i(t)\delta_i(t)]L_t \\ &\quad +O(h(\bar{r}(t))+\bar{d}(t)). \end{aligned} \tag{42}$$

By the definition of L_t, we have

$$\begin{aligned} L_{t+1} &= \rho L_t + ||\boldsymbol{X}(t+1)||^2 \\ &\leq [\rho + C\sum_{i=1}^{N} \beta_i(t)\delta_i(t)]L_t \\ &\quad +O(h(\bar{r}(t))+\bar{d}(t)) \end{aligned} \tag{43}$$

where

$$C = \max(C_1, C_2, \cdots, C_N). \tag{44}$$

Step 3. Since $\rho \in (0,1)$, $\beta_i(t)\delta_i(t) \to 0$ as $t \to \infty$, together with Eq. (43), we know that for any $\varepsilon > 0$, there exists sufficient large $t_0 > 0$ such that

$$\rho + C\sum_{k=t}^{\infty} \beta_i(k)\delta_i(k) \in (\rho, \rho+\varepsilon) \quad \forall t \geq t_0 \tag{45}$$

By taking $\varepsilon \in (0, 1-\rho)$, we can get

$$L_{t+1} \leq (\rho+\varepsilon)L_t + O(h(\bar{r}(t))+\bar{d}(t)) \tag{46}$$

Then, by iterating the above inequality, we obtain that

$$L_{t+1} = O(h(\bar{r}(t))+\bar{d}(t)) \tag{47}$$

Thus, by the definition of L_t, we can obtain that

$$||\boldsymbol{X}(t+1)||^2 \leq L_{t+1} = O(h(\bar{r}(t))+\bar{d}(t)) \quad a.s. \tag{48}$$

Consequently, by Eq. (33) and Eq. (38), we have

$$\begin{aligned} |u_i(t)|^2 &= O(h(\bar{r}(t))+\bar{d}(t)) \\ ||\boldsymbol{\phi}_i(t)||^2 &= O(h(\bar{r}(t))+\bar{d}(t)). \end{aligned} \tag{49}$$

Step 4. By using Eq. (30), Eq. (31) and Eq. (49), we have

$$\begin{aligned} R_i(t) &= \sum_{k=0}^{t} |x_i(k+1) - x_i^*(k+1) - w_i(k+1)|^2 \\ &= \sum_{k=0}^{t} (\boldsymbol{\phi}_i^\tau(k)\tilde{\theta}_i(k) + (\boldsymbol{\phi}_i^0(k) - \boldsymbol{\phi}_i(k))^\tau \boldsymbol{\theta}_i)^2 \\ &= \sum_{k=0}^{t} (\beta_i(k)\delta_i(k)||\boldsymbol{\phi}_i(k)||^2) + O(\alpha_i^{-1}(t)) \\ &= O(\max_{0\leq k\leq t} ||\boldsymbol{\phi}_i(k)||^2) + O(\alpha_i^{-1}(t)) \\ &= O(h(\bar{r}(t))+\bar{d}(t)) \quad a.s. \end{aligned} \tag{50}$$

Therefore, to prove Eq. (27), we need only to prove $\bar{r}(t) = O(t)$. In fact, by Assumption 3.1, we have

$$\sum_{k=0}^{t} |w_i(k+1)|^2 = O(t). \tag{51}$$

Noting that $\bar{d}(t) = O(t)$, $x_i^*(t) = O(1)$, by Eq. (50), we obtain that

$$\begin{aligned} &\sum_{k=0}^{t} |x_i(k+1)|^2 \\ \leq\ & R_i(t) + \sum_{k=0}^{t} |w_i(k+1)|^2 + \sum_{k=0}^{t} |x_i^*(k+1)|^2 \\ =\ & O(t) + O(h(\bar{r}(t))). \end{aligned} \tag{52}$$

Then, by Assumption 3.2, we can obtain that

$$\sum_{k=0}^{t} (u_i(k))^2 = O(t) + O(h(\bar{r}(t))) \tag{53}$$

and

$$\begin{aligned} \sum_{k=0}^{t} (\hat{w}_i(k))^2 &= \sum_{k=0}^{t} ((\hat{w}_i(k) - w_i(k)) + w_i(k))^2 \\ &\leq 2\sum_{k=0}^{t} [(\hat{w}_i(k) - w_i(k))^2 + (w_i(k))^2] \\ &= O(h(\bar{r}(t))) + O(t) \quad a.s. \end{aligned} \tag{54}$$

Therefore, by the definition of $\bar{r}(t)$, $\boldsymbol{\phi}_i(t)$ and together with Eq. (13), we can obtain

$$\begin{aligned} \bar{r}(t) &= \max\{r_i(t),\ 1 \leq i \leq N\} \\ &= \max\{||P_i^{-1}(0)|| + \sum_{k=0}^{t} ||\boldsymbol{\phi}_i(k)||^2,\ 1 \leq i \leq N\} \\ &= O(h(\bar{r}(t))) + O(t). \end{aligned} \tag{55}$$

Finally, from the definition of $h(\cdot)$, we can obtain that $\bar{r}(t) = O(t)$. This completes the proof of the theorem. □

V. Simulation Study

For the decentralized adaptive controller based on decentralized WLS algorithm, we have given a rigorous proof of the closed-loop system stability and optimality in last section. In this section, we shall illustrate several simulation results to verify the theoretical results obtained.

In the simulation, for simplicity, the number of agents is taken as $N = 4$ and parameters of agents are randomly taken such that Assumptions 3.2 and 3.4 satisfied. And the noise sequences $w_i(t)$ $(i = 1, 2, \cdots, N)$ are all i.i.d. taken from Gaussian distribution $N(0, 1)$, which satisfy $|w_i(t)|^2 = O(t)$ a.s. The objective of the control for Agent i $(i = 1, 2, \cdots, N)$ is to track a local reference signal $x_i^*(t) = 10\sin\frac{\pi t}{20} + 5\sin\frac{\pi t}{10}$.

In the simulation, the parameters of agents are shown in Table I, and to save place, only the results of the first two agents are illustrated in Fig. 1 and Fig. 2, respectively. In each figure, the states $x_i(t)$, the reference signals $x_i^*(t)$ and the tracking errors $e_i(t)$ are plotted in the top-left subfigure; the accumulated squared errors, the control $u_i(t)$ and the driving noise $w_i(t)$ are plotted in the top-right subfigure, bottom-left subfigure, and bottom-right subfigure, respectively.

From the simulation results, we can see that, by using the local adaptive controllers, all agents can track their local reference signals successfully, which are consistent with our theoretical results in Theorem 4.1.

VI. Conclusion

In this paper, we have studied an adaptive control problem for multi-agent complex dynamical system with various uncertainties, where the agents are supposed to use decentralized weighted least-square (WLS) algorithm as the local estimators of parametric uncertainties and the agents can design their local adaptive controllers based on decentralized WLS algorithm by "certainty equivalence" principle. Generally speaking, it is difficult to analyze such problems because

TABLE I
SIMULATION SETTINGS

Agent No.	$\boldsymbol{a}_i$	$\boldsymbol{b}_i$	$\boldsymbol{c}_i$	$\boldsymbol{g}_i$	$\mathcal{N}_i$
1	-0.861724 -0.183611	-1.963973 -0.407137	0.932109	0.679022 0.544034	4 2
2	-0.516674 0.495204	-1.080332 0.713995	0.413996	0.968700 0.148386	3 4
3	-1.258505 -2.246897	1.399799 -0.451906	0.531898	0.220984 0.035710	3 1
4	0.603717 0.578080	1.063012 0.341797	0.680615	0.203578 0.552013	1 2

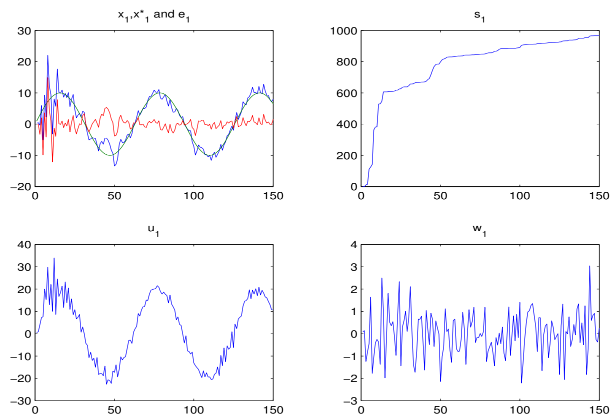

Fig. 1. Simulation result for agent 1

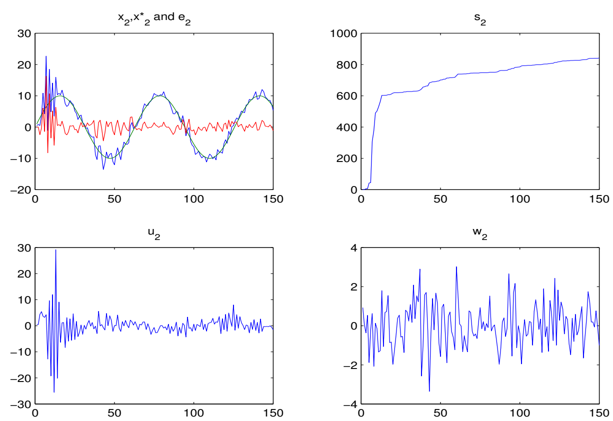

Fig. 2. Simulation result for agent 2

of the complexity caused by existence of the couplings between the agents and the high nonlinearity of the closed-loop system. For the sake of theoretical analysis, basic properties of WLS algorithm are utilized in this paper as key technical tools. The self-convergence property of WLS algorithm brings certain benefits for the WLS-based decentralized adaptive controller. To resolve the complex coupling problem involved, the philosophy of individual analysis plus global analysis is adopted to yield a rigorous proof on stability and optimality of closed-loop system. And the simulation examples also verified our theoretical results.

REFERENCES

[1] R. Albert, H. Jeong, and A. L. Barabäsi. Error and attack tolerance of complex networks. *Nature*, 406:378–382, 2000.

[2] M. E. J. Newman. The structure and function of complex networks. *SIAM Rev.*, 45:167–256, 2003.

[3] E. Ravasz and A.-L. Barabäsi. Hierarchical organization in complex networks. *Phys. Rev. E*, 67:026112., 2003.

[4] S. H. Strogatz. Exploring complex network. *Nature*, 410:268–276, 2001.

[5] J.H. Holland. *Hidden Order: How Adaptation Builds Complexity*. Addison-Wesley, New York, 1996.

[6] Y. D. Landau. *Adaptive control: The model reference approach*. Dekker, New York, 1979.

[7] G. Goodwin and K. Sin. *Adaptive Filtering, Prediction and Control*. Prentice-Hall, Englewood Cliffs, NJ, 1984.

[8] K.J. Aström and B. Wittenmark. *Adaptive Control*. Addison-Wesley Pupl. Comp., 1989.

[9] P. A. Ioannou and J. Sun. *Robust adaptive control*. Prentice Hall, Englewood, Cliffs, NJ, 1996.

[10] H. F. Chen and L. Guo. *Identification and Stochastic Adaptive Control*. Birkhäuser, Boston, MA, 1991.

[11] C. Wen. Decentralized robust control of class unknown interconnected systems. *Automatica*, 30(3):543–544, 1994.

[12] G. H. Yang and S. Y. Zhang. Decentralized robust control for interconnected systems with time-varying uncertainties. *Automatica*, 31(11):1603–1608, 1996.

[13] Y. Y. Wang, G. X. Guo, and D. J. Hill. Robust decentralized nonlinear controller design for interconnected systems for multimachine power systems. *Automatica*, 33(9):1725–1733, 2007.

[14] M. S. Mahmoud and S. Bingulac. Robust design of stabilizing controllers for interconnected time-delay systems. *Automatica*, 34(6):795–800, 1998.

[15] Z. H. Guan, G. R. Chen, X. H. Yu, and Y. Qin. Robust decentralized stabilization for a class of large-scale time-delay uncertain impulsive dynamical systems. *Automatica*, 28:2075–2084, 2002.

[16] P. Ioannou. Decentralized adaptive control of interconnected systems. *IEEE Transactions on Automatic Control*, 31:291–298, 1986.

[17] D. J. Hill, C. Wen, and G. C. Goodwin. Stability analysis of decentralised robust adaptive control. *Syst. Control Lett.*, 11(4):277–284, 1988.

[18] D. T. Gavel and D. D. Siljak. Decentralized adaptive control: structural conditions for stability. *IEEE Transactions on Automatic Control*, 34:413–26, 1989.

[19] C. Wen and D. J. Hill. Global boundedness of discrete-time adaptive control just using estimator projection. *Automatica*, 28:1143–1157, 1992.

[20] S. J. Liu, J. F. Zhang, and Z. P. Jiang. Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems. *Automatica*, 43(2):238–251, 2007.

[21] C. Wen and J. Zhou. Decentralized adaptive stabilization in the presence of unknown backlash-like hysteresis. *Automatica*, 43(3):426–440, 2007.

[22] H. B. Ma. Decentralized Adaptive Synchronization of a Stochastic Discrete-Time Multiagent Dynamic Model. *SIAM Journal on Control and Optimization*, 48(2):859-880, 2009.

[23] H. B. Ma. *Capability and Limitation of Feedback Mechanism in Dealing with Uncertainties of Some Discrete-time Control Systems*. Phd thesis, Graduate School of Chinese Academy of Sciences, Beijing, China, June 2006.

[24] H. B. Ma, K. Y. Lum, and S. S. Ge. Decentralized Åström-Wittenmark self-tuning regulator of a multi-agent uncertain coupled armax system. In *Proceedings of the 2007 IEEE Multi-conference on Systems and Control*, Suntec City Convention Centre, Singapore, October 1-3 2007.

[25] L. Guo. Self-convergence of weighted least-squares with applications to stochastic adaptive control. *IEEE Transactions on Automatic Control*, 41(1):79–89, January 1996.